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S-LIMITED SHIFTS

Abstract

In this paper, we explore the construction and dynamical properties of \mathcal{S} -limited shifts. An \mathcal{S} -limited shift is a subshift defined on a finite alphabet $\mathcal{A} = \{1, \dots, p\}$ by a set $\mathcal{S} = \{S_1, \dots, S_p\}$, where $S_i \subseteq \mathbb{N}$ describes the allowable lengths of blocks in which the corresponding letter may appear. We give conditions for which an \mathcal{S} -limited shift is a subshift of finite type or sofic. We give an exact formula for finding the entropy of such a shift and show that an \mathcal{S} -limited shift and its factors must be intrinsically ergodic. Finally, we give some conditions for which two such shifts can be conjugate, and additional information about conjugate \mathcal{S} -limited shifts.

1 Introduction

The use of symbolic spaces started in the study of geodesics on surfaces of negative curvature. Symbolic spaces are often used to describe the behavior of discrete dynamical systems. Symbolic spaces on the alphabet $\mathcal{A} = \{0, 1\}$ gave rise to the study of information theory and are often used to analyze data storage and transmission. With its rise in popularity, the study of symbolic dynamics has grown as a math field of its own, in which one investigates the properties and behavior of a space of infinite strings from a finite alphabet equipped with a shift map. Various restrictions on the types of strings included in the space lead us to the study of different classes of subshifts.

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One particular class of subshifts, called the S -gap shifts, are defined on the alphabet $\mathcal{A} = \{0, 1\}$ by a set $S \subseteq \mathbb{N}_0$ (where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$), which describes the allowable number of 0s that can separate two 1s in a string in the space. S -gap shifts and their dynamical properties have been and continue to be studied thoroughly and used in applications ([5], [6], [1]). In [6], Jung proved that an S -gap shift $X(S)$ is mixing if and only if $\gcd\{n+1 : n \in S\} = 1$, along with other necessary and sufficient conditions for which $X(S)$ satisfies various specification properties. In [1], Dastjerdi and Jangjoo established a collection of dynamical and topological properties of S -gap shifts, including a characterization of the S -gap shifts that are subshifts of finite type or sofic. In [2], Dastjerdi and Jangjooye introduced a broader class of shift spaces, called (S, S') -gap shifts, which again are defined on the alphabet $\{0, 1\}$ by two sets $S, S' \subseteq \mathbb{N}_0$, which define the allowable lengths of the blocks of 0s and 1s, respectively. They established many properties of this class of shift spaces, including results about the entropy of an (S, S') -gap shift, and some specific conditions for conjugacy.

In this paper, we investigate a new class of shift spaces, called \mathcal{S} -limited shifts, which includes all S -gap and (S, S') -gap shifts. These subshifts are defined on an alphabet $\{1, \dots, p\}$ by a finite set $\mathcal{S} = \{S_1, \dots, S_p\}$ with $S_i \subseteq \mathbb{N}$ for $1 \leq i \leq p$, and each S_i describes the allowable lengths of blocks of the corresponding letter in a string in the shift space. For the majority of the paper, we restrict the order in which the blocks may appear. In the more general setting in which the order is unrestricted, we refer to the shift as a generalized \mathcal{S} -limited shift. In both cases, we study important dynamical properties of these shift spaces. In particular, we prove the following result about the entropy of an \mathcal{S} -limited shift.

Theorem. *Let $\mathcal{S} = \{S_1, S_2, \dots, S_p\}$ such that $S_i \subseteq \mathbb{N}$ for $1 \leq i \leq p$. Then, the entropy of the \mathcal{S} -limited shift $X(\mathcal{S})$ is $\log \lambda$, where λ is the unique positive solution to*

$$\sum_{\omega \in G_{\mathcal{S}}} x^{-|\omega|} = 1,$$

where $G_{\mathcal{S}} = \{1^{m_1} 2^{m_2} \dots p^{m_p} : m_i \in S_i \text{ for } 1 \leq i \leq p\}$.

We note that this theorem is an extension of a result in [2], which states that the entropy of an (S, S') -gap shift is given by $\log(\lambda)$, where λ is the unique non-negative solution to

$$\sum_{s+s' \in \{\{S+S'\}\}} x^{-(s+s'+2)} = 1,$$

where $\{\!\{S + S'\}\!\} = \{s + s' : s \in S, s' \in S'\}$ and values of multiplicities are included (that is, if $s_1 + s'_1 = s_2 + s'_2$ but $s_1 \neq s_2$ then both $s_1 + s'_1$ and $s_2 + s'_2$ are included in $\{\!\{S + S'\}\!\}$). Their method of proof requires a Fischer cover, a concept that we will not use in this paper. Instead, we use a combinatorial approach with ideas similar to those used in [9] and an approach for S -gap shifts posted online by V. Climenhaga in 2014.

Next, we obtain the following results regarding conjugacy between two S -limited shifts.

Theorem. *Let $\mathcal{S} = \{S_1, \dots, S_p\}$ and $\mathcal{T} = \{T_1, \dots, T_q\}$ and suppose $X(\mathcal{S})$ is conjugate to $X(\mathcal{T})$. Then, where $G_{\mathcal{S}}$ and $G_{\mathcal{T}}$ denote the collections $\{1^{s_1} \dots p^{s_p} : s_i \in S_i\}$ and $\{1^{t_1} \dots q^{t_q} : t_i \in T_i\}$ respectively, for all $l \in \mathbb{N}$, $|\{x \in G_{\mathcal{S}} : |x| = l\}| = |\{y \in G_{\mathcal{T}} : |y| = l\}|$.*

Theorem. *Let $\mathcal{S} = \{S_1, \dots, S_p\}$ and $\mathcal{T} = \{T_1, \dots, T_p\}$ where for all i , $S_i, T_i \subseteq \mathbb{N}$. Let s_i^m denote the m -th element of S_i sorted in increasing order, and define t_i^m similarly. If for all $i_1, i_2, \dots, i_p \in \mathbb{N}$,*

$$\sum_{k=1}^p s_k^{i_k} = \sum_{k=1}^p t_k^{i_k},$$

then $X(\mathcal{S})$ is conjugate to $X(\mathcal{T})$.

Both theorems are generalizations of existing results for (S, S') -gap shifts from [2]. The former is a generalization of a theorem that states if $X(S, S')$ and $X(T, T')$ are conjugate (S, S') -gap shifts, then $\{\!\{S + S'\}\!\} = \{\!\{T + T'\}\!\}$. The latter is a generalization of a theorem that assumes $\{\!\{S + S'\}\!\} = \{\!\{T + T'\}\!\}$ and gives specific cases for which $X(S, S')$ is conjugate to $X(T, T')$. While the overall approach to our proofs of these theorems is similar to that taken in [2]—that being, splitting elements of the potentially conjugate subshifts into finite words and then providing a sliding block code that maps between those finite words from the two subshifts—the added complexity of having more than 2 letters in the alphabet required new techniques to yield the generalized result. However, for the case of $p = 2$, our techniques would lead to the same sliding block code that is used to prove the conjugacy of (S, S') -gap shifts.

2 Background and Definitions

Let $\mathcal{A} = \{1, \dots, p\}$ be an alphabet, and let X be a shift space on \mathcal{A} . Define $B_n(X) = \{\omega_1 \dots \omega_n : \omega_i \in \mathcal{A} \text{ for } 1 \leq i \leq n\}$ to be the collection of all words

of length n and let $\mathcal{L}(X) = \bigcup_{n \geq 1} B_n(X)$ denote the **language** of X . If the shift space is understood in context, we will use B_n and \mathcal{L} . A shift space X is called **irreducible** if for all $\omega, \tau \in \mathcal{L}(X)$, there exists some $\xi \in \mathcal{L}(X)$ such that $\omega\xi\tau \in \mathcal{L}(X)$. A word $\xi \in \mathcal{L}(X)$ is called **synchronizing** if for any $\omega\xi, \xi\tau \in \mathcal{L}(X)$, the word $\omega\xi\tau \in \mathcal{L}(X)$. An irreducible shift space X with a synchronizing word is called a **synchronized system**. A shift space X is called **mixing** if for all $\omega, \tau \in \mathcal{L}(X)$, there exists some N such that for each $n \geq N$, there is a $\xi \in B_n(X)$ such that $\omega\xi\tau \in \mathcal{L}(X)$.

Let X denote the full shift on \mathcal{A} . A **subshift of finite type (SFT)** is a subshift $X_F \subseteq X$ defined by a finite collection $F \subset \mathcal{L}(X)$ of forbidden words, meaning that

$X_F = \{\omega \in X : \tau \text{ appears nowhere in } \omega \text{ for all } \tau \in F\}$. For any given SFT, the list of forbidden words define both an adjacency matrix and finite graph presentation that represent the shift space. The finite graph presentation consists of finitely many vertices, and infinite walks correspond to infinite strings in the shift space. Similarly, one can consider a broader class, called **sofic** subshifts, which includes all subshifts that have a finite graph presentation. As an adjacency matrix can be defined from a finite graph, each sofic subshift has an associated adjacency matrix. For more information on SFTs and sofic subshifts, see [3].

