SURVEY

**Real Analysis Exchange**
Vol. 41(2), 2016, pp. 233–262

John C. Morgan II, Department of Mathematics and Statistics, California State Polytechnic University, Pomona, CA 91764, U.S.A.
email: `jcmorganii@yahoo.com`

# COMPLETION FROM AN ABSTRACT PERSPECTIVE

**Abstract**

A general conception of a completion is developed with illustrations, primarily crafted from Euclidean material, accompanied by a comprehensive survey of characterizations of completeness encountered in analysis.

## Contents



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Mathematical Reviews subject classification: Primary: 26A03, 54D35; Secondary: 54D80
Key words: Ascoli completion, compactification, filter space, real induction
Received by the editors August 15, 2015
Communicated by: P. D. Humke

## 1 Introduction

The expansion of the rational number line to include the multitude of irrational numbers which lay hidden in kinds of clouds at infinity, the subsequent formation of the extended real number line by the addition of negative and positive infinite numbers, the compactification of the Euclidean plane via adjunction of an infinitely remote entity to form the Riemann sphere, and the creation of projective space through the appending of points, lines, and planes at infinity to Euclidean space are completion processes. A space $\mathfrak{Y}$ is conceptualized as a completion of a space $\mathfrak{X}$ if there exists a system $\mathfrak{S}$ composed of cascades (certain types of nets) of nonempty open sets in $\mathfrak{X}$, an embedding of $\mathfrak{X}$ in $\mathfrak{Y}$, and an interrelated set mapping transforming the system $\mathfrak{S}$ into a system of cascades of nonempty open sets in $\mathfrak{Y}$ which are convergent to points in $\mathfrak{Y}$. The foregoing examples, as well as Niemytzki's tangent disc space and Alexandrov's two circle compactification, are discerned to be completions in this sense.

A general procedure, exemplified by Epstein's approach to Carathéodory's compactification of a bounded simply connected planar region by the aggregation of prime ends, is found to underlie the construction of copies of the cited completions. This procedure also plays a fundamental role in affirming that Alexandrov's one point compactification, Wallman's compactification, the Stone–Čech compactification, the Hewitt–Nachbin realcompactification, Morita's completion of generalized uniform spaces, Hausdorff's completion of metrized spaces, and certain ordered spaces are completions in the presented sense.

Each of the two systems involved in the completion procedure begets a framework of descending cascades of open covers of the respective spaces relative to that pair of which the embedding is a uniform isomorphism. In certain situations it will actually be an isometry or order isomorphism. The observation that a generalized form of Bolzano's convergence principle is valid with respect to the associated frameworks of numerous completions prompts the investigation of related forms and implications of that principle in various settings. Detailed verification of assertions, oftentimes tedious, is omitted.

## 2 Abstract spaces

A *space* $\mathfrak{X} = (X, \mathcal{D})$ consists of a nonempty set $X$ of *points*, and a family $\mathcal{D}$ of nonempty sets, designated *domains*, with $X = \bigcup \mathcal{D}$. The domains containing a given point and their supersets are called *neighborhoods* of that point. Unions of families of domains are termed *open sets*. Spaces having the same open sets are said to be *basically equivalent spaces*. A subset of $X$ is *dense* in $\mathfrak{X}$ when it contains at least one point in each domain. A *subspace* $\mathfrak{P} = (P, \mathcal{D}_P)$ of a space $\mathfrak{X} = (X, \mathcal{D})$ consists of a nonempty subset $P$ of $X$ with $\mathcal{D}_P = \{D \cap P : D \in \mathcal{D} \text{ and } D \cap P \neq \emptyset\}$.

*Continuous functions* and *homeomorphisms* are defined in the usual neighborly manner. An *embedding* of a space $\mathfrak{X}$ in a space $\mathfrak{Y} = (Y, \mathcal{E})$ is a homeomorphism $h : X \to Y$ with $h(X)$ dense in $\mathfrak{Y}$.

Product spaces are defined as follows: Let $\mathbb{T}$ be a nonempty set, let $\Xi = (\mathfrak{X}_t : t \in \mathbb{T})$ be a nonempty collection of spaces $\mathfrak{X}_t = (X_t, \mathcal{D}_t)$, called *coordinate spaces*, and let $\mathcal{M}$ be a family of nonempty subsets of $\mathbb{T}$ with $\bigcup \mathcal{M} = \mathbb{T}$. Set $W = \prod(X_t : t \in \mathbb{T})$ and define $\mathcal{A}$ to be the family of all the product sets $A = \prod(A_t : t \in \mathbb{T})$ such that, for some $M \in \mathcal{M}$, the set $A_t \in \mathcal{D}_t$ whenever $t \in M$ and $A_t = X_t$ whenever $t \in \mathbb{T} \setminus M$. The space $\mathfrak{W} = (W, \mathcal{A})$ is the *product space determined* by $\Xi$ and $\mathcal{M}$. Particular cases considered below are the *Cartesian product spaces of order* $n$ ($\mathbb{T} = \{1, \ldots, n\}$ and $\mathcal{M} = \{\mathbb{T}\}$ for each $n \in \mathbb{N}$, the set of natural numbers), the *Tihonov product spaces* ($\mathbb{T}$ is an infinite set and $\mathcal{M}$ is the family of nonempty finite subsets of $\mathbb{T}$), and the *generalized Tihonov product spaces* ($\mathbb{T}$ is a set of regular cardinality $\aleph_\alpha$ and $\mathcal{M}$ is the family of nonempty subsets of $\mathbb{T}$ having cardinality less than $\aleph_\alpha$).

Limit points and concepts based thereon are defined relative to a hypothetical cardinal number $\aleph_\alpha$, while condensation points, separability, the hierarchy of Borel sets, and Baire's classification of functions are defined relative to a stipulated cardinal number $\aleph_\beta$. The familiar denumerably based $\mathcal{G}_\delta$–sets become $\mathcal{G}_\Delta$–sets (i.e., intersections of families having cardinality at most $\aleph_\beta$ comprised of open sets).

## 3 Index schemes

In his inaugural dissertation, Vietoris introduced the basic convergence theory of what are presently known as nets and filter bases. (See [52, pp. 184–186] and

[44, 45, 53].) Here a cardinality modification of his definition of an oriented set with no last element is adopted.

An *index scheme* $\mathfrak{I} = (\mathbb{I}, \prec, \aleph_\alpha)$ consists of a nonempty *index set* $\mathbb{I}$ on which is defined an irreflexive, transitive relation $\prec$ having the property that if $K$ is a subset of $\mathbb{I}$ with cardinality $|K|$ less than a stipulated cardinal number $\aleph_\alpha$, then there exists an index $\lambda$ such that $\kappa \prec \lambda$ for every $\kappa \in K$. We denote by $\mathbb{W}_\alpha$ the set of non-zero numbers less than the initial ordinal number $\omega_\alpha$ of cardinality $\aleph_\alpha$. Relative to an assigned index scheme for a space, the point set theoretic concepts mentioned above pertain to the stipulated cardinal number $\aleph_\alpha$ and the cardinality $\aleph_\beta$ of the index set. Following are the main *types* of index schemes.

