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BEST L^p-APPROXIMANT PAIR ON SMALL INTERVALS

Abstract

In this paper, we study the behavior of best L^p-approximations by algebraic polynomial pairs on unions of intervals when the measure of those intervals tends to zero.

1 Introduction

Let $X = \{x_j\}_{j=1}^k \subset \mathbb{R}$, $k \in \mathbb{N}$, and let $\{B_j\}_{j=1}^k$ be pairwise disjoint closed intervals centered at x_j of radius 1. Let $n, m \in \mathbb{N} \cup \{0\}$ and suppose that $n + m + 1 = kq + r$ with $q \in \mathbb{N} \cup \{0\}$, $0 < r < k$. For $s \in \mathbb{N} \cup \{0\}$, we let $\mathcal{C}^s(I)$ denote the space of real functions defined on $I := \bigcup_{j=1}^k B_j$ which are continuously differentiable up to order s on I. For simplicity, we write $\mathcal{C}(I)$ instead of $\mathcal{C}^0(I)$.

If $\|\cdot\|$ is a norm defined on $\mathcal{C}(I)$ and $h \in \mathcal{C}(I)$, then for each $0 < \epsilon \leq 1$, we write $\|h\|_\epsilon = \|h^\epsilon\|$, where $h^\epsilon(x) = h(\epsilon(x - x_j) + x_j)$, $x \in B_j$. We put

$$\|h\| = \left(\int_I |h(x)|^p \frac{dx}{|I|} \right)^{\frac{1}{p}}, \quad 1 < p < \infty,$$

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where $|I|$ is the Lebesgue measure of I. For $I_\epsilon := \bigcup_{j=1}^k [x_j - \epsilon, x_j + \epsilon]$, we observe that $(\mathcal{C}(I_\epsilon), \|\cdot\|_\epsilon)$ is a normed space and

$$\|h\|_\epsilon = \left(\int_{I_\epsilon} |h(x)|^p \frac{dx}{|I_\epsilon|} \right)^{\frac{1}{p}}.$$

We define $\|h\|_\infty := \max_{x \in I} |h(x)|$ and $\|h\|_{B_j} := \left(\int_{B_j} |h(x)|^p dx \right)^{\frac{1}{p}}$, $1 \leq j \leq k$.

Let Π^n be the class of algebraic polynomials with real coefficients of degree at most n. We consider the set

$$\mathcal{H}_m^n := \{(P, Q) \in \Pi^n \times \Pi^m : \|Q\|_\infty = 1\}.$$

Given $(P, Q), (U, V) \in \mathcal{H}_m^n$, we identify (P, Q) with (U, V) if and only if $P = \lambda U$, $Q = \lambda V$, $|\lambda| = 1$. We denote it briefly by $(P, Q) \equiv (U, V)$.

Let $f \in \mathcal{C}(I)$ and $0 < \epsilon \leq 1$. We say that $(P_\epsilon, Q_\epsilon) \in \mathcal{H}_m^n$ is a *best approximant pair* of f from $\mathcal{H}_m^n$ with respect to $\|\cdot\|_\epsilon$ if

$$\|fQ_\epsilon - P_\epsilon\|_\epsilon = \inf_{(P,Q) \in \mathcal{H}_m^n} \|fQ - P\|_\epsilon. \tag{1}$$

It is easy to see that the pair (P_ϵ, Q_ϵ) always exists.

Given $q > 0$, $f \in \mathcal{C}^{q-1}(I)$ and $(P, Q) \in \Pi^n \times \Pi^m$, if

$$(fQ - P)^{(s)}(x_j) = 0, \quad 0 \leq s \leq q-1, \quad 1 \leq j \leq k, \tag{2}$$

then (P, Q) is said to be a *Padé approximant pair of* f at X. If $Q \neq 0$ and

$$\left(f - \frac{P}{Q} \right)^{(s)} (x_j) = 0, \quad 0 \leq s \leq q-1, \quad 1 \leq j \leq k,$$

then the rational function $\frac{P}{Q}$ is called a *Padé approximant of* f at X.

We define

$$\mathcal{W}_m^n(f, X) := \{(P, Q) \in \mathcal{H}_m^n : (P, Q) \text{ is a Padé approximant pair of } f \text{ at } X\}.$$

If $q = 0$, then no constraint over the pair is assumed and $\mathcal{W}_m^n(f, X) = \mathcal{H}_m^n$.

Clearly, $\mathcal{W}_m^n(f, X) \neq \emptyset$. In fact, let $x_{k+1} \in I - X$, and we consider the system (2) with constrains $(fQ - P)^{(s)}(x_{k+1}) = 0$, $0 \leq s \leq r-1$. This system always has a nontrivial solution for (P, Q), since it is a homogeneous system of $n+m+1$ equations in $n+m+2$ unknowns. Now, if $Q = 0$, then $P = 0$ because $P \in \Pi^n$, a contradiction. So, $Q \neq 0$ and $\left(\frac{P}{\|Q\|_\infty}, \frac{Q}{\|Q\|_\infty} \right) \in \mathcal{W}_m^n(f, X)$.

We say that $(P_0, Q_0) \in \mathcal{W}_m^n(f, X)$ is a *best Padé approximant pair of* f *at* X if

$$\sum_{j=1}^{k} \left| (fQ_0 - P_0)^{(q)}(x_j) \right|^p \leq \sum_{j=1}^{k} \left| (fQ - P)^{(q)}(x_j) \right|^p \tag{3}$$

for all $(P, Q) \in \mathcal{W}_m^n(f, X)$. If $(P, Q) \in \Pi^n \times \Pi^m$, $Q \neq 0$, then $\frac{P}{Q}$ is said to be *normal* if it is irreducible and either $\deg P = n$ or $\deg Q = m$. (The null rational function $\frac{0}{Q}$ is normal if and only if $\deg Q = 0$.)