One of the simplest constructions of a subshift that is neither sofic nor an SFT on the alphabet $\mathcal{A} = \{0, 1\}$ is the prime gap shift, which includes all strings in which any two adjacent 1s are separated by a prime number of 0s. This shift space is an example of an S -gap shift. An **S -gap shift** is a subshift of the form

$$X_S = \overline{\{\dots 10^{n-1} 10^{n_0} 10^{n_1} 1\dots : n_i \in S\}},$$

where $S \subseteq \mathbb{N}_0$. While the prime gap shift provides an example of a non-sofic subshift, there exist S -gap shifts that are SFTs or sofic as well. However, we cannot guarantee the existence of a finite graph presentation or adjacency matrix for a general S -gap shift, and hence new techniques must be used for exploring properties of these shift spaces. A more general class of subshifts that includes all S -gap shifts is the (S, S') -gap shifts. An **(S, S') -gap shift** is a subshift on the alphabet $\mathcal{A} = \{0, 1\}$ of the form:

$$X(S, S') = \overline{\{\dots 1^{m-2} 0^{n-1} 1^{m-1} 0^{n_0} 1^{m_0} 0^{n_1} 1^{m_1} \dots : n_i \in S, m_i \in S' \text{ for all } i \in \mathbb{Z}\}},$$

where $S, S' \subseteq \mathbb{N}$. Again, it is the case that (S, S') -gap shifts can be SFT, sofic, or neither. To clarify some of these characterizations, we consider the following examples.

Example 2.1 (Golden mean shift). Let $\mathcal{A} = \{0, 1\}$. The golden mean shift is an SFT with forbidden word list $F = \{11\}$. It has the following graph presentation and adjacency matrix:



The golden mean shift is also an S -gap shift with $S = \mathbb{N}$.

Example 2.2 (Even Shift). Let $\mathcal{A} = \{0, 1\}$. The even shift is a sofic subshift that contains all infinite strings in which any two adjacent 1s are separated by an even number of 0s. It has the following graph presentation and adjacency matrix:



It is an S -gap shift with $S = \{2n : n \in \mathbb{N}_0\}$.

For the remainder of this paper, unless otherwise specified, we fix our alphabet $\mathcal{A} = \{1, 2, \dots, p\}$. Building upon the ideas of an S -gap and (S, S') -gap shift, we arrive at our definition of an \mathcal{S} -limited shift. As a preliminary step, we will introduce our collection of limiting sets $\mathcal{S} = \{S_1, \dots, S_p\}$ with $S_i \subseteq \mathbb{N}$ for $1 \leq i \leq p$ and a collection of finite blocks

$$G_{\mathcal{S}} = \{1^{m_1} 2^{m_2} \dots p^{m_p} : m_i \in S_i \text{ for } 1 \leq i \leq p\}$$

called the **core set**. An **\mathcal{S} -limited shift** is a subshift on \mathcal{A} defined by a shift-invariant set of the form

$$X(\mathcal{S}) = \overline{\{\dots x_{-1} x_0 x_1 \dots : x_i \in G_{\mathcal{S}} \text{ for all } i \in \mathbb{Z}\}}$$

and equipped with the shift map. In the case that any of the S_i are infinite, we note that taking the closure is a non-trivial step in defining the subshift, as we now consider bi-infinite strings ending or beginning with an infinite string of a single letter.

While most of our results concern \mathcal{S} -limited shifts, one can extend a few results to a more general setting. So far, we have been restricting the order in

which the blocks of letters can appear in \mathcal{S} -limited shifts. Let \mathcal{A} and \mathcal{S} be as above. A **generalized \mathcal{S} -limited shift**, denoted $X_{\mathcal{S}}$, is a subshift defined by the shift-invariant closure of a set of the form

$$\{\dots\omega_{-1}^{\alpha_{-1}}\omega_0^{\alpha_0}\omega_1^{\alpha_1}\dots \mid \omega_i \in \mathcal{A}, \omega_i \neq \omega_{i+1}, \text{ and } \alpha_i \in S_{\omega_i} \text{ for all } i\}.$$

The first difference between \mathcal{S} -limited shifts and generalized \mathcal{S} -limited shifts is that the full shift on p letters is a generalized \mathcal{S} -limited shift with $S_i = \mathbb{N}$ for $1 \leq i \leq p$. However, for $p > 2$, the full shift is not an \mathcal{S} -limited shift. The most significant difference between these two classes of shift spaces is the absence of an analogous core set $G_{\mathcal{S}}$ in the case of a generalized \mathcal{S} -limited shift.

As mentioned in the introduction, we will discuss the entropy of \mathcal{S} -limited shifts. Although the definition of entropy is more complex in a general setting (see [7] for details), the definition reduces nicely in the symbolic dynamics setting. The **(topological) entropy** of a subshift X , denoted by $h(X)$, is given by

$$h(X) = \lim_{n \rightarrow \infty} \frac{\log(B_n(X))}{n}.$$

Entropy can be used to define another class of subshifts. A subshift X is **almost sofic** if given $\varepsilon > 0$, there exists an SFT $Y \subset X$ such that $h(Y) > h(X) - \varepsilon$. We are using the definition from [8], but analogous definitions are used in other sources, such as [3].

Entropy is widely used as a conjugacy invariant in the study of dynamical systems. We say that two subshifts X and Y are conjugate if there exists a homeomorphism $\varphi : X \rightarrow Y$ such that $\varphi \circ \sigma = \sigma \circ \varphi$, where σ denotes the shift map. In searching for conjugacy between two shift spaces, we often look for an invertible sliding block code. A **sliding block code** with memory m and anticipation n is a map $\varphi : X \rightarrow \mathcal{A}^{\mathbb{Z}}$ defined by $y = \varphi(x)$ with $y_i = \Phi(x_{[i-m, i+n]})$, where $\Phi : \mathcal{B}_{m+n+1}(X) \rightarrow \mathcal{A}$ and $x_{[i-m, i+n]} = x_{i-m}x_{i-m+1} \cdots x_{i+n} \in \mathcal{B}_{m+n+1}(X)$.

3 Dynamical Properties

For the remainder of this paper, we will fix our alphabet $\mathcal{A} = \{1, \dots, p\}$, and let $\mathcal{S} = \{S_1, \dots, S_p\}$ with $S_i \subseteq \mathbb{N}$. We will use $X(\mathcal{S})$ to denote the \mathcal{S} -limited shift and $X_{\mathcal{S}}$ to denote a generalized \mathcal{S} -limited shift. As we first turn our attention to some of the properties of \mathcal{S} -limited shifts, we first note some unique features of these subshifts. In the case that $S_i = \mathbb{N}$ for each $1 \leq i \leq p$, the corresponding \mathcal{S} -limited shift is an SFT with forbidden word

list $F = \{nm : m \neq n, m \neq n + 1, 1 \leq n < p\} \cup \{pi : 1 < i < p\}$. Given that a maximal \mathcal{S} -limited shift is an SFT, we consider the following proposition, which gives necessary and sufficient conditions for which an \mathcal{S} -limited shift is an SFT.

Proposition 3.1. *$X(\mathcal{S})$ is an SFT if and only if S_i is finite or cofinite for every $S_i \in \mathcal{S}$.*

PROOF. First, recall that we are restricting the order in which blocks may appear. Hence,

$$F_0 = \{nm : m \neq n, m \neq n + 1, 1 \leq n < p\} \cup \{pi : 1 < i < p\}$$

is forbidden for any $X(\mathcal{S})$. Next, assume S_i is finite. Then, if S_i is finite,

$$F_i = \{ai^nb : a, b \in \mathcal{A} \setminus \{i\}, n \in \{1, 2, \dots, \max S_i\} \setminus S_i\} \cup \{i^{1+\max S_i}\}$$

is a finite list of forbidden words associated with the set S_i . If, on the other hand, S_i is cofinite, then

$$F_i = \{ai^nb : a, b \in \mathcal{A} \setminus \{i\}, n \in \mathbb{N} \setminus S_i\}$$

is a finite list of forbidden words associated with the set S_i . Hence, $F = \bigcup_{i=0}^n F_i$ is a finite list of forbidden words, and so $X(\mathcal{S})$ is an SFT.

Now, assume that $X(\mathcal{S})$ is an SFT. Notice that if there exists an S_i that is neither finite nor cofinite, then the forbidden word list F_i associated with that set must be infinite. Hence, each $S_i \in \mathcal{S}$ must be finite or cofinite. \square

Proposition 3.2. *A generalized \mathcal{S} -limited shift $X_{\mathcal{S}}$ is an SFT if and only if every S_i is finite or cofinite for every $S_i \in \mathcal{S}$.*

The proof follows exactly as in Proposition 3.1, except we can discard the forbidden word list F_0 since we are no longer restricting the order in which the blocks appear.