(I1) The *sequential scheme* $\mathfrak{I} = (\mathbb{N}, <, \aleph_0)$.

(I2) The more general $\omega_\alpha$*–sequential scheme* $\mathfrak{I} = (\mathbb{W}_\alpha, <, \aleph_\alpha)$ for a regular cardinal number $\aleph_\alpha$.

(I3) The *Cartesian scheme* $\mathfrak{I} = (\mathbb{H}^n, \prec, \aleph_\alpha)$ of order $n \in \mathbb{N}$ for a fixed index scheme $(\mathbb{H}, <, \aleph_\alpha)$ with indices $\langle \xi_1, \ldots, \xi_n \rangle \prec \langle \eta_1, \ldots, \eta_n \rangle$ if and only if $\xi_j < \eta_j$ for all $j = 1, \ldots, n$.

(I4) *Moore's index scheme* $\mathfrak{I} = (\mathbb{I}, \prec, \aleph_0)$ for an infinite set $\mathbb{T}$ with $\mathbb{I}$ the family of nonempty finite subsets of $\mathbb{T}$ and $\prec$ proper set inclusion.

(I5) The *transordinal scheme* $\mathfrak{I} = (\mathbb{I}, \prec, \aleph_\alpha)$ for a set $\mathbb{T}$ of cardinality at least $\aleph_\alpha$, where $\aleph_\alpha$ is a regular cardinal number, the set $\mathbb{I}$ consists of the functions $\lambda : \mathbb{T} \to \mathbb{W}_\alpha \cup \{0\}$ for which the set $N(\lambda) = \{t \in \mathbb{T} : \lambda(t) \neq 0\}$ satisfies the condition $0 < |N(\lambda)| < \aleph_\alpha$, and the relation $\prec$ is defined by $\lambda \prec \mu$ if and only if $\lambda(t) < \mu(t)$ for all $t \in N(\lambda)$ (see [39]).

The latter three schemes are employed for the respective product spaces in Section 2. An index scheme is *well-directed* if $|\{\kappa \in \mathbb{I} : \kappa \prec \lambda\}| < \aleph_\alpha$ for all $\lambda \in \mathbb{I}$, *well-founded* if every nonempty subset of $\mathbb{I}$ has at least one minimal element, and *well-endowed* when it is well-directed and well-founded.

Functions defined on an index set $\mathbb{I}$ are termed *cascades*. A cascade $\hat{F}$ of sets or families of sets is a *descending cascade* if $F(\mu) \subset F(\lambda)$ for all indices $\lambda, \mu \in \mathbb{I}$ with $\lambda \prec \mu$. A *cascade of sets or points converges to a point* in a space if its terms are eventually contained in any neighborhood of the point, in which case, the point is called a *limit* of the cascade.

A *framework* for a space is a descending cascade $\hat{\mathcal{R}}$ of coverings $\mathcal{R}(\lambda)$ of the space comprised of nonempty open sets. For the *real line* $\mathfrak{R} = (\mathbb{R}, \mathcal{D})$, whose domains are the intervals $(a, b)$ with $a < b$, the *primary sequential framework* is that for which $\mathcal{R}(\lambda)$ is the family of those intervals having length at most $\lambda^{-1}$. The assignment of a framework to a space induces a notion of "smallness." Each subset of a set in a given family $\mathcal{R}(\lambda)$ is called a $\lambda$–*small set.* A family of sets in the space is said to *contain arbitrarily small sets* if it contains at least one nonempty $\lambda$–small set for every $\lambda \in \mathbb{I}$. Generalizing the concept of a strong measure zero set, a set is designated a *minuscule set* when, for each cascade $\hat{\lambda}$ of indices, there exists a cascade $\hat{M}$ of sets $M(\xi) \in \mathcal{R}(\lambda(\xi))$ covering that set.

Given spaces $\mathfrak{X}, \mathfrak{Y}$ with assigned frameworks $\hat{\mathcal{R}}$, $\tilde{\mathcal{S}}$ having index sets $\mathbb{I}$, $\mathbb{J}$, respectively, and a set $P \subset X$, a function $f : P \to Y$ is called a *uniform function* relative to $\langle \hat{\mathcal{R}}, \tilde{\mathcal{S}} \rangle$ if, for every $\nu \in \mathbb{J}$, there exists $\lambda \in \mathbb{I}$ such that the image of each $\lambda$–small subset of P is a $\nu$–small subset of Y. Minuscule sets remain such under uniform functions for spaces with frameworks of the same type. See [49]. As evidenced by Dirichlet's characteristic function of the set of rational numbers, a uniform function may be discontinuous. A one-to-one continuous uniform function whose inverse is also a continuous uniform function is termed a *uniform isomorphism.*

## 4 Completeness

The conception of completeness of the real line originating with Bolzano (see [10] and [46, p. 171]) amounts to distinguishing a certain system of sequences of points and deeming each of them convergent to some point. This stipulation is equivalent to the specification of a particular system of sequences of nonempty open sets on the line which are convergent to points. In an endeavor to unify constructions of completions and compactifications, the latter idea appears promising.

A *system* for a space is a nonempty set of cascades for a fixed index scheme whose terms are nonempty open sets. The terminology *sequential system, Moore system*, etc., identifies the type of index scheme. A space is *complete relative to an assigned system* if each cascade in the system converges to some point in the space.

Let $\mathfrak{W}$ be a product space with systems $\mathfrak{S}_t$ of the same type assigned to each of the spaces $\mathfrak{X}_t$ in the collection $\Xi$. Under certain assumptions, a product system $\mathfrak{S}$ is defined for $\mathfrak{W}$. For a Cartesian product space of order $n$ it is presumed that all the index schemes $\mathfrak{I}_t = (\mathbb{H}_t, \prec_t, \aleph_\alpha)$ have a common type $(\mathbb{H}, <, \aleph_\alpha)$, the Cartesian index scheme $\mathfrak{I}$ is assigned to $\mathfrak{W}$, and $\mathfrak{S}$ consists of the cascades $\hat{C}$ of product sets $C(\lambda) = \prod(C_t(\lambda) : t \in \mathbb{T})$ satisfying the condition

$$(*) \text{ there is an assemblage } \Delta(\hat{C}) = (\check{S}_t : t \in \mathbb{T}) \text{ with } \check{S}_t \in \mathfrak{S}_t \text{ for each } t \in \mathbb{T}$$

such that, for all indices $\lambda = \langle \xi_1, \ldots, \xi_n \rangle \in \mathbb{I}$, the set $C_t(\lambda) = S_t(\xi_t)$ for every $t \in \mathbb{T}$.