In 1934, Walsh proved in [9] that the Taylor polynomial of degree n for an analytic function f can be obtained by taking the limit as $\epsilon \to 0$ of the best (Tchebychev) approximant from Π^n to f on the disk $|z| \leq \epsilon$. In [10], the author generalized this result to rational approximation. In [2], Chui, Shisha and Smith proved that the net of best (Tchebychev) aproximants pairs on $[0, \epsilon]$, from $\{(P, Q) \in \Pi^n \times \Pi^m : Q(0) = 1\}$, converges to the Padé approximant pair in the origin as $\epsilon \to 0$. Similar results for the L^2-norm can be seen in [3]. The case of a unique point in several variables was treated in [1] with the L^p-norms. Finally, the case of L^ϕ-approximation on k disjoint intervals, where $n + m + 1$ is divisible by k, was investigated in [6].

In Section 2, we show that there exists at least a best Padé approximant pair of f at X. In Section 3, we prove that, any cluster point of best approximant pairs $\{(P_\epsilon, Q_\epsilon)\}$ as $\epsilon \to 0$ is a best Padé approximant pair of f at X.

2 Existence of best Padé approximant pairs

Now, we establish an existence theorem of best Padé approximant pairs.

Theorem 1. *Let $f \in \mathcal{C}^q(I)$. Then there exists at least one best Padé approximant pair of f at X.*

PROOF. Let $\{(P_l, Q_l)\} \subset \mathcal{W}_m^n(f, X)$ be a sequence satisfying

$$\lim_{l \to \infty} \sum_{j=1}^{k} |(fQ_l - P_l)^{(q)}(x_j)|^p = \inf_{(P,Q) \in \mathcal{W}_m^n(f,X)} \sum_{j=1}^{k} |(fQ - P)^{(q)}(x_j)|^p =: E. \tag{4}$$

If $q > 0$, then $(fQ_l - P_l)^{(i)}(x_j) = 0$, $0 \leq i \leq q - 1$, $1 \leq j \leq k$. According to (4), there is constant $M > 0$ such that

$$|(fQ_l - P_l)^{(i)}(x_j)| \leq M, \quad 0 \leq i \leq q, \quad 1 \leq j \leq k, \quad l \in \mathbb{N}. \tag{5}$$

We observe that if $q = 0$, (5) is true also, by (4).

Let $(S,T) \in \mathcal{W}_m^n(f,X)$. Since $SQ_l - TP_l = T(fQ_l - P_l) - Q_l(fT - S)$, by the Leibniz rule for the ith derivative of a product of two factors,

$$|(SQ_l - TP_l)^{(i)}(x_j)| \leq N,\ 0 \leq i \leq q,\ 1 \leq j \leq k,\ l \in \mathbb{N},$$

for some constant $N > 0$. As $\|P\| := \max_{0\leq i\leq q} \max_{1\leq j\leq k} |P^{(i)}(x_j)|$ is a norm on $\Pi^{k(q+1)-1}$, the equivalence of the norms in $\Pi^{k(q+1)-1}$ implies that $\{SQ_l - TP_l\}$ is uniformly bounded on I, and consequently $\{TP_l\}$ is uniformly bounded on I. Since $\|P\|_T := \max_{t\in I} |TP(t)|$ is a norm on Π^n, we get that $\{P_l\}$ is uniformly bounded on I. So, there is a subsequence of $\{(P_l, Q_l)\}$, which we denote the same way, and $(P_0, Q_0) \in \Pi^n \times \Pi^m$ such that $P_l \to P_0$ and $Q_l \to Q_0$ uniformly on I. By (4), it is obvious that $\sum_{j=1}^k |(fQ_0 - P_0)^{(q)}(x_j)|^p = E$. On the other hand, $(P_0, Q_0) \in \mathcal{W}_m^n(f,X)$ because $(P_l, Q_l) \in \mathcal{W}_m^n(f,X)$ for all l. So, (P_0, Q_0) is a best Padé approximant pair of f at X. $\square$

Remark 2. *We observe that if (P,Q) is a best Padé approximant pair of f at X, then so is $(-P,-Q)$.*

3 Convergence of best approximant pairs

Let $q > 0$, $f \in \mathcal{C}^q(I)$ and $(S,T) \in \mathcal{W}_m^n(f,X)$. We denote by $M_{p,q} \in \Pi^{q-1}$ the best approximant of x^q from Π^{q-1} with respect to the norm

$$\|h\|_p = \left(\int_{-1}^{1} |h(t)|^p dt\right)^{\frac{1}{p}}.$$

If $x^q - M_{p,q}(x) = \prod_{s=0}^{q-1}(x - t_s)$, it is well known that $t_s \in (-1,1)$, $0 \leq s \leq q-1$ and $t_s \neq t_c$ if $s \neq c$; see [7, §5.10]. We put $\mathcal{K}_{pq} = \|x^q - M_{p,q}\|_p$. Let

$$z_{js}^{\epsilon} = \epsilon t_s + x_j \in [x_j - \epsilon, x_j + \epsilon], \quad 0 \leq s \leq q-1, \quad 1 \leq j \leq k, \quad 0 < \epsilon \leq 1,$$

and let $y_1, ..., y_r \notin I$ be such that $y_v \neq y_w$ if $v \neq w$ and $T(y_v) \neq 0$, $1 \leq v \leq r$.