Next, we aim to find conditions for which an \mathcal{S} -limited shift will be sofic. As a preliminary step, we introduce some terminology and an alternative definition for sofic subshifts. Let X be a subshift and $\omega \in \mathcal{L}(X)$. The **follower set** of ω , denoted $\mathcal{F}_X(\omega)$ is the set of the form $\mathcal{F}_X(\omega) = \{\tau \in \mathcal{L}(X) : \omega\tau \in \mathcal{L}(X)\}$. A subshift X is sofic if and only if it has a finite number of distinct follower sets. For more information on sofic subshifts, see [3]. We will also use the difference sets associated with each S_i . Let

$$\Delta(S_i) = \{s_0, d_1, d_2, d_3, \dots : d_i = s_i - s_{i-1} \text{ for } i \geq 1\}$$

be the sequence of differences between consecutive entries of S_i for $1 \leq i \leq p$.

Proposition 3.3. *$X(\mathcal{S})$ is sofic if and only if for every $S_i \in \mathcal{S}$, $\Delta(S_i)$ is eventually periodic.*

PROOF. First, notice that a follower set is determined by the last letter of the last block of letters in the finite word. More specifically, for $\omega \in \mathcal{L}(X(\mathcal{S}))$, $i \in \mathcal{A}$, and any $n \in \mathbb{N}$, we can see that $F(\omega ai^n) = F(ai^n)$, where $a = i - 1$ if $2 \leq i \leq p$ and $a = p$ if $i = 1$. Hence, we will only consider follower sets of words of the form ai^n or i .

Now, assume that $\Delta(S_i)$ is eventually periodic. If $S_i = \{s_0, s_1, \dots\}$, then $\Delta(S_i) = \{d_0, d_1, \dots, d_{k-1}, \overline{m_1, \dots, m_l}\}$. Then, assuming $i \in \mathcal{A}$ and $a = i - 1$ when $2 \leq i \leq p$ and $a = p$ when $i = 1$, the follower sets can take three forms:

$$\begin{aligned} F(i), & & 1 \leq i \leq p \\ F(ai^n), & & 1 \leq n \leq s_{k-1} \\ F(ai^{s_{k+i-2}+j_i}), & & 0 \leq j_i \leq m_i - 1 \text{ and } 1 \leq i \leq l \end{aligned}$$

Therefore, given our initial statement regarding the form of follower sets, there can only be at most finitely many follower sets for $X(\mathcal{S})$. Thus, $X(\mathcal{S})$ is sofic. Next, assume that $X(\mathcal{S})$ is sofic and so $X(\mathcal{S})$ has only finitely many follower sets. Then, there must exist some integers m and q with $m < q$ such that $F(ai^m) = F(ai^q)$. Choose the smallest such q for which there exists such an m . Now, let $S_i \cap (m, q] = \{n_1, \dots, n_t\}$ and $s_{max} = \max\{s \in S_i : s \leq m\}$. Then,

$$\Delta(S_i) = \{s_0, s_1 - s_0, \dots, n_1 - s_{max}, \overline{n_2 - n_1, \dots, n_t - n_{t-1}, q + n_1 - n_t}\},$$

and so $\Delta(S_i)$ is eventually periodic. Since this argument is not dependent on the choice of i , we conclude that each $\Delta(S_i)$ must be eventually periodic. □

Proposition 3.4. *A generalized \mathcal{S} -limited shift $X_{\mathcal{S}}$ is sofic if and only if $\Delta(S_i)$ is eventually periodic for every $S_i \in \mathcal{S}$.*

The proof follows from the proof of Proposition 3.3 with one small modification. When considering follower sets of the form $F(ai^n)$, we allow $a \in \mathcal{A} \setminus \{i\}$.

Next, we move on to properties involving the language of an \mathcal{S} -limited shift. First, notice that all \mathcal{S} -limited shifts are irreducible and synchronized with synchronizing words of the form $p1$ or $a(a + 1)$ where $1 \leq a < p$. The following example highlights some of the challenges we face when determining which \mathcal{S} -limited shifts are mixing.

Example 3.5. Let $\mathcal{A} = \{1, 2\}$ and let $S_1 = S_2 = \{2n + 1 : n \geq 0\}$. Notice that $21, 12 \in \mathcal{L}(X(\mathcal{S}))$. If $\omega \in \mathcal{L}(X(\mathcal{S}))$ is a word such that $21\omega 12 \in \mathcal{L}(X(\mathcal{S}))$, then $\ell(\omega) = 2l + 1$ for some $l \geq 0$. Hence, for any $N \geq 1$ and $2n \geq N$, there exists no word $\omega \in B_{2n}(X(\mathcal{S}))$ such that $21\omega 12 \in \mathcal{L}(X(\mathcal{S}))$.

Proposition 3.6. *An \mathcal{S} -limited shift $X(\mathcal{S})$ is mixing if and only if $\gcd\{s_1 + \dots + s_p : s_i \in S_i\} = 1$.*

PROOF. First, we will assume that $X(\mathcal{S})$ is mixing. Since $p1 \in \mathcal{L}(X(\mathcal{S}))$, then there exists some N such that for all $n \geq N$, there is a word $\omega \in B_n(X(\mathcal{S}))$ such that $p1\omega p1 \in \mathcal{L}(X(\mathcal{S}))$. Since ω must be of the form $1^{s_1-1}2^{s_2} \dots p^{s_p}\tau_1\tau_2 \dots \tau_n 1^{s'_1}2^{s'_2} \dots (p-1)^{s_{p-1}}p^{s'_p-1}$, where $\tau_k \in G_{\mathcal{S}}$ for $1 \leq k \leq n$ and $s_i, s'_i \in S_i$ for $1 \leq i \leq p$. Hence, there exist words of this form, say ω_1 and ω_2 , of lengths n and $n+1$, respectively. Thus, there exist words $1\omega_1 p$ and $1\omega_2 p$ of lengths $n+2$ and $n+3$ (respectively), where both $1\omega_1 p$ and $1\omega_2 p$ consist of concatenated blocks from $G_{\mathcal{S}}$. Hence, $\gcd\{s_1 + \dots + s_p : s_i \in S_i\} = 1$.

Next, we assume $\gcd\{s_1 + \dots + s_p : s_i \in S_i\} = 1$. Then, there exists some sufficiently large N such that for all $n \geq N$, there exists a word of length n of the form $\tau_1 \dots \tau_m$, where $\tau_i \in G_{\mathcal{S}}$ for all $1 \leq i \leq m$. Hence, $p\tau_1 \dots \tau_m 1 \in B_{n+2}(X(\mathcal{S}))$. Since $p1$ is synchronizing and $X(\mathcal{S})$ is irreducible, then $X(\mathcal{S})$ must be mixing. □

A generalized \mathcal{S} -limited shift does not have a core set analogous to the set $G_{\mathcal{S}}$ in the \mathcal{S} -limited shift setting. To parallel our use of $G_{\mathcal{S}}$ in the previous proof, we can think of words from $X_{\mathcal{S}}$ in terms of non-repeating blocks, that is, we consider finite words of the form $a_1^{n_1}\tau_1\tau_2 \dots \tau_m a_2^{n_2}$, where $a_1, a_2 \in \mathcal{A}$, $n_1, n_2 \in \mathbb{N}$, and $\tau_i = a_{i1}^{s_{i1}} a_{i2}^{s_{i2}} \dots a_{ik}^{s_{ik}}$, with $a_{ij} \neq a_{il}$ for $j \neq l$ and $s_{ij} \in S_{a_{ij}}$.

Proposition 3.7. *A generalized \mathcal{S} -limited shift $X_{\mathcal{S}}$ is mixing if and only if*

$$\gcd \left\{ \sum_{i=1}^k s_{a_i} : s_{a_i} \in S_{a_i}, a_i \neq a_j (\text{for } i \neq j), 2 \leq k \leq p \right\} = 1.$$

PROOF. First, assume $X_{\mathcal{S}}$ is mixing. Let $a_1, a_2 \in \mathcal{A}$ with $a_1 \neq a_2$. Then, $a_1 a_2 \in \mathcal{L}(X_{\mathcal{S}})$ and hence there exists N such that for all $n \geq N$, there is a word $\omega \in B_n(X_{\mathcal{S}})$ such that $a_1 a_2 \omega a_1 a_2 \in \mathcal{L}(X_{\mathcal{S}})$. Notice that ω must be of the form $\omega = a_2^{s_{a_1}-1} a_1^{s_{a_2}-1}$ or $\omega = a_2^{s_{a_1}-1} \tau_1 \tau_2 \dots \tau_m a_1^{s_{a_2}-1}$ where $\tau_i = a_i^{s_i}$ with $s_i \in S_{a_i}$ for $1 \leq i \leq m$. Hence, there exist words $\omega_1 \in B_n(X_{\mathcal{S}})$ and $\omega_2 \in B_{n+1}(X_{\mathcal{S}})$ of this form. Therefore, there exist words $a_2 \omega_1 a_1 \in B_{n+2}(X_{\mathcal{S}})$ and $a_2 \omega_2 a_1 \in B_{n+3}(X_{\mathcal{S}})$, and each word consists of concatenated blocks of i 's of length $s_i \in S_i$ where $i \in \mathcal{A}$. Since a_1 and a_2 were chosen arbitrarily, then $\gcd \left\{ \sum_{i=1}^k s_{a_i} : s_{a_i} \in S_{a_i}, a_i \neq a_j (\text{for } i \neq j), 2 \leq k \leq p \right\} = 1$.