For a Tihonov product space it is presumed that each system $\mathfrak{S}_t$ is comprised of sequences of proper subsets of the sets $X_t$, Moore's index scheme for $\mathbb{T}$ is assigned to $\mathfrak{W}$, and $\mathfrak{S}$ consists of the cascades of product sets satisfying the condition $(*)$ with the modification that $C_t(\lambda) = S_t(|\lambda|)$ for each $t \in \lambda$ and $C_t(\lambda) = X_t$ for each $t \in \mathbb{T} \setminus \lambda$, for all $\lambda \in \mathbb{I}$. For a generalized Tihonov product space it is presumed that each system $\mathfrak{S}_t$ is comprised of $\omega_\alpha$–sequences of proper subsets of the sets $X_t$, the transordinal index scheme is assigned to $\mathfrak{W}$, and $\mathfrak{S}$ consists of the cascades of product sets satisfying condition $(*)$ with the modification that $C_t(\lambda) = S_t(\sup(\lambda))$ whenever $t \in N(\lambda)$ and $C_t(\lambda) = X_t$ whenever $t \in \mathbb{T} \setminus N(\lambda)$, for all $\lambda \in \mathbb{I}$. The stated conditions ensure that a cascade $\hat{C} \in \mathfrak{S}$ converges to a point $w = (w_t : t \in \mathbb{T})$ in $\mathfrak{W}$ if and only if $\breve{S}_t \in \Delta(\hat{C})$ converges to $w_t$ in $\mathfrak{X}_t$ for all $t \in \mathbb{T}$ and that the open, continuous projection mappings are uniform functions. Ensuing theoretical statements pertaining to product spaces are valid for each of these product systems.

**Theorem 4.1.** *A product space is complete if and only if all coordinate spaces are complete.*

## 5 Completion formalized

Let $\mathfrak{X} = (X, \mathcal{D})$ and $\mathfrak{Y} = (Y, \mathcal{E})$ be spaces with $\mathcal{H}$ the family of nonempty open sets in $\mathfrak{Y}$. Denote by $\mathcal{B}$ the family of terms of cascades belonging to a system $\mathfrak{S}$ assigned to $\mathfrak{X}$. Given a mapping $\varphi : \mathcal{B} \to \mathcal{H}$, we define $\varphi(\mathfrak{S})$ to be the system of cascades $\varphi(\hat{S})$ with terms $\varphi(S(\lambda))$, for all cascades $\hat{S} \in \mathfrak{S}$.

A space $\mathfrak{Y}$ is a *completion of a space* $\mathfrak{X}$ if, for some system $\mathfrak{S}$ of cascades in $\mathfrak{X}$, there exists a mapping $\varphi : \mathcal{B} \to \mathcal{H}$ such that $\mathfrak{Y}$ is complete relative to $\varphi(\mathfrak{S})$ and an embedding $h$ of $\mathfrak{X}$ in $\mathfrak{Y}$ which are interrelated by the condition: for all $x \in X$ and all $B \in \mathcal{B}$, the point $x \in B$ precisely when the point $h(x) \in \varphi(B)$; more accurately, $\mathfrak{Y}$ *is a completion of* [$\mathfrak{X}$, $\mathfrak{S}$] *with respect to* $\langle \varphi, h \rangle$. A completion is called a *sequential completion, Moore completion,* etc. according to the nature of the index scheme for $\mathfrak{S}$. A completion $\mathfrak{Y}$ of a space $\mathfrak{X}$ in which $h(X)$ is an open set in $\mathfrak{Y}$, is called an *augmentation* of $\mathfrak{X}$. A completion where the embedding is the identity mapping is termed an *extension.* In that case, the elements of $X$ are called *ordinary points*, while those of $Y \setminus X$ are called *extraordinary points.*

We note that the interrelating condition can be replaced by the condition: $h(B) = \varphi(B) \cap h(X)$ for all $B \in \mathcal{B}$. It is seen that $\varphi$ is one-to-one and, for all

$A, B \in \mathcal{B}$, the inclusion $h(A) \subset h(B)$ implies that $A \subset B$; when the reverse implication is true, the mapping and completion are called *monotone*.

Defining connectedness in the usual manner, we have in harmony with [51]

**Theorem 5.1.** *A completion of a connected space is a connected space.*

**Theorem 5.2.** *A homeomorphic image of a completion is a completion of the same space.*

The following is valid for each product space.

**Theorem 5.3.** *If $\mathfrak{W}$ is a product space for a collection of spaces $\mathfrak{X}_t$ and $\mathfrak{Y}_t$ is a completion of $\mathfrak{X}_t$ for each $t \in \mathbb{T}$, then the product space for those completions is a completion of $\mathfrak{W}$.*

The following examples are noteworthy.

**Example 5.4.** *Let $\mathcal{Q} = (\mathbb{Q}, \mathcal{D})$ be the rational line with domains the open intervals $(a, b)$. Let $\mathfrak{S}$ be comprised of the descending sequences of domains whose $n^{th}$ term has length $n^{-1}$. The real line $\mathfrak{R}$ is a completion of $[\mathcal{Q}, \mathfrak{S}]$.*

*Denote by $(Y, <')$ the ordered set obtained by expansion of the natural ordering $<$ of $\mathbb{R}$ wherein each rational number $x$ is replaced by three consecutive elements $\alpha(x) <' \beta(x) <' \gamma(x)$. (The set $Y$ is order isomorphic to the subset of $\mathbb{R}$ resulting from adjoining to the Cantor set the midpoints of all contiguous open intervals and deleting $0$ and $1$.) Let $\mathcal{E}$ consist of the open intervals $(\beta(u), \beta(v))$ of $Y$ for $u, v \in \mathbb{Q}$ with $u < v$. The space $\mathfrak{Y} = (Y, \mathcal{E})$, referred to as the* tri-rational line*, is also a completion of $[\mathcal{Q}, \mathfrak{S}]$ which is not a Hausdorff space.*

We denote by $\mathfrak{R}^m = (\mathbb{R}^m, \mathcal{D})$ *Euclidean $m$–space* whose domains are the open balls with respect to the *Euclidean metric $\delta_m$*, for each $m \in \mathbb{N}$.