Lemma 3. *Under the above assumptions, for each $0 < \epsilon \leq 1$, there exists $(P_\epsilon, Q_\epsilon) \in \mathcal{H}_m^n$ such that*

$$\begin{aligned} P_\epsilon\left(z_{js}^{\epsilon}\right) &= f\left(z_{js}^{\epsilon}\right) Q_\epsilon\left(z_{js}^{\epsilon}\right) - \epsilon^q, \quad 0 \leq s \leq q-1, \quad 1 \leq j \leq k \\ P_\epsilon\left(y_v\right) &= \frac{S}{T}(y_v) Q_\epsilon\left(y_v\right) - \epsilon^q, \quad 1 \leq v \leq r \end{aligned} . \tag{6}$$

PROOF. Let $0 < \epsilon \le 1$. Clearly, there exists a nontrivial $(U_\epsilon, V_\epsilon) \in \Pi^n \times \Pi^m$ such that

$$\begin{aligned} U_\epsilon\left(z_{js}^\epsilon\right) &= f\left(z_{js}^\epsilon\right) V_\epsilon\left(z_{js}^\epsilon\right), \quad 0 \le s \le q-1, \quad 1 \le j \le k \\ U_\epsilon\left(y_v\right) &= \frac{S}{T}(y_v) V_\epsilon\left(y_v\right), \quad 1 \le v \le r \end{aligned} . \tag{7}$$

In fact, (7) is a homogeneous system of $n+m+1$ equations in $n+m+2$ unknowns and therefore always has a nontrivial solution. We observe that if $V_\epsilon = 0$, then $U_\epsilon = 0$, a contradiction; so $V_\epsilon \ne 0$. Now, taking $P_\epsilon = \frac{U_\epsilon}{\|V_\epsilon\|_\infty} - \epsilon^q$ and $Q_\epsilon = \frac{V_\epsilon}{\|V_\epsilon\|_\infty}$, we conclude that $(P_\epsilon, Q_\epsilon) \in \mathcal{H}_m^n$ satisfies (6). $\square$

Lemma 4. *Let $\{(P_\epsilon, Q_\epsilon)\} \subset \mathcal{H}_m^n$ be the net of Lemma 3. Then $\{P_\epsilon\}$ and $\{Q_\epsilon\}$ are uniformly bounded on compact sets as $\epsilon \to 0$. Moreover, if $\{P_\epsilon\}$ and $\{Q_\epsilon\}$ are subsequences convergent to P_* and Q_* respectively, then $P_*T - Q_*S = 0$.*

PROOF. Since $\|Q_\epsilon\|_\infty = 1$, $0 < \epsilon \le 1$, the net $\{Q_\epsilon\}$ is uniformly bounded on compact sets.

Let $0 \le i \le q-1$, $1 \le j \le k$ and $1 \le v \le r$. From (6), we get $(fQ_\epsilon - P_\epsilon)^\epsilon\left(z_{ji}^1\right) = \epsilon^q$, $0 < \epsilon \le 1$, and therefore

$$\left|(fQ_\epsilon - P_\epsilon)^\epsilon\left(z_{ji}^1\right)\right| = O(\epsilon^q) \quad \text{as} \quad \epsilon \to 0. \tag{8}$$

As $(S,T) \in \mathcal{W}_m^n(f,X)$, we have $(fT-S)^{(i)}(x_j) = 0$. Expanding $(fT-S)^\epsilon$ by its Taylor polynomial at x_j, $1 \le j \le k$, up to order $q-1$, for each $x \in B_j$, there exists $\xi(x) \in [x_j - \epsilon, x_j + \epsilon]$ such that

$$(fT-S)^\epsilon(x) = \frac{\epsilon^q}{q!}(fT-S)^{(q)}(\xi(x))(x-x_j)^q,$$

and consequently

$$\left|(fT-S)^\epsilon(z_{ji}^1)\right| = O(\epsilon^q) \quad \text{as} \quad \epsilon \to 0. \tag{9}$$

But

$$\left|(P_\epsilon T - Q_\epsilon S)^\epsilon(z_{ji}^1)\right| \le \left|T^\epsilon(z_{ji}^1)\right| \left|(fQ_\epsilon - P_\epsilon)^\epsilon(z_{ji}^1)\right| + \left|Q_\epsilon^\epsilon(z_{ji}^1)\right| \left|(fT-S)^\epsilon(z_{ji}^1)\right|,$$

so according to (8) and (9), we have $\left|(P_\epsilon T - Q_\epsilon S)(z_{ji}^\epsilon)\right| = \left|(P_\epsilon T - Q_\epsilon S)^\epsilon(z_{ji}^1)\right|$ $= O(\epsilon^q)$ as $\epsilon \to 0$.

On the other hand, (6) implies $|(P_\epsilon T - Q_\epsilon S)(y_v)| = O(\epsilon^q)$ as $\epsilon \to 0$, $1 \le v \le r$. So, there exist $0 < \epsilon_0 \le 1$ and $N > 0$, independent of i, j and v, such that

$$\left|(P_\epsilon T - Q_\epsilon S)(z_{ji}^\epsilon)\right| \le \epsilon^q N \quad \text{and} \quad |(P_\epsilon T - Q_\epsilon S)(y_v)| \le \epsilon^q N, \tag{10}$$

$0 \le i \le q-1$, $1 \le j \le k$, $1 \le v \le r$ and $0 < \epsilon \le \epsilon_0$.