Next, assume $\gcd \left\{ \sum_{i=1}^k s_{a_i} : s_{a_i} \in S_{a_i}, a_i \neq a_j \text{ (for } i \neq j), 2 \leq k \leq p \right\} = 1$. Then, there exists some sufficiently large N such that for all $n \geq N$, there exists a word $a_1^{s_{a_1}} a_2^{s_{a_2}} \dots a_m^{s_{a_m}} \in B_n(X_S)$, where $a_i \in \mathcal{A}$, $a_i \neq a_{i+1}$ for $1 \leq i < m$ and $s_{a_1} \in S_{a_1}$. Hence, $a_0 a_1^{s_{a_1}} a_2^{s_{a_2}} \dots a_m^{s_{a_m}} a_{m+1} \in B_{n+2}(X_S)$, where $a_0 \neq a_1$ and $a_m \neq a_{m+1}$. Notice that $a_i a_j$ is synchronizing for all $a_i \neq a_j$ with $a_i, a_j \in \mathcal{A}$. Since X_S is irreducible, then X_S must be mixing. □

4 Entropy

In the remaining sections, we will only work with \mathcal{S} -limited shifts. Our current methods would require significant modifications to be extended to the generalized \mathcal{S} -limited shift setting due to the fact that they rely on the existence of a good core set, G_S .

Entropy is a conjugacy invariant that is often sought in symbolic dynamics. While entropy of SFTs and sofic subshifts is well-understood (see [3]) due to the existence of adjacency matrices in these settings, entropy calculations can vary in more general shift space settings. Entropy calculations exist for both S -gap shifts and (S, S') -gap shifts (see [3] and [2]).

Theorem 4.1. *Let $\mathcal{S} = \{S_1, S_2, \dots, S_p\}$ with $S_i \subseteq \mathbb{N}$ for $1 \leq i \leq p$. Then, the entropy of $X(\mathcal{S})$ is $\log \frac{1}{\lambda}$, where λ is the unique positive solution to*

$$\sum_{\omega \in G_S} x^{|\omega|} = 1.$$

PROOF. Consider the generating function $H(z) = \sum_{m=1}^{\infty} (\#B_m)z^m$, where $\#B_m$ denotes the cardinality of $B_m(X(\mathcal{S}))$. First, we claim that the radius of convergence of $H(z)$ is $e^{-h(X(\mathcal{S}))}$.

To see this, we will show that $\lim_{m \rightarrow \infty} \sqrt[m]{\#B_m z^m} = 1$ when $z = e^{-h(X(\mathcal{S}))}$. First, notice that

$$e^{-h(X(\mathcal{S}))} = e^{-\lim_{m \rightarrow \infty} \frac{\log \#B_m}{m}} = \lim_{m \rightarrow \infty} \left(\frac{1}{\#B_m} \right)^{1/m}.$$

By letting $z = e^{-h(X(\mathcal{S}))}$, we obtain

$$\lim_{m \rightarrow \infty} (\#B_m z^m)^{1/m} = \lim_{m \rightarrow \infty} (\#B_m)^{1/m} \left(\frac{1}{\#B_m} \right)^{1/m} = 1.$$

Hence, $e^{-h(X(\mathcal{S}))}$ must be the radius of convergence of $H(z)$. Next, we define the following values which depend on m and k :

$$A_m^k = \#\{\omega \in B_m(X(\mathcal{S})) : \omega = \tau_1\tau_2 \cdots \tau_k, \text{ where } \tau_i \in G_{\mathcal{S}} \text{ for } 1 \leq i \leq k\}.$$

Using this, we can define the collection of functions $F_k(z) = \sum_{m=1}^{\infty} A_m^k z^m$. Notice

that $F_1(z) = \sum_{\omega \in G_{\mathcal{S}}} z^{|\omega|} = 1$. It remains to show that the unique positive

solution to $F_1(z) = 1$ is equal to the radius of convergence of $H(z)$, so that $e^{-h(X(\mathcal{S}))} = \lambda$ and hence $\log(\frac{1}{\lambda}) = h(X(\mathcal{S}))$.

First, we will show that $F_k(z)F_l(z) = F_{k+l}(z)$ for all $k, l \in \mathbb{N}$. Let $m \in \mathbb{N}$ and consider the coefficient on the z^m term of $F_{k+l}(z)$, which corresponds to all words ω formed by concatenating $k+l$ words from $G_{\mathcal{S}}$ such that $\ell(\omega) = m$. Set $\omega = xy$, where x is composed of k concatenated words from $G_{\mathcal{S}}$ and y is composed of l concatenated words from $G_{\mathcal{S}}$. Then, we know that $\ell(y) =$

$m - \ell(x)$. Therefore, the coefficient on z^m in $F_{k+l}(z)$ is $\sum_{i=0}^n A_i^k A_{m-i}^l$, which

is the coefficient on z^m in $F_k(z)F_l(z)$. Therefore, $F_k(z)F_l(z) = F_{k+l}(z)$ as

claimed. From this, we notice $\sum_{k=1}^{\infty} F_k(z) = \sum_{k=1}^{\infty} (F_1(z))^k$, which converges for

all z for which $F_1(z) < 1$, so its radius of convergence is equal to the positive value of z for which $F_1(z) = 1$.

Next, we look to establish a relation between $F_1(z)$ and $H(z)$. While it is clear that $\sum_{k \geq 1} A_m^k \leq \#B_m$, we find that words $\omega \in B_m(X(\mathcal{S}))$ can take

three forms, where in all cases if some word x is not specifically in $G_{\mathcal{S}}$, then it cannot have any subwords in $G_{\mathcal{S}}$:

1. $\omega = x$ where $|x| = n$
2. $\omega \in A_n^k$
3. $\omega = x\tau y$ where $|x| = i$, $|y| = j$, $|\tau| = n - i - j$ and τ is the concatenation of k words in $G_{\mathcal{S}}$

The first question that is raised is, how many allowable words of length m can have no subwords in $G_{\mathcal{S}}$. The first thing to notice is that there must be fewer than $2p$ transitions from one letter to another of a different value, otherwise a substring of $G_{\mathcal{S}}$ must appear since our alphabet contains p letters. Now, we can create an upper bound for the number of such words by summing

over the number of transitions between different letters. There are at most $p \sum_{t=0}^{2p} \binom{m}{t}$ such words, where p corresponds to the number of options for a first letter and t corresponds to the positions for the transitions. Then, we can see that is bounded above by $2p^2m^{2p}$. Now, we define the sequence W_m to be the number of allowable words of length m that have no subwords in $G_{\mathcal{S}}$, and based on this upper bound, we can use the root test to see that the radius of convergence of $\sum_{m=1}^{\infty} W_m z^m$ is at least 1.

Using this, we can construct the following inequality:

$$\sum_{k \geq 1} A_m^k \leq \#B_m \leq W_m + \sum_{k \geq 1} W_k \sum_{i \geq 0} W_i \sum_{j \geq 0} A_{m-i-j}^k.$$

Using that chain of inequalities, we obtain the following inequalities:

$$\sum_{k \geq 1} F_k(z) \leq H(z) \leq \sum_{m \geq 1} W_m z^m + \sum_{k \geq 1} F_k(z) \left(\sum_{i \geq 0} W_i z^i \right) \left(\sum_{j \geq 0} W_j z^j \right).$$

Here, we can see that $H(z)$ will converge if and only if $\sum_{k \geq 1} F_k(z)$ converges, since above we showed that happens only if $F_1(z) < 1$ and all of the other sums on the rightmost side of the inequality have a radius of convergence of 1 so we do not need to worry about their effect on convergence. Therefore, the radius of convergence of $H(z)$ is equal to the unique positive solution to $F_1(z) = 1$, which we have already shown proves our original claim. \square

Now that we have established entropy calculations for \mathcal{S} -limited shifts, we can classify all shifts of this type.