**Example 5.5.** *Let $\mathfrak{S}$ consist of the sequences $\hat{A}(x, \cdot)$ of intervals $A(x, n) = (x - (2n)^{-1}, x + (2n)^{-1})$ of $\mathfrak{R}^1$, for each $x \in \mathbb{R}$, together with the sequences $\hat{C}$ and $\hat{D}$ of rays $C(n) = (*, -n) = \{x \in \mathbb{R} : x < -n\}$ and $D(n) = (n, *) = \{x \in \mathbb{R} : x > n\}$. The extended real line with domains the nonempty open intervals of its order topology is a completion of $[\mathfrak{R}^1, \mathfrak{S}]$.*

**Example 5.6.** *Take $\mathfrak{X} = (X, \mathcal{D})$ to be the open unit disc of the Euclidean plane centered at $\theta$ with $\mathcal{D}$ the family of open discs having center $x \in X$ and radius $r < 1 - \delta_2(\theta, x)$. For each $x \in X$ and $n \in \mathbb{N}$, define $A(x, n)$ to be the open disc centered at $x$ with radius $[1 - \delta_2(\theta, x)] \cdot 2^{-n}$. For each half line $L$ emanating from $\theta$ and each $n \in \mathbb{N}$, define $C(L, n)$ as the open disc centered* *at the point on $L$ at distance $1 - 2^{-n}$ from $\theta$ with radius $2^{-n}$. Let $\mathfrak{S}$ be the totality of sequences $\hat{A}(x, \cdot)$ and $\hat{C}(L, \cdot)$.*

*Denote by $\mathbb{C}_1$ the unit circle and by $Y$ the closed unit disc. Let $\mathcal{E}$ consist of the open discs in $\mathcal{D}$ and the sets $C(L, n) \cup \alpha(L)$, where $\alpha(L) \in L \cap \mathbb{C}_1$, for all half lines $L$. The space $\mathfrak{Y} = (Y, \mathcal{E})$ is an augmentation of $\mathfrak{X}$, referred to as the* tangent disc space determined by $X$. *This completion is a rotund version of Niemytzki's tangent disc space for the upper half plane introduced in [41, p. 70].*

**Example 5.7.** *Let $Y$ be the union of the concentric circles $\mathbb{C}_1$ and $\mathbb{C}_2 = \{u \in \mathbb{R}^2 : \delta_2(\theta, u) = 2\}$. Let $\mathcal{E}$ be comprised of the singleton subsets of $\mathbb{C}_2$ and the sets of the form $I \cup \rho[I \setminus m(I)]$ where $I$ is an open arc on $\mathbb{C}_1$ with midpoint $m(I)$ and $\rho : \mathbb{C}_1 \to \mathbb{C}_2$ is the radial projection mapping. The compact space $\mathfrak{Y} = (Y, \mathcal{E})$ is* Alexandrov's two circle space. *See [2, pp. 13–15].*

*Define $\mathfrak{X} = (X, \mathcal{D})$ with $X = \mathbb{C}_2$ and $\mathcal{D}$ the family of nonempty subsets of $\mathbb{C}_2$. Let $\mathfrak{S}$ consist of the sequences $\hat{A}(x, \cdot)$ of singletons $A(x, n) = \{x\}$ and the sequences $\hat{C}(x, \cdot)$ of sets $C(x, n) = J(x, n) \setminus \{x\}$ where $J(x, n)$ is the open arc on $X$ centered at $x$ having arclength $n^{-1}$, for all $x \in X$. The space $\mathfrak{Y}$ is a completion of $[\mathfrak{X}, \mathfrak{S}]$ with respect to the identity function and the mapping $\varphi$ defined for all $x \in X$ and $n \in \mathbb{N}$ by $\varphi[A(x, n)] = A(x, n)$ and $\varphi[C(x, n)] = C(x, n) \cup \rho^{-1}[J(x, n)]$.*

## 6 Representation systems

A system for a space is designated a *representation system* if each point is representable as a limit of at least one cascade in the system all of whose terms contain that point. Each space possessing a representation system is necessarily a *topological space* (defined here as a space where intersections of pairs of neighborhoods of any point contain a neighborhood of the point). Every topological space $\mathfrak{X}$ has at least one Moore representation system. Namely, let $\tilde{M} = (M(\xi) : \xi \in \mathbb{W}_\gamma)$ be an arrangement of least cardinality of all nonempty open sets, one of which is repeated $\aleph_0$ times (to cover the finite case). Take $\mathfrak{I}$ to be Moore's index scheme for $\mathbb{T} = \mathbb{W}_\gamma$. For each $x \in X$ and $\lambda \in \mathbb{I}$, set $S(x, \lambda) = \bigcap\{M(\xi) : \xi \in \lambda \text{ and } x \in M(\xi)\}$. Then $\mathfrak{S} = \{\hat{S}(x, \cdot) : x \in X\}$ is a representation system for $\mathfrak{X}$.

Product systems are representation systems if and only if all the coordinate systems are representation systems. If $\mathfrak{P}$ is a subspace of a space $\mathfrak{X}$ having an assigned system $\mathfrak{S}$, then the *subspace system* assigned to $\mathfrak{P}$ is the system of the same type as $\mathfrak{S}$ composed of the cascades $\hat{T}$ of sets in $\mathfrak{P}$ for which there exists a cascade $\hat{S} \in \mathfrak{S}$ such that $T(\lambda) = S(\lambda) \cap P \neq \emptyset$ for all $\lambda \in \mathbb{I}$. Subspace systems determined by representation systems are representation systems, and we have

**Theorem 6.1.** *If a space is complete relative to a representation system, then each closed subspace is complete relative to its subspace system.*

A completion $\mathfrak{Y}$ of a pair $[\mathfrak{X}, \mathfrak{S}]$ with respect to $\langle\varphi, h\rangle$ is called a *representable completion* when $\varphi(\mathfrak{S})$ is a representation system for $\mathfrak{Y}$. The topological identity of completions of a common pair is actualized in the following situation.

**Theorem 6.2.** *If $\mathfrak{Y}$ and $\mathfrak{Z}$ are regular Kolmogorov spaces which are monotone, representable completions of the same pair $[\mathfrak{X}, \mathfrak{S}]$ with respect to $\langle\varphi, h\rangle$ and $\langle\psi, i\rangle$, then there is a homeomorphism $j$ from $\mathfrak{X}$ onto $\mathfrak{Z}$ with $j \circ h = i$.*

Each representation system $\mathfrak{S}$ generates the framework $\hat{\mathcal{R}}\langle\mathfrak{S}\rangle$ whose terms $(\hat{\mathcal{R}}\langle\mathfrak{S}\rangle)(\lambda)$ are comprised of the sets occurring in the $\lambda$–*tail* $\{S(\mu) : \mu \succcurlyeq \lambda\}$ of some cascade $\hat{S} \in \mathfrak{S}$, leading to the deduction of a general fact.