Let $w_\epsilon(x) = \prod_{c=1}^{k} \prod_{l=0}^{q-1} (x - z_{cl}^\epsilon) \prod_{u=1}^{r} (x - y_u)$. It is easy to check that

$$w'_\epsilon(z) = \begin{cases} \epsilon^{q-1} \prod\limits_{\substack{c=1\\c\neq j}}^{k} \prod\limits_{l=0}^{q-1} (z_{ji}^\epsilon - z_{cl}^\epsilon) \prod\limits_{\substack{s=0\\s\neq i}}^{q-1} (t_i - t_s) \prod\limits_{u=1}^{r} (z_{ji}^\epsilon - y_u) & \text{if} \quad z = z_{ji}^\epsilon \\ \prod\limits_{c=1}^{k} \prod\limits_{l=0}^{q-1} (y_v - z_{cl}^\epsilon) \prod\limits_{\substack{u=1\\u\neq v}}^{r} (y_v - y_u) & \text{if} \quad z = y_v \end{cases},$$

and for $x \in I$,

$$\frac{w_\epsilon(x)}{x-z} = \begin{cases} \prod\limits_{\substack{c=1\\c\neq j}}^{k} \prod\limits_{l=0}^{q-1} (x - z_{cl}^\epsilon) \prod\limits_{\substack{s=0\\s\neq i}}^{q-1} (x - z_{js}^\epsilon) \prod\limits_{u=1}^{r} (x - y_u) & \text{if} \quad z = z_{ji}^\epsilon \\ \prod\limits_{c=1}^{k} \prod\limits_{l=0}^{q-1} (x - z_{cl}^\epsilon) \prod\limits_{\substack{u=1\\u\neq v}}^{r} (x - y_u) & \text{if} \quad z = y_v \end{cases}.$$

Therefore, there is $M > 0$, independent of i, j and v, satisfying

$$\lim_{\epsilon\to 0} \frac{\left|w'_\epsilon(z_{ji}^\epsilon)\right|}{\epsilon^{q-1}} = \prod_{\substack{c=1\\c\neq j}}^{k} |x_j - x_c|^q \prod_{\substack{s=0\\s\neq i}}^{q-1} |t_i - t_s| \prod_{u=1}^{r} |x_j - y_u| \ge \frac{1}{M},$$

$$\lim_{\epsilon\to 0} |w'_\epsilon(y_v)| = \prod_{c=1}^{k} |y_v - x_c|^q \prod_{\substack{u=1\\u\neq v}}^{r} |y_v - y_u| \ge \frac{1}{M},$$

and $\left|\frac{w_\epsilon(x)}{x-z}\right| \le M$, $x \in I$ and $z = z_{ji}^\epsilon, y_v$. Hence, (10) implies that there exists $0 < \epsilon_1 \le \epsilon_0$ such that

$$\left|\frac{(P_\epsilon T - Q_\epsilon S)(z_{ji}^\epsilon) w_\epsilon(x)}{w'_\epsilon(z_{ji}^\epsilon)(x - z_{ji}^\epsilon)}\right| \le \epsilon N M^2 \text{ and } \left|\frac{(P_\epsilon T - Q_\epsilon S)(y_v) w_\epsilon(x)}{w'_\epsilon(y_v)(x - y_v)}\right| \le \epsilon^q N M^2,$$

$x \in I$, $0 \le i \le q-1$, $1 \le j \le k$, $1 \le v \le r$ and $0 < \epsilon \le \epsilon_1$. Now, using the Lagrange interpolation formula,

$$\begin{aligned} &|(P_\epsilon T - Q_\epsilon S)(x)| \\ &= \left| \sum_{j=1}^{k} \sum_{i=0}^{q-1} \frac{(P_\epsilon T - Q_\epsilon S)(z_{ji}^\epsilon) w_\epsilon(x)}{w'_\epsilon(z_{ji}^\epsilon)(x - z_{ji}^\epsilon)} + \sum_{v=1}^{r} \frac{(P_\epsilon T - Q_\epsilon S)(y_v) w_\epsilon(x)}{w'_\epsilon(y_v)(x - y_v)} \right| \qquad (11) \\ &\le \epsilon N M^2 \end{aligned}$$

for $x \in I$, $0 < \epsilon \le \epsilon_1$. Since

$$\|P_\epsilon\|_T \le NM^2 + \|S\|_\infty, \quad 0 < \epsilon \le \epsilon_1,$$

from the equivalence of the norms in Π^n, we conclude that $\{P_\epsilon\}$ is uniformly bounded on compact sets as $\epsilon \to 0$.

Finally, if $\{P_\epsilon\}$ and $\{Q_\epsilon\}$ are subsequences convergent to P_* and Q_* respectively, by (11) we get $P_*T - Q_*S = 0$. $\square$

Lemma 5. *Let $\{(P_\epsilon, Q_\epsilon)\} \subset \mathcal{H}_m^n$ be the net of Lemma 3. If $\frac{S}{T}$ is normal, then there exist $\alpha \in \{1, -1\}$ and a subsequence of $\{(P_\epsilon, Q_\epsilon)\}$, which we denote the same way, such that $\lim_{\epsilon \to 0} P_\epsilon = \alpha S$ and $\lim_{\epsilon \to 0} Q_\epsilon = \alpha T$ uniformly on I. Moreover,*

$$[(fQ_\epsilon - P_\epsilon)^\epsilon]\left[z_{j0}^1, z_{j1}^1, \cdots, z_{js}^1\right] = 0 \tag{12}$$

for $0 \le s \le q-1$, $1 \le j \le k$, $0 < \epsilon < 1$.