Proposition 4.2. *All \mathcal{S} -limited shifts are almost sofic.*

PROOF. Let $X(\mathcal{S})$ be an \mathcal{S} -limited shift with $\mathcal{S} = \{S_1, \dots, S_p\}$. Consider the \mathcal{S} -limited shift $X(\mathcal{S}|_n)$ defined by $\mathcal{S}|_n = \{S_1|_n, S_2|_n, \dots, S_p|_n\}$, where $S_i|_n$ consists of the first n entries of S_i for $1 \leq i \leq p$. By Proposition 3.1, $X(\mathcal{S}|_n)$ is an SFT. By Theorem 4.1, entropy of an \mathcal{S} -limited shift is given by $\sum_{\omega \in G_{\mathcal{S}}} x^{|\omega|} = 1$, and so we can choose n large enough such that $h(X(\mathcal{S}|_n))$ is arbitrarily close to $h(X(\mathcal{S}))$. \square

As mentioned earlier, (topological) entropy has a simplified definition for symbolic dynamic spaces. The measure-theoretic entropy of σ is denoted by $h_\mu(\sigma)$ where $\mu \in \mathcal{M}_\sigma(X)$ and $\mathcal{M}_\sigma(X)$ denotes the space of shift invariant probability measures on X . While we will not state the full definition of measure-theoretic entropy, we will establish the connection between $h(X)$ and $h_\mu(\sigma)$. The variational principle gives the following relationship:

$$h(X) = \sup\{h_\mu(\sigma) : \mu \in \mathcal{M}_\sigma(X)\}.$$

A shift space X is **intrinsically ergodic** if there exists a unique measure $\mu \in \mathcal{M}_\sigma(X)$ that attains the supremum in the variational principle. For more information on entropy and intrinsically ergodic systems, see [7].

Theorem 4.3. *Every subshift factor of an S -limited shift is intrinsically ergodic.*

PROOF. Consider the following decomposition of $\mathcal{L}(X(S))$:

$$\begin{aligned} G_S^* &= \{\tau_1\tau_2 \dots \tau_n : \tau_i \in G_S \text{ for } 1 \leq i \leq n\} \\ \mathcal{C}^P &= \{l^n(l+1)^{s_{l+1}} \dots p^{s_p} : 1 < l \leq p, s_i \in S_i \text{ for } l < i \leq p, n \in \mathbb{N}\} \\ &\quad \cup \{1^n 2^{s_2} \dots p^{s_p} : n \in \mathbb{N} \setminus S_1, s_i \in S_i \text{ for } 2 \leq i \leq p\} \\ \mathcal{C}^S &= \{1^{s_1} \dots (k-1)^{s_{k-1}} k^n : 1 \leq k \leq p, s_i \in S_i \text{ for } 1 \leq i < k, n \in \mathbb{N}\} \\ &\quad \cup \{1^{s_1} 2^{s_2} \dots (p-1)^{s_{p-1}} p^n : n \in \mathbb{N} \setminus S_p, s_i \in S_i \text{ for } 1 \leq i \leq p-1\} \end{aligned}$$

Notice that $\mathcal{L} = \mathcal{C}^P G_S^* \mathcal{C}^S$, i.e. for every $\omega \in \mathcal{L}$, $\omega = \xi_P \xi_G \xi_S$, where $\xi_P \in \mathcal{C}^P$, $\xi_G \in G_S^*$, and $\xi_S \in \mathcal{C}^S$. In order to use the results from [4], we show that this decomposition satisfies three properties:

- (i) For any $\tau_1, \dots, \tau_m \in G_S^*$, $\tau_1\tau_2 \dots \tau_m \in G_S^*$, so G_S^* has specification.
- (ii) We will show that $\tilde{h}(\mathcal{C}^P \cup \mathcal{C}^S) = 0$, where $\tilde{h}(\cdot)$ denotes the growth rate $\tilde{h}(X) = \frac{1}{n} \log(B_n(X))$. We claim that the cardinalities of both \mathcal{C}^S and \mathcal{C}^P are bounded above by a polynomial of degree 2^{p-2} , where $p = |\mathcal{A}|$. We will calculate the upper bound by considering the case where $S_i = \mathbb{N}$ for all $1 \leq i \leq p$. Notice that when $p = 2$, there is exactly 1 block of n 1's and $(n-1)$ blocks of the form $1^m 2^{n-m}$. So, $|B_n(\mathcal{C}^P)| = n$, and similarly $|B_n(\mathcal{C}^S)| = n$.

If we assume that $|B_n(\mathcal{C}^P)| \approx n^{2^{p-2}}$ for $p > 2$, notice that $|B_n(\mathcal{C}^P)| \approx (n^{2^{p-2}})^2 = n^{2^{p-1}}$ for $(p+1)$ symbols. Hence, our claim holds. Therefore, we obtain

$$\tilde{h}(\mathcal{C}^P \cup \mathcal{C}^S) = \lim_{n \rightarrow \infty} \frac{1}{n} \log(n^{2^{p-2}}) = 0.$$

- (iii) Given our decomposition, for $M \in \mathbb{N}$, define the collections of words $G_{\mathcal{S}}(M)$ to be

$$G_{\mathcal{S}}(M) = \{\xi_P \xi_G \xi_S : \xi_P \in \mathcal{C}^P, \xi_G \in G_{\mathcal{S}}^*, \xi_S \in \mathcal{C}^S, |\xi_P| \leq M, |\xi_S| \leq M\}.$$

Given some $\xi_P \xi_G \xi_S \in G_{\mathcal{S}}(M)$, notice that ξ_P and ξ_S determine the length of word u and v such that $u \xi_P \xi_G \xi_S v \in G_{\mathcal{S}}$. Since there are only finitely many such ξ_P and ξ_S , then there exists some t such that there exists words u and v with $|u| \leq t$ and $|v| \leq t$ for which $u \xi_P \xi_G \xi_S v \in G_{\mathcal{S}}$.

Since our decomposition satisfies these three properties, then by [4], an \mathcal{S} -limited shift and all of its subfactors must be intrinsically ergodic. \square

5 Conjugacy

Next, we address the question of conjugacy of \mathcal{S} -limited shifts through a generalization of the results in [2]. The benefit of restricting the order in which the blocks appear is that we can now split the elements of an \mathcal{S} -limited shift up into its building blocks from $G_{\mathcal{S}}$, since we know that every time we begin a run of 1s a new element of $G_{\mathcal{S}}$ has begun, which is the only time that can happen. Therefore, we can consider elements of \mathcal{S} -limited shifts in terms of the elements of $G_{\mathcal{S}}$ they consist of and the order in which those elements appear.

In the following proof, we consider two \mathcal{S} -limited shifts, say $X(\mathcal{S})$ and $X(\mathcal{T})$. We will denote the core sets associated with the shifts $X(\mathcal{S})$ and $X(\mathcal{T})$ by $G_{\mathcal{S}}$ and $G_{\mathcal{T}}$, respectively.

Theorem 5.1. *Let $\mathcal{S} = \{S_1, \dots, S_p\}$ and $\mathcal{T} = \{T_1, \dots, T_q\}$ and suppose $X(\mathcal{S})$ is conjugate to $X(\mathcal{T})$. Then, for all $l \in \mathbb{N}$,*

$$\left| \{x \in G_{\mathcal{S}} : |x| = l\} \right| = \left| \{y \in G_{\mathcal{T}} : |y| = l\} \right|.$$

PROOF. Let $X(\mathcal{S})$ be conjugate to $X(\mathcal{T})$. Suppose, to the contrary, that there exists some $l \in \mathbb{N}$ such that $\left| \{x \in G_{\mathcal{S}} : |x| = l\} \right| \neq \left| \{y \in G_{\mathcal{T}} : |y| = l\} \right|$. In particular, let l be the smallest such length and suppose without loss of generality that $\left| \{x \in G_{\mathcal{S}} : |x| = l\} \right| > \left| \{y \in G_{\mathcal{T}} : |y| = l\} \right|$. Now, consider the number of points of period l in $X(\mathcal{S})$ and $X(\mathcal{T})$. There must be the same number of periodic points that are built using elements of $G_{\mathcal{S}}$ and $G_{\mathcal{T}}$ of length less than l because we can construct a length-preserving bijection between the elements of $G_{\mathcal{S}}$ and $G_{\mathcal{T}}$ that are shorter than l letters. However, there will be $\left| \{x \in G_{\mathcal{S}} : |x| = l\} \right|$ elements in $X(\mathcal{S})$ of the form \bar{x} where $x \in G_{\mathcal{S}}$ and

$|x| = l$ and $\left| \{y \in G_{\mathcal{T}} : |y| = l\} \right|$ elements in $X(\mathcal{T})$ of the form \bar{y} where $y \in G_{\mathcal{T}}$ and $|y| = l$, and those consist of the remaining words of period l in the two subshifts respectively. Therefore, there are more words of period l in $X(\mathcal{S})$ than there are in $X(\mathcal{T})$. However, conjugacy preserves the number of periodic points in a subshift, so we have a contradiction. Thus, for all $l \in \mathbb{N}$, we must have $\left| \{x \in G_{\mathcal{S}} : |x| = l\} \right| = \left| \{y \in G_{\mathcal{T}} : |y| = l\} \right|$. \square

As shown in [2] for an (S, S') -gap shift (that is, the case where $p = q = 2$), even though this is necessary, it is far from sufficient. Next, we can provide a sufficient case for sufficiency which is a generalization of their result for (S, S') -gap shifts.