**Theorem 6.3.** *If $\mathfrak{Y}$ is a completion of $[\mathfrak{X}, \mathfrak{S}]$ with $\mathfrak{S}$ a representation system, then the families of sets $\mathcal{T}(\lambda) = h[(\hat{\mathcal{R}}\langle\mathfrak{S}\rangle](\lambda)$ constitute a framework $\hat{\mathcal{T}}$ for the subspace of $\mathfrak{Y}$ determined by $h(X)$, and the embedding is a uniform isomorphism relative to $\langle\hat{\mathcal{R}}\langle\mathfrak{S}\rangle, \hat{\mathcal{T}}\rangle$.*

A representation system all of whose cascades are descending is called a *determinant system.* Subspace and product systems of determinant systems are determinant systems.

## 7 Ascoli completions

An *Ascoli system* is a determinant system each of whose cascades converges to a point belonging to all its terms (see [6, pp. 1064–1065]). A space complete relative to such a system is said to be *Ascoli complete.* A completion $\mathfrak{Y}$ of a pair $[\mathfrak{X}, \mathfrak{S}]$ with respect to $\langle\varphi, h\rangle$ for which $\varphi(\mathfrak{S})$ is an Ascoli system is designated an *Ascoli completion.*

**Theorem 7.1.** *A subspace of an Ascoli complete Hausdorff space is Ascoli complete if and only if it is a closed subspace.*

**Theorem 7.2.** *A product space is Ascoli complete if and only if all coordinate spaces are Ascoli complete.*

The existence of Ascoli completions is guaranteed by

**Theorem 7.3.** *Each Kolmogorov space $\mathfrak{X}$ with a prescribed determinant system $\mathfrak{S}$ has a monotone Ascoli completion which is a Kolmogorov space.*

The construction employed, referred to as the *Ascoli Completion Process* and abbreviated **ACP**, is as follows (see [18, 20, 56]): Define $X^+$ to be the set of equivalence classes of cascades $\hat{S}$, $\hat{T} \in \mathfrak{S}$ which are interlaced (i.e., for every $\lambda \in \mathbb{I}$, there is index $\mu \succ \lambda$ such that $T(\mu) \subset S(\lambda)$ and, for every $\mu \in \mathbb{I}$, there is an index $\nu \succ \mu$ such that $S(\nu) \subset T(\mu)$). For each set $B$ belonging to the family $\mathcal{B}$ of all terms of cascades in $\mathfrak{S}$, define $\varphi(B)$ as the set of elements $x^+ \in X^+$ such that each cascade in $x^+$ has a term included in $B$ and set $\mathcal{D}^+ = \varphi(\mathcal{B})$. For each $x \in X$, define $h(x)$ to be the equivalence class containing a cascade in $\mathfrak{S}$ of neighborhoods of $x$ converging to $x$. The space $\mathfrak{X}^+ = (X^+, \mathcal{D}^+)$ is the desired completion.

It ensues that if $\hat{B}$ is any cascade of sets in $\mathcal{B}$ converging to a point $x$ in $\mathfrak{X}$, then $\varphi(\hat{B})$ converges to $h(x)$ in $\mathfrak{X}^+$. It is also follows that the embedding maps dense sets to dense sets.

In order that $\mathfrak{X}^+$ be a Hausdorff space, it is necessary and sufficient that $\mathfrak{S}$ satisfy the following condition.

(H) Every pair of noninterlaced cascades has at least one pair of disjoint terms.

The validity of this condition in general settings is hereditary and productive provided that the coordinate systems are comprised of descending cascades. A system satisfying condition (H) is called a *Hausdorff system*.

It is noted that the hypothesis that $\mathfrak{X}$ be a Kolmogorov space is only utilized to establish that the function $h$ is one-to-one and that $\mathfrak{X}^+$ is a Kolmogorov space. Absent that assumption, the constructed space is referred to as a *pseudocompletion*.

## 8 Illustrations

Application of the **ACP**to each of the pairs $[\mathfrak{X}, \mathfrak{S}]$ of Examples 5.4–5.7 yields (homeomorphic) copies of the completions $\mathfrak{Y}$ defined therein. We provide some additional informative examples.

In connection with Example 5.4, the question arises as to whether the **ACP**can be utilized directly to enlarge the ordered set of rational numbers through interposition of new entities corresponding to the irrational numbers, whose nebulous nature has been noted by Michael Stifel (see [23, pp. 74–75] and [32, pp. 251–252] ). An affirmative answer is obtained by either of the following approaches, akin to Dedekind's order-theoretic method and Cantor's arithmetic method. The use of the **ACP**in both constructions enhances Hausdorff's observation ([22, p. 316]) of the existence of an analogy between those procedures. See also Theorems 14.2 and 14.5.

**Example 8.1.** *Let* $\mathfrak{X} = (\mathbb{Q}, \mathcal{D})$ *with* $\mathcal{D}$ *the family of sets* $(*, q) = \{x \in \mathbb{Q} : x < q\}$ *for all* $q \in \mathbb{Q}$*, and let* $\mathfrak{S}$ *consist of the descending sequences of domains whose intersection is a nonempty set which is not a domain. Let* $\mathfrak{X}^+ = (X^+, \mathcal{D}^+)$ *be the completion of* $[\mathfrak{X}, \mathfrak{S}]$ *determined by the* **ACP***. The relation* $<^+$*, defined for* $w^+, x^+ \in X^+$ *by* $w^+ <^+ x^+$ *if and only if there are non-equivalent sequences* $\hat{E} \in w^+$ *and* $\hat{F} \in x^+$ *such that, for each* $m \in \mathbb{N}$*, there exists* $n \in \mathbb{N}$ *with* $E(n) \subset F(m)$*, is an order relation relative to which* $X^+$ *has neither initial nor final elements and the Infimum Principle (see page 253) is valid. The family* $\mathcal{B} = \mathcal{D}$ *and, for each set* $B = (*, q) \in \mathcal{B}$*, the set* $\varphi(B) = \{w^+ \in X^+ : w^+ <^+ h(q)\}$*. The embedding* $h$ *is an order isomorphism and* $h(\mathbb{Q})$ *is ordinally dense in* $X^+$*. We refer to this completion as the* Dedekind completion of the ordered set $\mathbb{Q}$.

**Example 8.2.** *Take* $\mathfrak{X}$ *to be the rational line. Define*

$$\mathbb{D} = \{k \cdot 2^{-n} : k \text{ is any integer and } n \in \mathbb{N}\}.$$

*The system* $\mathfrak{S}$ *of descending sequences of domains with endpoints in* $\mathbb{D}$ *whose* $n^{th}$ *term contains at most one of the fractions* $j \cdot 2^{-m}$ *in* $\mathbb{D}$ *with* $m \leq n$ *and has none of these fractions as an endpoint is a Hausdorff determinant system. The relation* $<^+$ *defined for* $w^+, x^+ \in X^+$ *by* $w^+ <^+ x^+$ *when and only when there exist* $\hat{E} \in w^+$*,* $\hat{F} \in x^+$*, and* $p \in \mathbb{N}$ *such that* $E(p) < F(p)$ *(i.e.,* $u < v$ *for all* $u \in E(p)$ *and* $v \in F(p)$*) satisfies the order-theoretic axioms characterizing the real line. The embedding* $h$ *is an order isomorphism and* $\varphi(B) = \{x^+ \in X^+ : h(a) <^+ x^+ <^+ h(b)\}$ *for* $B = \{u \in X : a < u < b\} \in \mathcal{B}$*. This example is referred to as the* Cantorian completion of the ordered set $\mathbb{Q}$.