PROOF. By Lemma 4, there is a subsequence of $\{(P_\epsilon, Q_\epsilon)\}$, which we denote the same way, and $P_0 \in \Pi^n$, $Q_0 \in \Pi^m$ such that $P_\epsilon \to P_0$ and $Q_\epsilon \to Q_0$ uniformly on I as $\epsilon \to 0$. Moreover,

$$P_0T = Q_0S. \tag{13}$$

For $1 \le j \le k$, let $K_j = \left\{i : 0 \le i \le m \text{ and } Q_0^{(i)}(x_j) \ne 0\right\}$. Since $\|Q_\epsilon\|_\infty = 1$, $0 < \epsilon \le 1$, we have $\|Q_0\|_\infty = 1$, and thus $K_j \ne \emptyset$. Set $k_j = \min(K_j)$. By hypothesis, $T(x_j) \ne 0$, so (13) implies that there are $P_1 \in \Pi^n$ and $Q_1 \in \Pi^m$ satisfying

$$P_0(x) = \prod_{c=1}^{k}(x-x_c)^{k_c}P_1(x), \quad Q_0(x) = \prod_{c=1}^{k}(x-x_c)^{k_c}Q_1(x) \quad \text{and} \quad Q_1(x_c) \ne 0, \tag{14}$$

$x \in I$, $1 \le c \le k$. Using (13) again, we obtain

$$P_1T = Q_1S.$$

Since $\frac{S}{T}$ is normal, either $\deg T = m$ or $\deg S = n$. If $\deg T = m$, then $\deg P_1 \le \deg S$. But $\frac{S}{T}$ is irreducible, so $\deg P_1 = \deg S$ and $\deg Q_1 = \deg T$. Therefore, there exists $\alpha \ne 0$ such that $P_1 = \alpha S$ and $Q_1 = \alpha T$. Now, according to (14), we have $k_c = 0$, $1 \le c \le k$, and consequently, $P_0 = \alpha S$ and $Q_0 = \alpha T$. But $\|Q_0\|_\infty = \|T\|_\infty = 1$, so $|\alpha| = 1$ and

$$P_0 = \alpha S \quad \text{and} \quad Q_0 = \alpha T. \tag{15}$$

If $\deg S = n$, in the same manner we can see (15).

Finally, let $0 \le s \le q-1$, $1 \le j \le k$ and $\epsilon > 0$. From Lemma 3,

$$\epsilon^q = (fQ_\epsilon - P_\epsilon)(z_{js}^\epsilon) = (fQ_\epsilon - P_\epsilon)(\epsilon(z_{js}^1 - x_j) + x_j) = (fQ_\epsilon - P_\epsilon)^\epsilon(z_{js}^1). \quad (16)$$

So, (12) immediately follows. $\square$

Lemma 6. *Suppose that $\frac{S}{T}$ is normal, and let $\{(P_\epsilon, Q_\epsilon)\} \subset \mathcal{H}_m^n$ be the subsequence of Lemma 4. Then for each $\epsilon > 0$ and $x \in B_j$, $1 \le j \le k$, $x \ne z_{js}^1$, $0 \le s \le q-1$, there exists $\xi_\epsilon(x) \in (x_j - \epsilon, x_j + \epsilon)$ satisfying*

$$\frac{1}{\epsilon^q}(fQ_\epsilon - P_\epsilon)^\epsilon(x) = \frac{1}{q!}(fQ_\epsilon - P_\epsilon)^{(q)}(\xi_\epsilon(x)) \prod_{l=0}^{q-1}(x - z_{jl}^1). \quad (17)$$

Proof. Let $\epsilon > 0$. It is well known that the $(q-1)$th Lagrange interpolation polynomial for $(fQ_\epsilon - P_\epsilon)^\epsilon$ with respect to $z_{j0}^1, z_{j1}^1, \cdots, z_{j(q-1)}^1$ can be expressed as

$$W_\epsilon(x) = \sum_{s=0}^{q-1}[(fQ_\epsilon - P_\epsilon)^\epsilon][z_{j0}^1, z_{j1}^1, \cdots, z_{js}^1] \prod_{l=0}^{s-1}\left(x - z_{jl}^1\right).$$

By Lemma 5, we have $W_\epsilon = 0$. Let $x \in B_j$, $1 \le j \le k$, $x \ne z_{js}^1$, $0 \le s \le q-1$. From [8, Th. 3, p. 309], we get

$$(fQ_\epsilon - P_\epsilon)^\epsilon(x) = [(fQ_\epsilon - P_\epsilon)^\epsilon]\left[z_{j0}^1, z_{j1}^1, \cdots, z_{j(q-1)}^1, x\right] \prod_{l=0}^{q-1}\left(x - z_{jl}^1\right). \quad (18)$$

Since $f \in \mathcal{C}^q(I)$, [8, Th. 4, p. 310] implies that there exists $\zeta_\epsilon(x) \in (x_j - 1, x_j + 1)$ such that

$$\begin{aligned} [(fQ_\epsilon - P_\epsilon)^\epsilon]\left[z_{j0}^1, z_{j1}^1, \cdots, z_{j(q-1)}^1, x\right] &= \frac{1}{q!}((fQ_\epsilon - P_\epsilon)^\epsilon)^{(q)}(\zeta_\epsilon(x)) \\ &= \frac{\epsilon^q}{q!}(fQ_\epsilon - P_\epsilon)^{(q)}(\epsilon(\zeta_\epsilon(x) - x_j) + x_j). \end{aligned}$$

So, according to (18), we have (17) with $\xi_\epsilon(x) = \epsilon(\zeta_\epsilon(x) - x_j) + x_j$. $\square$

Theorem 7. *Let $q > 0$, $f \in \mathcal{C}^q(I)$ and let $(S,T) \in \mathcal{W}_m^n(f, X)$ be such that $\frac{S}{T}$ is normal. Then there exists a sequence $\{(P_\epsilon, Q_\epsilon)\} \subset \mathcal{H}_m^n$ such that*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^q}\|(fQ_\epsilon - P_\epsilon)^\epsilon\|_{B_j} = \frac{1}{q!}|(fT - S)^{(q)}(x_j)|\, \mathcal{K}_{pq}, \quad 1 \le j \le k. \quad (19)$$