Theorem 5.2. *Let $\mathcal{S} = \{S_1, \dots, S_p\}$ and $\mathcal{T} = \{T_1, \dots, T_p\}$ where $S_i, T_i \subseteq \mathbb{N}$ for $1 \leq i \leq p$. Let s_i^m denote the m -th element of S_i sorted in increasing order, and define t_i^m similarly. If for all $i_1, i_2, \dots, i_p \in \mathbb{N}$,*

$$\sum_{k=1}^p s_k^{i_k} = \sum_{k=1}^p t_k^{i_k},$$

then $X(\mathcal{S})$ is conjugate to $X(\mathcal{T})$.

In order to prove this theorem, however, we first need the following result to allow us to—as done in the case for sufficiency—think in terms of elements of $G_{\mathcal{S}}$.

Definition 5.3. Suppose $\mathcal{S} = \{S_1, \dots, S_p\}$ and $\mathcal{T} = \{T_1, \dots, T_q\}$. Then, we say the sliding block code $\varphi : X(\mathcal{S}) \rightarrow X(\mathcal{T})$ induced by the block map Φ of memory m and anticipation n **induces** $\psi : G_{\mathcal{S}} \rightarrow G_{\mathcal{T}}$ if for all x and y of length m and n respectively and for all $z \in G_{\mathcal{S}}$, $\varphi(xzy) = \psi(z)$.

Example 5.4 (Induced Conjugacy). Let $\mathcal{S} = \{S_1, S_2, S_3\}$ where $S_1 = \mathbb{N}$, $S_2 = \{2n : n \in \mathbb{N}\}$, and $S_3 = \{3, 5\}$. Also, let $\mathcal{T} = \{T_1, T_2, T_3\}$ where $T_1 = \mathbb{N}$, $T_2 = \{2n + 1 : n \in \mathbb{N}\}$, and $T_3 = \{2, 4\}$. If we say s_i^m and t_i^m are the m th smallest element of s_i and t_i respectively, we notice that $s_1^m = t_1^m$, $s_2^m + 1 = t_2^m$, and $s_3^m - 1 = t_3^m$. Then, we can see that $\psi : G_{\mathcal{S}} \rightarrow G_{\mathcal{T}}$ where $\psi(1^n 2^m 3^p) = 1^n 2^{m+1} 3^{p-1}$ is a conjugacy. If we take the block code $\Phi : \{1, 2, 3\}^2 \rightarrow \{1, 2, 3\}$ with memory 1 and anticipation 0 where

$$\Phi(xy) = \begin{cases} x & \text{if } xy = 23 \\ y & \text{otherwise} \end{cases}$$

we can see that the sliding block code induced by Φ will induce ψ .

Not all elements of $X(\mathcal{S})$, however, will be the concatenation of elements of $G_{\mathcal{S}}$. Specifically, that will be the case when some element of $X(\mathcal{S})$ begins or ends with an infinite block of the same letter. Therefore, before we can speak about the action of sliding block codes that induce bijections from some $G_{\mathcal{S}}$ to some other $G_{\mathcal{T}}$, we must consider what happens when an element it would act on begins or ends with infinitely many of the same letter.

Lemma 5.5. *Let $\mathcal{S} = \{S_1, \dots, S_p\}$, $\mathcal{T} = \{T_1, \dots, T_q\}$, and $\varphi : X(\mathcal{S}) \rightarrow \{1, 2, \dots, q\}^{\mathbb{Z}}$ be a sliding block code induced by the block map Φ of memory m and anticipation n that induces a bijection $\psi : G_{\mathcal{S}} \rightarrow G_{\mathcal{T}}$. Also, let $I = \{i \in \mathbb{N} : |S_i| = |\mathbb{N}|\}$ and $J = \{j \in \mathbb{N} : |T_j| = |\mathbb{N}|\}$. Then, $\pi : I \rightarrow \{1, 2, \dots, q\}$ where for all $i \in I$, $\Phi(i^r) = \pi(i)$ for $r = m + n + 1$ must have image J .*

PROOF. Let $\mathcal{S} = \{S_1, \dots, S_p\}$, $\mathcal{T} = \{T_1, \dots, T_q\}$, and $\varphi : X(\mathcal{S}) \rightarrow \{1, 2, \dots, q\}^{\mathbb{Z}}$ be a sliding block code induced by the block map Φ of memory m and anticipation n that induces a bijection $\psi : G_{\mathcal{S}} \rightarrow G_{\mathcal{T}}$, and set $r = m + n + 1$. Also, let $I = \{i \in \mathbb{N} : |S_i| = |\mathbb{N}|\}$ and $J = \{j \in \mathbb{N} : |T_j| = |\mathbb{N}|\}$. From this, we can define a function $\pi : \{1, 2, \dots, p\} \rightarrow \{1, 2, \dots, q\}$ by $\pi(i) = \Phi(i^r)$. First we wish to show for any $j \in J$ there must exist some $i \in I$ so that $\pi(i) = j$, after which we will show $\pi(I) \subseteq J$.

Assume, to the contrary, that there exists $j \in J$ such that there does not exist any $i \in I$ so that $\pi(i) = j$. Now, let x and y be strings over $\{1, 2, \dots, p\}$ of length m and n respectively, and consider the longest block of js that can occur in the image of $\varphi(xzy)$ where $z \in G_{\mathcal{S}}$ with any block code Φ_j with memory m and anticipation n where $\Phi_j(i^r) \neq j$ for all $i \in I$. This will occur when $\varphi_j(xzy) = j^{|z|}$, so let us consider the largest z where this can be the case. For each letter in $\{1, 2, \dots, p\} \setminus I$, there is a maximum number of consecutive instances of that letter that can occur in any $z \in G_{\mathcal{S}}$. Clearly, for the z we are constructing, we wish to have as many of each of these letters as possible. Let us say that s is the largest of all of those values, then the length of z contributed by these letters is at most $p * s$. For the letters in I , of which there are at most p , we can have at most $r - 1$ consecutive instances of that letter. Therefore, the contribution to the length of z from letters in I is less than $p * r$. Thus, we cannot have more than $p(r + s)$ consecutive js in the image of any word in $G_{\mathcal{S}}$ under φ_j . Then, since Φ_j will output a j whenever Φ does, there cannot be more than $p(r + s)$ consecutive js in the image of any word in $G_{\mathcal{S}}$ under φ . However, T_j is infinite and $p(r + s)$ is finite, so there exists a word in $G_{\mathcal{T}}$ with more than $p(r + s)$ consecutive js . Thus, φ cannot induce a bijection from $G_{\mathcal{S}}$ to $G_{\mathcal{T}}$, giving us a contradiction. Thus, $\pi(I) \supseteq J$.

Now, suppose that $i \in I$. We wish to show $\pi(i) \in J$. To see this, let k be

the maximum value in any set in \mathcal{T} that is not infinite, and fix some $i \in I$. Because $i \in I$, we know that S_i is infinite, so there must be some value $l \in S_i$ where $l > k + r$. Therefore, there must exist $w \in G_{\mathcal{S}}$ that has l consecutive is . Then, $\varphi(w)$ has a block of $l - r > k$ consecutive $\pi(i)$ s. Therefore, $\pi(i) \in J$ because otherwise $\varphi(w)$ could not be in $G_{\mathcal{T}}$, which would contradict the fact that φ induces a bijection from $G_{\mathcal{S}}$ to $G_{\mathcal{T}}$. Thus $\pi(I) \subseteq J$. Combining that with the result above that $\pi(I) \supseteq J$, we have $\pi(I) = J$. \square

Using Lemma 5.5, we can establish a link between sliding block codes that induce bijections and conjugacies.

Lemma 5.6. *Let $\mathcal{S} = \{S_1, \dots, S_p\}$ and $\mathcal{T} = \{T_1, \dots, T_q\}$ and $\varphi : X(\mathcal{S}) \rightarrow \{1, 2, \dots, q\}^{\mathbb{Z}}$ be a sliding block code that induces a bijection $\psi : G_{\mathcal{S}} \rightarrow G_{\mathcal{T}}$. Then, φ is a conjugacy from $X(\mathcal{S})$ to $X(\mathcal{T})$.*

PROOF. Let $\mathcal{S} = \{S_1, \dots, S_p\}$ and $\mathcal{T} = \{T_1, \dots, T_q\}$ and suppose $\varphi : X(\mathcal{S}) \rightarrow \{1, 2, \dots, q\}^{\mathbb{Z}}$ is a sliding block code that induces a bijection $\psi : G_{\mathcal{S}} \rightarrow G_{\mathcal{T}}$. First, we will show that the image of φ is $X(\mathcal{T})$. Let $x \in X(\mathcal{S})$, and suppose that it neither begins nor ends with an infinite block of the same letter. Then, using the definition of an \mathcal{S} -limited shift, we can rewrite $x = \dots x_{-1}x_0x_1\dots$ where each $x_i \in G_{\mathcal{S}}$. Because φ induces ψ , we can see $\varphi(x) = \dots \psi(x_{-1})\psi(x_0)\psi(x_1)\dots$. Then, since each $\psi(x_i) \in G_{\mathcal{T}}$, we have $\varphi(x) \in X(\mathcal{T})$.