**Example 8.3.** *The space* $\mathfrak{X} = (\mathbb{R}, \mathcal{D})$ *whose domains are the intervals* $[a, b)$ *is called* Alexandrov's line *(see [2, pp. 6 and 76–78] and [12, 13]). Take* $\mathfrak{S}$ *to consist of the sequences* $\hat{A}(x, \cdot)$*,* $\hat{C}(x, \cdot)$*,* $\hat{D}$*, and* $\hat{E}$ *of sets* $A(x, n) = [x - n^{-1}, x)$*,* $C(x, n) = [x, x + n^{-1})$*,* $D(n) = (*, -n)$*, and* $E(n) = [n, *)$*. The completion* $\mathfrak{X}^+$ *derived via the* **ACP***is a compactification of* $\mathfrak{X}$ *where the set* $X^+$*, ordered as in the preceding example, has order type* $1 + 2\lambda + 1$*; i.e., like the extended real line with each real number* $x$ *replaced by consecutive elements* $\alpha(x) <^+ \beta(x)$*. The four varieties of sequences in* $\mathfrak{S}$ *correspond respectively to the elements* $\alpha(x)$*,* $\beta(x)$*, the minimal element, and the maximal element of* $X^+$*. The embedding is an order isomorphism with* $h(x) = \beta(x)$ *for each*

*$x \in \mathbb{R}$. The family $\varphi(\mathcal{B})$ is comprised of the sets*

$$\begin{aligned}\varphi[(A(x,n)] &= \{z^+ \in X^+ : \alpha(x - n^{-1}) <^+ z^+ <^+ \beta(x)\},\\ \varphi[C(x,n)] &= \{z^+ \in X^+ : \alpha(x) <^+ z^+ <^+ \beta(x + n^{-1})\},\\ \varphi[D(n)] &= \{z^+ \in X^+ : z^+ <^+ \beta(-n)\}, \text{ and}\\ \varphi[(E(n)] &= \{z^+ \in X^+ : \alpha(n) <^+ z^+\}\end{aligned}$$

*which constitute a basis for the order topology on $X^+$. (In the terminology of [2, p. 7], $\mathfrak{X}^+$ is basically equivalent to a space $T_\Theta$ with $\Theta = 1 + 2\lambda + 1$.)*

*It is noted that Alexandrov's two arrow space, defined on the same set $X^+$ having neighborhoods $\{\alpha(x)\} \cup \{z^+ \in X^+ : \beta(x) <^+ z^+ <^+ \alpha(y)\}$ and $\{z^+ \in X^+ : \beta(x) <^+ z^+ <^+ \alpha(y)\} \cup \{\beta(y)\}$ with $x < y$, is not basically equivalent to the space $\mathfrak{X}^+$, contrary to what a remark in [2, p. 77] seems to indicate.*

**Example 8.4.** *Let $\mathfrak{X}$ be Euclidean space $\mathfrak{R}^3$. For each $x \in X$, let $\hat{A}(x,\cdot)$ be the sequence of open balls $A(x,n)$ with center $x$ and radius $(2n)^{-1}$. Let $\mathcal{L}$ be the family of lines in $\mathfrak{X}$. For each $L \in \mathcal{L}$, select a point $o(L)$ on $L$ and, for each $n \in \mathbb{N}$, define $\pi(L,n)$ and $\rho(L,n)$ to be the planes perpendicular to $L$ containing the points $p(L,n)$ and $r(L,n)$ on $L$ at length $n$ from $o(L)$. The set $X \setminus [\pi(L,n) \cup \rho(L,n)]$ is composed of three disjoint open regions, the union of those two of which not containing $o(L)$ is denoted by $C(L,n)$. Take the system $\mathfrak{S}$ to consist of the sequences $\hat{A}(x,\cdot)$ for all $x \in X$ and the sequences $\hat{C}(L,\cdot)$ for all $L \in \mathcal{L}$. The completion $\mathfrak{X}^+$ of $[\mathfrak{X},\mathfrak{S}]$ determined by the* **ACP** *is a prototype for the classical projective space.*

*Denoting by $\omega(L)$ the equivalence class containing $\hat{C}(L,\cdot)$ for each $L \in \mathcal{L}$, we have $\omega(L) = \omega(M)$ precisely when $L$ and $M$ are coincident or parallel lines and the set $X^+ = h(X) \cup \{\omega(L) : L \in \mathcal{L}\}$. (A natural choice for a set of representative elements in the equivalence classes is the set of sequences $\hat{C}(L,\cdot)$ with $L$ belonging to the bundle of lines passing through one fixed point.) Denoting by $\mathcal{P}$ the family of planes of $\mathfrak{X}$, the* lines *of $\mathfrak{X}^+$ are the sets $h(L) \cup \{\omega(L)\}$, for all $L \in \mathcal{L}$, together with the sets $\lambda(P) = \{\omega(L) : L \in \mathcal{L}$ and $L \subset P\}$ for each $P \in \mathcal{P}$. The* planes *of $\mathfrak{X}^+$ are the sets $h(P) \cup \lambda(P)$ for all $P \in \mathcal{P}$ and the set $\Pi = \{\omega(L) : L \in \mathcal{L}\}$. The separation relation is defined by certain linear arrangements of points or circular arrangements of lines in $\mathfrak{X}$. (See [11, 17].)*

**Application 8.5.** *Taking $\mathfrak{S}$ as the system of descending sequences of nonempty open sets in a first countable Kolmogorov topological space, the* **ACP** *yields the fact that such a space can be embedded in a first countable Baire space (i.e., a space every nonempty open subset of which is a second category set).*

**Example 8.6.** *Assume that $\mathfrak{X}$ is a non-compact, locally compact Kolmogorov topological space. (A space is* locally compact *if, for each point $x$, we can single out an open neighborhood $N(x)$ of $x$ whose closure is a compact set.) Assign to $\mathfrak{X}$ the Moore scheme for the family $\mathbb{T}$ of nonempty open sets with compact closure. Let $\mathfrak{S}$ be the cascades $\hat{A}(x,\cdot)$ of sets $A(x,\lambda) = N(x) \cap [\bigcap\{G \in \lambda : x \in G\}]$ for all $x \in X$ and the cascade $\hat{C}$ of sets $C(\lambda) = X \setminus \mathrm{Cl}(\bigcup \lambda)$. The completion $\mathfrak{X}^+$ obtained via the* **ACP***is a compactification of $\mathfrak{X}$ with $X^+ = h(X) \cup \{\omega\}$, where $\omega$ is the equivalence class containing $\hat{C}$. If $\mathfrak{X}$ is a Hausdorff space, then $\mathfrak{X}^+$ is likewise; the open sets being the sets $h(G)$ for $G$ open in $\mathfrak{X}$, together with the sets $X^+ \setminus h(K)$ for each compact set $K$ in $\mathfrak{X}$. In this case, $\mathfrak{X}^+$ is identified as Alexandrov's one point compactification of $\mathfrak{X}$. See [1, 2 pp. 68–72, 22 p. 285].*

We formulate a generalized version of Alexandrov's two circle space.