PROOF. By Lemma 5, there exist $\alpha \in \{1,-1\}$ and a sequence $\{(P_\epsilon, Q_\epsilon)\} \subset \mathcal{H}_m^n$ such that

$$\lim_{\epsilon\to 0} P_\epsilon = \alpha S \quad \text{and} \quad \lim_{\epsilon\to 0} Q_\epsilon = \alpha T \quad \text{uniformly on } I. \tag{20}$$

From Lemma 6, for each ϵ and $x \in B_j$, $1 \le j \le k$, $x \ne z_{js}^1$, $0 \le s \le q-1$, there is $\xi_\epsilon(x) \in (x_j - \epsilon, x_j + \epsilon)$ satisfying

$$\frac{1}{\epsilon^q}(fQ_\epsilon - P_\epsilon)^\epsilon(x) = \frac{1}{q!}(fQ_\epsilon - P_\epsilon)^{(q)}(\xi_\epsilon(x)) \prod_{l=0}^{q-1}\left(x - z_{jl}^1\right). \tag{21}$$

Since (20) implies that $\lim_{\epsilon\to 0}(fQ_\epsilon - P_\epsilon)^{(q)}(\xi_\epsilon(x)) = \alpha(fT - S)^{(q)}(x_j)$, we have

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^q}(fQ_\epsilon - P_\epsilon)^\epsilon(x) = \frac{\alpha}{q!}(fT - S)^{(q)}(x_j) \prod_{l=0}^{q-1}\left(x - z_{jl}^1\right), \tag{22}$$

$x \in B_j$, $1 \le j \le k$, $x \ne z_{js}^1$, $0 \le s \le q-1$.

On the other hand, by (20) we see that $\{P_\epsilon\}$ and $\{Q_\epsilon\}$ are uniformly bounded on I as $\epsilon \to 0$. Hence, there exist $M > 0$ and $\epsilon_1 > 0$ such that $|(fQ_\epsilon - P_\epsilon)^{(q)}(x)| \le q!M$, $x \in I$, $0 < \epsilon < \epsilon_1$. So, from (21) we deduce that

$$\left|\frac{1}{\epsilon^q}(fQ_\epsilon - P_\epsilon)^\epsilon(x)\right| \le 2^q M, \quad x \in B_j, \quad x \ne z_{js}^1, \quad 0 < \epsilon < \epsilon_1.$$

According to (22) and the Lebesgue Convergence Theorem, we get

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^q}\|(fQ_\epsilon - P_\epsilon)^\epsilon\|_{B_j} = \frac{1}{q!}|(fT - S)^{(q)}(x_j)| \left\|\prod_{l=0}^{q-1}\left(x - z_{jl}^1\right)\right\|_{B_j}.$$

Now, substituting $x - x_j$ by t into the above equality gives

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{1}{\epsilon^q}\|(fQ_\epsilon - P_\epsilon)^\epsilon\|_{B_j} &= \frac{1}{q!}|(fT - S)^{(q)}(x_j)|\|t^q - M_{pq}(t)\|_p \\ &= \frac{1}{q!}|(fT - S)^{(q)}(x_j)|\mathcal{K}_{pq}.\end{aligned}$$

□

Theorem 8. *Let $f \in \mathcal{C}^q(I)$ and let $\{(S_\epsilon, T_\epsilon)\} \subset \mathcal{H}_m^n$ be a net of best approximant pairs of f from $\mathcal{H}_m^n$ with respect to $\|\cdot\|_\epsilon$. Then $\{S_\epsilon\}$ and $\{T_\epsilon\}$ are uniformly bounded on compact sets as $\epsilon \to 0$.*

PROOF. Since $\|T_\epsilon\|_\infty = 1$, $0 < \epsilon \le 1$, the net $\{T_\epsilon\}$ is uniformly bounded on compact sets.

Let $(S,T) \in \mathcal{W}_m^n(f,X)$. Then for each $1 \le j \le k$,

$$\begin{aligned}
\frac{\|(S_\epsilon T - T_\epsilon S)^\epsilon\|_{B_j}}{\epsilon^q} &= \frac{1}{\epsilon^q}\,\|(-T(fT_\epsilon - S_\epsilon) + T_\epsilon(fT - S))^\epsilon\|_{B_j} \\
&\le \frac{(2k)^{1/p}}{\epsilon^q}\,\|{-T}(fT_\epsilon - S_\epsilon) + T_\epsilon(fT - S)\|_\epsilon \\
&\le \frac{(2k)^{1/p}}{\epsilon^q}\,(\|T(fT_\epsilon - S_\epsilon)\|_\epsilon + \|T_\epsilon(fT - S)\|_\epsilon) \qquad (23)\\
&\le \frac{(2k)^{1/p}}{\epsilon^q}\,(\|fT_\epsilon - S_\epsilon\|_\epsilon + \|fT - S\|_\epsilon) \\
&\le \frac{2(2k)^{1/p}}{\epsilon^q}\,\|fT - S\|_\epsilon\,.
\end{aligned}$$

If $q = 0$, then

$$|(S_\epsilon T - T_\epsilon S)(x_j)| = O(1) \quad \text{as} \quad \epsilon \to 0, \quad 1 \le j \le k. \tag{24}$$