Now, we must handle the case where x begins and/or ends with an infinite block of the same letter. Suppose $x \in X(\mathcal{S})$ ends with an infinite block of the letter j . Then, through possibly reindexing the x_i s, we can say $x = \dots x_{-2}x_{-1}x_0yj^{\mathbb{N}}$ where each $x_i \in G_{\mathcal{S}}$, $y = 1^{n_1}2^{n_2}\dots(j-1)^{n_{j-1}}$ where $n_i \in S_i$ for all $i \in \{1, 2, \dots, j-1\}$. Then, because x ends with infinitely many js , S_j must be infinite, so there must be some $z \in G_{\mathcal{S}}$ so that z begins yj^r where r is 1 plus the sum of the memory and anticipation of Φ . Then, by Lemma 5.5, if we say $\Phi(j^r) = k$, we know that we can have a word in $X(\mathcal{T})$ end in a block of infinitely many ks . Therefore, we have that $\varphi(x) = \dots \psi(x_{-2})\psi(x_{-1})\psi(x_0)wk^{\mathbb{N}}$ where w is the first $|y|$ letters of $\psi(z)$, from which we can see $\varphi(x) \in X(\mathcal{T})$. All of the remaining cases can be shown using the same approach of breaking as much of x as possible up into elements of $G_{\mathcal{S}}$, and then viewing the remaining part as a substring of an element of $G_{\mathcal{S}}$ that begins and/or ends with a sufficiently long string of the letter that begins and/or ends x . Then, we can use the fact that our sliding block code induces a bijection to see what it will map that word to, and we find that it is in $X(\mathcal{T})$. Therefore, $\varphi(x) \in X(\mathcal{T})$ for all $x \in X(\mathcal{S})$.

Now, we will show that $\varphi : X(\mathcal{S}) \rightarrow X(\mathcal{T})$ is a bijection. For that, first we will show that φ is onto. Suppose $y \in X(\mathcal{T})$. Again, we first consider the case where $y = \cdots y_{-1}y_0y_1 \cdots$ where each $y_i \in G_{\mathcal{T}}$. Then, since ψ is a bijection, we can construct $x \in X(\mathcal{S})$ where $x = \cdots x_{-1}x_0x_1 \cdots$ where for every $i \in \mathbb{Z}$, $x_i = \psi^{-1}(y_i)$. Then, using the fact that φ induces ψ , we can see $\varphi(x) = y$.

Suppose now that $y \in X(\mathcal{T})$ begins and/or ends with an infinite block of the same letter—as above, without loss of generality, say $y = \cdots y_{-2}y_{-1}y_0zj^\infty$ where each $y_i \in G_{\mathcal{T}}$ and $z = 1^{n_1}2^{n_2} \cdots (j-1)^{n_{j-1}}$ where $n_i \in T_i$ for all $i \in \{1, 2, \dots, j-1\}$. Now, for each $l \in \mathbb{N}$, let $w_l \in G_{\mathcal{T}}$ so that w_l begins yj^{l+r} —note that this exists for every l because this can be followed by more j s and T_j is infinite. We wish to find a sufficiently large l so the first $|yj^l|$ letters of $\psi^{-1}(w_l)$ ends with i^r where $\Phi(i^r) = j$ with S_i infinite. If we let $l_0 = q(r+m)$ where m is the largest element of any finite set in \mathcal{T} , then by similar reasoning to that employed in the proof of Lemma 5.5, this must be the case for w_{l_0} . Thus, if $x = \cdots x_{-2}x_{-1}x_0w_i^\infty$ where $x_i = \psi^{-1}(y_i)$ for all $i \leq 0$, and w is the first $|yj^l|$ letters of $\psi^{-1}(w_{l_0})$, then $\varphi(x) = y$. Therefore, φ is onto.

Next, suppose $x, y \in X(\mathcal{S})$ with $x \neq y$. There are three ways in which this could happen: either x and y consist of a different sequence of words from $G_{\mathcal{S}}$, $x = \sigma^n(y)$ for some $n \neq 0$, or x and y consist of the concatenation of the same complete elements of $G_{\mathcal{S}}$ but are not equal (and therefore must begin and/or end with infinite blocks of some letter). In the second case, $\varphi(x) = \varphi(\sigma^n(y)) = \sigma^n(\varphi(y))$. Thus, for us to have $\varphi(x) = \varphi(y)$, then $\varphi(x)$ must have a period that divides n . However, this means that x and $\varphi(x)$ would have different periods because $x \neq y$ so x cannot have a period that divides n , which gives us a contradiction. Therefore, if $x = \sigma^n(y)$, then $\varphi(x) \neq \varphi(y)$. If x and y consist of a different sequence of words from $G_{\mathcal{S}}$, then, because φ induces the bijection φ , we must have $\varphi(x)$ and $\varphi(y)$ consist of a different sequence of words from $G_{\mathcal{T}}$. Thus, we must have $\varphi(x) \neq \varphi(y)$. Finally, suppose that x and y consist of the concatenation in the same order of the same complete elements of $G_{\mathcal{S}}$ but still $x \neq y$. Suppose, without loss of generality, that their difference comes after the end of the rightmost complete element of $G_{\mathcal{S}}$, and say that the parts of the words after the last complete element of $G_{\mathcal{S}}$ until there is an infinite block of some letter are x' and y' respectively, and that the letters repeated infinitely many times are i and j respectively. Notice that we must have $x' \neq y'$. Therefore, there must exist words x'' and y'' in $G_{\mathcal{S}}$ so that x'' begins with $x'i^r$ and y'' begins with $y'j^r$, and clearly $x_1 \neq y_1$. We can have it so that x'' and y'' have the same number of every letter following the smaller of i and j . Then, since ψ is induced by a sliding block code, we can see that $\psi(x'')$ and $\psi(y'')$ must disagree on the section that is the image

of the part up to and including the block of the smaller of the is and js . We can also see that the section of $\psi(x'')$ that is the image of the part of x'' up to the block of is and the section of $\psi(y'')$ that is the image of the part of y'' up to the block of js must be in $\varphi(x)$ and $\varphi(y)$ respectively, so we must have $\varphi(x) \neq \varphi(y)$. Therefore, we have $\varphi(x) = \varphi(y)$ if and only if $x = y$, so φ is one-to-one. Thus, φ is an invertible factor map, so it is a conjugacy. \square

Before proving Theorem 5.2, we will introduce one last definition to keep our notation as simple as possible.

Definition 5.7. Let $\mathcal{S} = \{S_1, \dots, S_p\}$ and $\mathcal{T} = \{T_1, \dots, T_q\}$ and suppose $\varphi : X(\mathcal{S}) \rightarrow X(\mathcal{T})$ is a sliding block code. Then, we will say that a **transition point** is any index i so that, if $\varphi(x) = y = \dots y_{-1}y_0y_1 \dots$ (where here $y_j \in A$ for all j), $y_i \neq y_{i+1}$. If $y_i + 1 = y_{i+1}$, we will call this an internal transition point, and if $y_{i+1} = 1$, we will call this an external transition point.

The motivation for the distinction between internal and external transition points comes from thinking about elements of $X(\mathcal{S})$ as the concatenation of words from $G_{\mathcal{S}}$, since the internal transition points will be those within a word in $G_{\mathcal{S}}$ and the external transition points will be the last index within one word in $G_{\mathcal{S}}$, and so will be the transition between that word and the next one. Using this, we can finally turn to our proof.

PROOF OF THEOREM 5.2. Let $\mathcal{S} = \{S_1, \dots, S_p\}$ and $\mathcal{T} = \{T_1, \dots, T_p\}$ with $S_i, T_i \subseteq \mathbb{N}$ for $1 \leq i \leq p$. Let s_i^m denote the m th element of S_i sorted in increasing order, and define t_i^m similarly. Suppose that for all $i_1, i_2, \dots, i_p \in \mathbb{N}$,

$$\sum_{k=1}^p s_k^{i_k} = \sum_{k=1}^p t_k^{i_k}.$$

As an outline of the rest of the proof, we will first construct a bijection $\psi : G_{\mathcal{S}} \rightarrow G_{\mathcal{T}}$. Then, we will construct a block map Φ that induces a sliding block code φ so that φ induces ψ . Finally, from there, we will use Lemma 5.6 to show that φ is a conjugacy.