**Example 8.7.** *Let $\mathfrak{W} = (W, \mathcal{C})$ be a compact Hausdorff topological space having no isolated points with $\mathcal{C}$ closed under finite intersections and satisfying the condition: If $J, K \in \mathcal{C}$, $u \in J$, $v \in K$ and the inclusion $J \setminus \{u\} \subset K \setminus \{v\}$ holds, then $J \subset K$. Let $\mathfrak{J}$ be Moore's scheme for $\mathbb{T} = \mathcal{C}$. For each $w \in W$ and each $\lambda \in \mathbb{I}$ , select a neighborhood $N(w) \in \mathcal{C}$, define $J(w,\lambda) = N(w) \cap [\bigcap\{C \in \lambda : w \in C\}]$, and define $G(w,\lambda) = J(w,\lambda) \setminus \{w\}$. Take $\mathfrak{X} = (X, \mathcal{D})$ to be the discrete space with $\mathcal{D} = \{\{w\} : w \in W\}$, and let $\mathfrak{S}$ be comprised of the constant cascades $\hat{A}(x,\cdot)$ with terms $A(x,\lambda) = \{x\}$ and the cascades $\hat{G}(x,\cdot)$ for all $x \in X$. The completion $\mathfrak{X}^+$ of $[\mathfrak{X}, \mathfrak{S}]$ determined by the* **ACP***is a Hausdorff compactification of $\mathfrak{X}$ with $X^+ \setminus h(X)$ homeomorphic to the space $\mathfrak{W}$ via the bijection $k(w) = \{\hat{G}(w,\cdot)\}$. It is seen here that $\varphi[A(x,\lambda)] = h(x)$ and $\varphi[G(x,\lambda)] = h[G(x,\lambda)] \cup k[J(x,\lambda)]$ for all $x \in X$ and $\lambda \in \mathbb{I}$.*

## 9 Filter spaces

A general method originating with Alexandrov has been employed in the construction of various compactifications (see [7, 19, 26, 33, 50, 54]). A specific collection $\mathbb{O}$ of open filters is assigned to a topological space $\mathfrak{X}$, which includes all neighborhood filters and has the property that none of the filters is a subfamily of any other filter. The existence of such a set of filters necessitates that $\mathfrak{X}$ be a $T_1$–topological space. A filter space $\mathfrak{Y} = (\mathbb{O}, \mathcal{E})$ which is a $T_1$–topological space is then defined by specifying that $\mathcal{E}$ be comprised of the sets of the form $\{\mathcal{F} \in \mathbb{O} : B \in \mathcal{F}\}$ where $B$ varies over all nonempty open subsets of $\mathfrak{X}$. All Hausdorff compactifications of non-compact, completely regular Hausdorff topological spaces are obtainable by this method, and these compactifications have been depicted as completions of certain uniformized spaces (see [3, 4, 5, 50]). Wallman's $T_1$–compactification of a $T_1$–topological space is also constructible by this method (see [33, 50]). As a matter of fact, every space arising in this manner is a completion in the general sense and derivable via the **ACP**.

**Theorem 9.1.** *The filter space* $\mathfrak{Y} = (\mathbb{O}, \mathcal{E})$ *is a Moore completion of* $\mathfrak{X}$.

PROOF. As before, let $\tilde{M} = \{M(\xi) : \xi \in \mathbb{W}_\gamma\}$ be an arrangement of least cardinality of all nonempty open sets, one of which is repeated $\aleph_0$ times, and let $\mathfrak{J}$ be the Moore index scheme for $\mathbb{T} = \mathbb{W}_\gamma$. Define $\mathfrak{S}\langle\mathbb{O}\rangle$ as the system of cascades $\hat{S}(\mathcal{F}, \cdot)$ with $S(\mathcal{F}, \lambda) = \bigcap(\mathcal{F} \cap \{\mathcal{M}(\xi) : \xi \in \lambda\})$ for all $\mathcal{F} \in \mathbb{O}$ and all $\lambda \in \mathbb{I}$. [When $\mathfrak{X}$ is infinite, one can simply take $\mathbb{T}$ to be the family of all nonempty open sets and $S(\mathcal{F}, \lambda) = \bigcap(\mathcal{F} \cap \lambda)$.] A point $w$ is a limit for a cascade $\hat{S}(\mathcal{F}, \cdot)$ precisely when it is a limit for the filter $\mathcal{F}$. This system is thus a determinant system for $\mathfrak{X}$ with each point representable as a limit of the cascade determined by its neighborhood filter and $\mathcal{B}$ is the family of nonempty open sets.

For the completion $\mathfrak{X}^+ = (X^+, \varphi(\mathcal{B}))$ of $[\mathfrak{X}, \mathfrak{S}\langle\mathbb{O}\rangle]$ given by the **ACP**, the set $X^+$ consists of the singleton sets $\{\hat{S}(\mathcal{F}, \cdot)\}$ for all $\mathcal{F} \in \mathbb{O}$. Let $q : X^+ \to \mathbb{O}$ be the bijection $q(\{\hat{S}(\mathcal{F}, \cdot)\}) = \mathcal{F}$. Let $i = q \circ h$ and $\psi = q \circ \varphi$. The space $\mathfrak{Y} = (\mathbb{O}, \psi(\mathcal{B}))$ is a completion of $[\mathfrak{X}, \mathfrak{S}\langle\mathbb{O}\rangle]$ with respect to $\langle\psi, i\rangle$ whose domains are deduced to have the form $\psi(B) = \{\mathcal{F} \in \mathbb{O} : B \in \mathcal{F}\}$. □

We turn to the filter space completion for generalized uniform structures discussed by Morita in [38]. A topological space $\mathfrak{X}$ is said to be *parauniformized* by a nonempty collection $\Phi$ of open covers if the following two conditions are satisfied.