In otherwise, as $(fT - S)^{(l)}(x_j) = 0$, $0 \le l \le q-1$, $1 \le j \le k$, expanding $(fT-S)^\epsilon$ by its Taylor polynomial at x_j up to order $q-1$, it follows that for each $x \in B_j$, there exists $\xi_\epsilon(x) \in [x_j - \epsilon, x_j + \epsilon]$ such that $(fT - S)^\epsilon(x) = \frac{\epsilon^q}{q!}(fT-S)^{(q)}(\xi_\epsilon(x))(x - x_j)^q$. So, $\|(fT-S)^\epsilon\|_{B_j} = O(\epsilon^q)$ as $\epsilon \to 0$, and consequently

$$\|fT - S\|_\epsilon = O(\epsilon^q) \quad \text{as} \quad \epsilon \to 0. \tag{25}$$

Therefore, by (23), we get $\|(S_\epsilon T - T_\epsilon S)^\epsilon\|_{B_j} = O(\epsilon^q)$ as $\epsilon \to 0$, $1 \le j \le k$. Since $(S_\epsilon T - T_\epsilon S)^\epsilon \in \Pi^{n+m}$ on B_j, according to Lemma 2.2 in [5], we have

$$\left|(S_\epsilon T - T_\epsilon S)^{(i)}(x_j)\right| = O(\epsilon^{q-i}) \quad \text{as} \quad \epsilon \to 0, \tag{26}$$

$1 \le j \le k$, $0 \le i \le q$. Since $n + m + 1 < k(q+1)$, from (24) and (26) we show that $\{S_\epsilon T - T_\epsilon S\} \subset \Pi^{n+m}$ is uniformly bounded on I as $\epsilon \to 0$; i.e., there exist $M > 0$ and $\epsilon_1 > 0$ such that

$$|(S_\epsilon T - T_\epsilon S)(x)| \le M, \quad x \in I, \quad 0 < \epsilon < \epsilon_1.$$

As $|T_\epsilon S(x)| \le \|S\|_\infty$, $x \in I$, $0 < \epsilon < \epsilon_1$, we have $\|S_\epsilon\|_T = \max_{x\in I}|(S_\epsilon T)(x)| \le \|S\|_\infty + M$, $0 < \epsilon < \epsilon_1$. Finally, by the equivalence of the norms in Π^n, we conclude that $\{S_\epsilon\}$ is uniformly bounded on compact sets as $\epsilon \to 0$. $\square$

Theorem 9. *Let $f \in \mathcal{C}^q(I)$ and let $\{(S_\epsilon, T_\epsilon)\}$ be a net of best approximant pairs of f from $\mathcal{H}_m^n$ with respect to $\|\cdot\|_\epsilon$. Suppose that there exists a best Padé approximant pair of f at X, say (S,T), such that $\frac{S}{T}$ is normal. Then any cluster point of $\{(S_\epsilon, T_\epsilon)\}$ as $\epsilon \to 0$ is a best Padé approximant pair of f at X.*

PROOF. According to Theorem 8, it follows that the set of cluster points of the net $\{(S_\epsilon, T_\epsilon)\}$ as $\epsilon \to 0$ is nonempty. Now, it is sufficient to prove that if (S_*, T_*) is the limit point of $\{(S_{\epsilon_l}, T_{\epsilon_l})\}$ as $\epsilon_l \to 0$, then (S_*, T_*) is a best Padé approximant pair of f at X. If $q = 0$, then the result is obvious, because

$$\begin{aligned}\sum_{j=1}^{k} |(fT_* - S_*)(x_j)|^p &= \lim_{\epsilon_l \to 0} k\|fT_{\epsilon_l} - S_{\epsilon_l}\|_{\epsilon_l}^p \\ &\le \lim_{\epsilon_l \to 0} k\|fT - S\|_{\epsilon_l}^p \\ &= \sum_{j=1}^{k} |(fT - S)(x_j)|^p .\end{aligned}$$

Now assume $q > 0$. Let $1 \le j \le k$, $0 \le i \le q-1$. As in the proof of Theorem 8, we have

$$\left| (S_{\epsilon_l} T - T_{\epsilon_l} S)^{(i)}(x_j) \right| = O(\epsilon_l^{q-i}) \quad \text{as} \quad \epsilon_l \to 0. \tag{27}$$

Therefore, $(S_* T - T_* S)^{(i)}(x_j) = 0$. Since

$$S_* T - T_* S = -T(fT_* - S_*) + T_*(fT - S)$$

and $(S,T) \in \mathcal{W}_m^n(f,X)$, using the Leibniz rule we get $(T(fT_* - S_*))^{(i)}(x_j) = 0$, and thus

$$(fT_* - S_*)^{(i)}(x_j) = 0, \tag{28}$$

because $T(x_j) \ne 0$. As i and j are arbitrary, (S_*, T_*) is a Padé approximant pair of f at X, and so $(S_*, T_*) \in \mathcal{W}_m^n(f,X)$ since $\|T_*\|_\infty = 1$.

Expanding $(S_{\epsilon_l} T - T_{\epsilon_l} S)^{\epsilon_l}$ and $T_{\epsilon_l}(fT - S)^{\epsilon_l}$ by their Taylor polynomials at x_j up to order $q-1$, it follows that for each $x \in B_j$, there exist