Let $x \in G_{\mathcal{S}}$. Then, by the definition of $G_{\mathcal{S}}$, we can write $x = 1^{n_1} 2^{n_2} \dots p^{n_p}$ where each $n_i \in S_i$. Using the indexing of each S_i , we can rewrite that as $x = 1^{s_1^{i_1}} 2^{s_2^{i_2}} \dots p^{s_p^{i_p}}$. Using that parameterization, we can construct $\psi : G_{\mathcal{S}} \rightarrow G_{\mathcal{T}}$ by letting $\psi(1^{s_1^{i_1}} 2^{s_2^{i_2}} \dots p^{s_p^{i_p}}) = 1^{t_1^{i_1}} 2^{t_2^{i_2}} \dots p^{t_p^{i_p}}$, which will be our bijection. Notice that, by the assumption that the sum of elements from the S_i s and T_i s with the same indices will be equal in in the statement of the theorem, for all $x \in G_{\mathcal{S}}$, $|x| = |\psi(x)|$, which is one of the properties of $X(\mathcal{S})$ and $X(\mathcal{T})$ that will allow us to construct a sliding block code that induces ψ .

The other main property we will use also comes from that assumption, which is that for all j , there exists some $n_j \in \mathbb{Z}$ so that, for all i , $s_j^i + d_j = t_j^i$. Specifically, we can see that $d_j = t_j^1 + s_j^1$ by subtracting the sum from the statement of the sum where all the indices are 1 from the sum where all the indices are 1 except for the index for the j th set, and that index is i . Using this, we can rewrite ψ so that it is expressed solely in terms of the sets S_j and the constants d_j . In order to do that, we notice that $1^{t_1^{i_1}} 2^{t_2^{i_2}} \dots p^{t_p^{i_p}}$ can be rewritten as $1^{s_1^{i_1} + d_1} 2^{s_2^{i_2} + d_2} \dots p^{s_p^{i_p} + d_p}$. Therefore, we can say $\psi(1^{s_1^{i_1}} 2^{s_2^{i_2}} \dots p^{s_p^{i_p}}) = 1^{s_1^{i_1} + d_1} 2^{s_2^{i_2} + d_2} \dots p^{s_p^{i_p} + d_p}$.

Now, we can use this to figure out where the transition points in $x \in X(\mathcal{S})$ must be in any sliding block code that induces ψ . Because every element of $G_{\mathcal{S}}$ is mapped to an element of $G_{\mathcal{T}}$ of the same length, the external transition points must be the indices i such that $x_i = n$ and $x_{i+1} = 1$. Turning to the internal transition points, we can assume that we are only looking at some word $y \in G_{\mathcal{S}}$ because internal transition points only occur completely within one of those words. For any such y , there will be $p - 1$ internal transition points: the change from ks to $(k + 1)s$ for all $k \in \{1, 2, \dots, p - 1\}$. Then, from the parameterization of ψ in terms of only the S_j s and the d_j s, we can see that the internal transition point from k to $k + 1$ will occur at the letter indexed

$\sum_{j=1}^k s_j^{i_j} + \sum_{j=1}^k d_j$. In terms of creating a sliding block code, the value that is of

more importance than that, however, is the distance between the change from ks to $(k + 1)s$ in a word in $G_{\mathcal{S}}$ and the transition point that switches from k to $k + 1$ in the image of that word under ψ . Using the previous indexing of the

transition points, we can see that distance will be $\sum_{j=1}^k d_j$, which we will refer

to as r_k . Using those, we will define the value $r = 1 + \max_k r_k$, which will be

both the memory and anticipation of the block map Φ we create.

To construct $\Phi : B_{2r+3}(X(\mathcal{S})) \rightarrow \{1, 2, \dots, p\}$, suppose $x \in B_{2r+3}(X(\mathcal{S}))$ where

$x = x_{-r-1}x_{-r} \dots x_0 \dots x_r x_{r+1}$. First, find the transition point as described in the previous paragraph that is the closest to x_0 , breaking ties to favor transition points that have negative indices in x . In order to do so, using the index offsets described, there are three cases for this we must address:

1. If there does not exist a transition point, then we will say $\Phi(x) = x_0$, because we know all the transition points are far enough from x_0 so that in Ψ (as well as any sliding block code that induces Ψ), we must have that the letters with that index are the same in both the domain and

the image.

2. If the closest transition point comes before x_0 , suppose it is a transition point from k to l , where $l = k + 1$ for any internal transition point and $l = 1$ for any external transition point. Then, we know that in the image of any word in $X(\mathcal{S})$ of which x is a substring, and in any sliding block code inducing Ψ , the letter in the image of that word with the same index as x_0 must be $k + 1$, so $\Phi(x) = l$.
3. If the closest transition point comes on or after x_0 , suppose it is a transition point from k to l , where $l = k + 1$ for any internal transition point and $l = 1$ for any external transition point. Then, for similar reasoning as in the previous case, $\Phi(x) = k$.

Now, let φ be the sliding block code induced by Φ . By the construction of Φ , we can see φ will induce ψ . Therefore, by Lemma 5.6, φ is a conjugacy from $X(\mathcal{S})$ to $X(\mathcal{T})$. □

Having established this proof, we can use that same technique to create a sliding block code that induces a bijection between the core sets of two specific \mathcal{S} -limited shifts.

Example 5.8 (\mathcal{S} -limited Shift Conjugacy). As in Example 5.4, let $\mathcal{S} = \{S_1, S_2, S_3\}$ where $S_1 = \mathbb{N}$, $S_2 = \{2n : n \in \mathbb{N}\}$, and $S_3 = \{3, 5\}$, and also $\mathcal{T} = \{T_1, T_2, T_3\}$ where $T_1 = \mathbb{N}$, $T_2 = \{2n + 1 : n \in \mathbb{N}\}$, and $T_3 = \{2, 4\}$. We can see that that $X(\mathcal{S})$ and $X(\mathcal{T})$ satisfy the conditions for Theorem 5.2, so $X(\mathcal{S})$ is conjugate to $X(\mathcal{T})$. In the previous example, we constructed a conjugacy $\psi : G_{\mathcal{S}} \rightarrow G_{\mathcal{T}}$, and then constructed a sliding block code that induced that conjugacy. While we could use that, along with Lemma 5.6, to show $X(\mathcal{S})$ is conjugate to $X(\mathcal{T})$, with Theorem 5.2, we can show the conjugacy by just noticing $s_1^m = t_1^m$, $s_2^m + 1 = t_2^m$, and $s_3^m - 1 = t_3^m$.

6 Conclusion

In this paper, we have extended existing results about dynamical properties of \mathcal{S} -gap and $(\mathcal{S}, \mathcal{S}')$ -gap shifts to a more general class, the \mathcal{S} -limited shifts. The full extension of these gap shifts is the generalized \mathcal{S} -limited shift, in which the order of the blocks of a single letter is not restricted. The subshifts in this space behave in a more chaotic way than an \mathcal{S} -limited shift, and therefore our current techniques did not allow us to extend all of our results. While we proved some properties (like conditions for which the shift is sofic or mixing)

for the generalized \mathcal{S} -limited shifts using modified versions of our techniques for the \mathcal{S} -limited shifts, the main problem with extending our other results stems from the fact that we currently cannot identify an analogous core set, $G_{\mathcal{S}}$, in this setting. We relied heavily on the existence of such a set in our proofs regarding entropy, conjugacy, and intrinsic ergodicity, and so our results do not immediately carry over to the generalized \mathcal{S} -limited shift setting without significant adjustment to our current techniques or a creative construction of such a set.

All of this is not to say that there are no remaining questions about \mathcal{S} -limited shifts. Most glaringly, there is still a gap between the necessary condition we have proven for conjugacy and the sufficient one. We have not even closed that gap for two \mathcal{S} -limited shifts with the same number of letters, and have very little we can say about the cases with different numbers of letters. Consider, for example, the following \mathcal{S} -gap shifts:

Example 6.1 (\mathcal{S} -limited Shifts of Different Letters). Let $\mathcal{S} = \{S_1, S_2, S_3\}$ where $S_1 = \{1, 2\}$, $S_2 = \{1\}$, and $S_3 = \{1\}$, and also, let $\mathcal{T} = \{T_1, T_2\}$ where $T_1 = \{2, 3\}$ and $T_2 = \{1\}$. Then, we have $G_{\mathcal{S}} = \{123, 1123\}$ and $G_{\mathcal{T}} = \{112, 1112\}$. Now, we can define a block map Φ of memory 0 and anticipation 0 with $\Phi(1) = 1$, $\Phi(2) = 1$, and $\Phi(3) = 2$, and let φ be the sliding block code induced by Φ . By inspection, we can see that φ induces a bijection $\psi : G_{\mathcal{S}} \rightarrow G_{\mathcal{T}}$ with $\psi(123) = 112$ and $\psi(1123) = 1112$. Therefore, by Lemma 5.6, $X(\mathcal{S})$ is conjugate to $X(\mathcal{T})$.

This example suggests another class of conjugate \mathcal{S} -limited shifts where there are two adjacent sets in \mathcal{S} where each every number in \mathbb{N} can be written as a sum of one element from each in at most one way, and \mathcal{T} has the same sets in the same order as \mathcal{S} except that those two adjacent sets are replaced by all sums consisting of one element from each. While that case can be shown using the same application of Lemma 5.6 as in the example, it raises the question of what other types of conjugacy between two \mathcal{S} -limited shifts over alphabets of different sizes can exist.

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