$\xi_{\epsilon_l}(x), \eta_{\epsilon_l}(x) \in [x_j - \epsilon_l, x_j + \epsilon_l]$ such that

$$\begin{aligned}
T^{\epsilon_l}(x)\frac{1}{\epsilon_l^q}\left(fT_{\epsilon_l} - S_{\epsilon_l}\right)^{\epsilon_l}(x) &= \frac{1}{\epsilon_l^q}\left(T(fT_{\epsilon_l} - S_{\epsilon_l})\right)^{\epsilon_l}(x) \\
&= \frac{1}{\epsilon_l^q}\left(-\left(S_{\epsilon_l}T - T_{\epsilon_l}S\right)^{\epsilon_l}(x) + \left(T_{\epsilon_l}(fT - S)\right)^{\epsilon_l}(x)\right) \\
&= -\sum_{i=0}^{q-1}\frac{\epsilon_l^{i-q}\left(S_{\epsilon_l}T - T_{\epsilon_l}S\right)^{(i)}(x_j)}{i!}(x - x_j)^i \\
&\quad - \frac{\left(S_{\epsilon_l}T - T_{\epsilon_l}S\right)^{(q)}\left(\xi_{\epsilon_l}(x)\right)}{q!}(x - x_j)^q \\
&\quad + \frac{1}{q!}\sum_{s=0}^{q}\binom{q}{s}\left(fT - S\right)^{(s)}\left(\eta_{\epsilon_l}(x)\right)T_{\epsilon_l}^{(q-s)}(\eta_{\epsilon_l}(x))(x - x_j)^q.
\end{aligned} \tag{29}$$

As $T(x_j) \neq 0$, from (27) there exist a subsequence of $\{\epsilon_l\}$, which we denote the same way, and $a_{ij} \in \mathbb{R}$, $0 \leq i \leq q-1$, $1 \leq j \leq k$, such that

$$\lim_{\epsilon_l \to 0} \epsilon_l^{i-q}\left(S_{\epsilon_l}T - T_{\epsilon_l}S\right)^{(i)}(x_j) = T(x_j)a_{ij}. \tag{30}$$

According to Theorem 8, we have

$$\lim_{\epsilon_l \to 0}\frac{\left(S_{\epsilon_l}T - T_{\epsilon_l}S\right)^{(q)}\left(\xi_{\epsilon_l}(x)\right)}{q!} = \frac{(S_*T - T_*S)^{(q)}(x_j)}{q!}, \tag{31}$$

so (28)-(31) imply that

$$\begin{aligned}
&\lim_{\epsilon_l \to 0}\frac{1}{\epsilon_l^q}\left(fT_{\epsilon_l} - S_{\epsilon_l}\right)^{\epsilon_l}(x) \\
&\quad = \frac{1}{q!\,T(x_j)}\Big(-(S_*T - T_*S)^{(q)}(x_j) + (fT - S)^{(q)}T_*(x_j)\Big)(x - x_j)^q \\
&\quad\quad - \sum_{i=0}^{q-1} a_{ij}(x - x_j)^i \\
&\quad = \frac{1}{q!}\left(fT_* - S_*\right)^{(q)}(x_j)\,(x - x_j)^q - \sum_{i=0}^{q-1} a_{ij}(x - x_j)^i
\end{aligned}$$

uniformly on B_j. Therefore, substituting $x - x_j$ by t into the following inequality gives

$$\begin{aligned}
&\lim_{\epsilon_l \to 0} \frac{1}{\epsilon_l^q} \left\| (fT_{\epsilon_l} - S_{\epsilon_l})^{\epsilon_l} \right\|_{B_j} \\
&\quad = \left\| \frac{1}{q!} (fT_* - S_*)^{(q)} (x_j) (x - x_j)^q - \sum_{i=0}^{q-1} a_{ij} (x - x_j)^i \right\|_{B_j} \\
&\quad \geq \left| \frac{1}{q!} (fT_* - S_*)^{(q)} (x_j) \right| \left\| t^q - M_{pq}(t) \right\|_p \\
&\quad = \frac{1}{q!} \left| (fT_* - S_*)^{(q)} (x_j) \right| \mathcal{K}_{pq},
\end{aligned} \tag{32}$$

$1 \leq j \leq k$. Since $\frac{S}{T}$ is normal, from Theorem 7, there exists a subsequence of $\{\epsilon_l\}$, which we denote the same way again, such that $\{(P_{\epsilon_l}, Q_{\epsilon_l})\} \subset \mathcal{H}_m^n$ and

$$\lim_{\epsilon_l \to 0} \frac{1}{\epsilon_l^q} \left\| (fQ_{\epsilon_l} - P_{\epsilon_l})^{\epsilon_l} \right\|_{B_j} = \frac{1}{q!} \left| (fT - S)^{(q)} (x_j) \right| \mathcal{K}_{pq}, \tag{33}$$

$1 \leq j \leq k$. But $\{(S_{\epsilon_l}, T_{\epsilon_l})\}$ is a net of best approximant pairs of f from $\mathcal{H}_m^n$, so (32) and (33) imply

$$\sum_{j=1}^{k} \left| (fT_* - S_*)^{(q)} (x_j) \right|^p \leq \sum_{j=1}^{k} \left| (fT - S)^{(q)} (x_j) \right|^p.$$

Finally, by (28), (S_*, T_*) is a best Padé approximant pair. $\square$

We say that the best Padé approximant pair of f at X is unique if, whenever $(P, Q), (U, V) \in \mathcal{W}_m^n(f, X)$ satisfy (3), then $(P, Q) \equiv (U, V)$.

The next corollary immediately follows.

Corollary 10. *Let $f \in \mathcal{C}^q(I)$, $q > 0$, and suppose that there exists a unique best Padé approximant pair of f at X, say (S, T), such that $\frac{S}{T}$ is normal. Then $\frac{S}{T}$ is a Padé approximant of f at X. In addition, if $\{(S_\epsilon, T_\epsilon)\}$ is a net of best approximant pairs of f from $\mathcal{H}_m^n$ with respect to $\| \cdot \|_\epsilon$, then $\frac{S_\epsilon}{T_\epsilon}$ converges to $\frac{S}{T}$ uniformly on some neighborhood of X as $\epsilon \to 0$.*

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