($\alpha$) The infimum of each pair of covers in $\Phi$ has a refinement in $\Phi$,

($\beta$) $\{\mathrm{St}(x, \mathcal{U}) : \mathcal{U} \in \Phi\}$ is a neighborhood base for each point $x$.

It is called *semiuniformized* if the conditions ($\alpha$), ($\beta$), and the following additional condition are satisfied.

($\gamma$) Every cover in $\Phi$ has a local star refinement in $\Phi$.

And $\mathfrak{X}$ is said to be *uniformized* by $\Phi$ if the conditions ($\alpha$), ($\beta$), and the following additional condition are satisfied.

($\delta$) Every cover in $\Phi$ has a star refinement in $\Phi$.

We refer to the collection $\Phi$ accordingly as a *parauniform, semiuniform*, or *uniform structure* for $\mathfrak{X}$. The topological spaces which can be parauniformized, semiuniformized, or uniformized coincide respectively with the weakly regular, regular, and completely regular spaces.

A filter $\mathcal{F}$ on $\mathfrak{X}$ is called a *weak star-filter* if it has the following two properties.

($\mu$) Every cover in $\Phi$ contains a superset of some set in $\mathcal{F}$,

($\nu$) For each $F \in \mathcal{F}$, there is a cover $\mathcal{U} \in \Phi$ with $\bigcup(\mathcal{F} \cap \mathcal{U}) \subset F$.

Completeness is defined in [38] by the requirement that every weak star-filter converge. (Alternative characterizations of this definition of completeness and related matters are found in [8, 9, 24, 48].)

If $\mathbb{O}$ consists of the weak star-filters on a $T_1$–topological space $\mathfrak{X}$ parauniformized by a collection $\Phi$, then the space $\mathfrak{Y}$ is parauniformized by the collection $\psi(\Phi) = \{\psi(\mathcal{U}) : \mathcal{U} \in \Phi\}$ and every weak star-filter with respect to $\psi(\Phi)$ converges in $\mathfrak{Y}$. If $\mathfrak{X}$ is semiuniformized (respectively, uniformized) by $\Phi$, then $\mathfrak{Y}$ is semiuniformized (respectively, uniformized) by $\psi(\Phi)$.

## 10 Bolzano's convergence condition

Of especial interest are completions with respect to whose generated frameworks a generalized form of Bolzano's convergence principle is valid. We formulate that and related conditions for an arbitrary framework.

Let $\hat{\mathcal{R}}$ be a prescribed framework for a space $\mathfrak{X}$. A cascade $\hat{M}$ of sets $M(\lambda) \in \mathcal{R}(\lambda)$ for each $\lambda \in \mathbb{I}$ is called a *regular cascade*. We say that a cascade $\hat{Q}$ of sets is *dominated by a cascade* $\hat{M}$ of sets if the inclusion $Q(\lambda) \subset M(\lambda)$ holds for all $\lambda \in \mathbb{I}$. A *diminishing cascade* is one comprised of nonempty sets which is dominated by a regular cascade. Generalizing the finite intersection property, *a family of sets has the intersection property* if each subfamily with cardinality less than $\aleph_\alpha$ has a nonempty intersection. A *cascade of sets has the intersection property* if its terms constitute a family having the intersection property. A regular cascade having the intersection property is designated a *principal cascade*. A cascade $\hat{x}$ of points is termed a *fundamental cascade* if there is a regular cascade $\hat{M}$ satisfying the condition that, for each $\lambda \in \mathbb{I}$, there exists $\mu \in \mathbb{I}$ such that $x(\pi) \in M(\lambda)$ for all indices $\pi \succcurlyeq \mu$. In the case of the primary sequential framework for the real line this condition is equivalent to *Bolzano's condition* that, for all positive real numbers $\epsilon$, there exists a natural number $n$ such that, for all natural numbers $p > n$, the quantity $|x(n) - x(p)| < \epsilon$ (see [10, 46 p.171]). Fundamental cascades remain fundamental cascades under uniform functions for spaces with frameworks of the same type.

By a *filter* on $X$, in the present context, we mean a nonempty family of nonempty subsets of $X$ which has the intersection property and includes all supersets contained in $X$ of each of its members. A *Cauchy filter* is a filter on $X$ containing a regular cascade of sets or, equivalently, containing arbitrarily small sets. As usual, a filter converges to a point in the space, if all neighborhoods of the point belong to the filter.

We consider the following statements.

(L1) Each regular cascade of pairwise intersecting sets converges.

(L2) Every principle cascade of sets converges.

(L3) All Cauchy filters converge.

(L4) Every fundamental cascade of points converges.

(L5) Each family of closed sets which has the intersection property and contains arbitrarily small sets has a nonempty intersection.

A space with a framework $\hat{\mathcal{R}}$ satisfying condition (L2) is said to be *principally complete relative to* $\hat{\mathcal{R}}$. In the vernacular of Section 5, a space is principally complete relative to $\hat{\mathcal{R}}$ if it is complete with respect to the system $\mathfrak{S}\langle\hat{\mathcal{R}}\rangle$ of principal cascades for $\hat{\mathcal{R}}$. A completion $\mathfrak{Y}$ of a pair $[\mathfrak{X}, \mathfrak{S}]$ with $\varphi(\mathfrak{S})$ a representation system is designated a *principal completion* when it is principally complete with respect to $\hat{\mathcal{R}}\langle\varphi(\mathfrak{S})\rangle$.

**Theorem 10.1.** *The implications* (*L*1) $\Rightarrow$ (*L*2) $\Rightarrow$ (*L*3) $\Rightarrow$ (*L*4) *and* (*L*2) $\Rightarrow$ (*L*5) *subsist. Whenever* $\aleph_\alpha$ *is a regular cardinal number, the implication* (*L*3) $\Rightarrow$ (*L*5) *is valid.*

We say that a *representation system* $\mathfrak{S}$ *satisfies Bolzano's Convergence Principle* (abbreviated **BCP**) if condition (L4) is satisfied by the framework $\hat{\mathcal{R}}\langle\mathfrak{S}\rangle$.

**Theorem 10.2.** *If a representation system satisfies* **BCP**, *then so do all closed subspace systems.*

It is seen that **BCP** is satisfied by the system $\varphi(\mathfrak{S})$ for many of the above completions, including the tri-rational line, the Dedekind completion (which is not a principal completion), the Cantorian completion (which is a principal completion), and Niemytzki's tangent disc space. This is not true of Alexandrov's two circle space and the given compactification of Alexandrov's line. Equivalent characterizations of **BCP** are discussed in the following sections.

## 11 Localization structures

A space $\mathfrak{X}$ has a *localization structure* if there exists a framework $\hat{\mathcal{R}}$ for $\mathfrak{X}$ satisfying the condition

(X) If $N$ is any neighborhood of a point $x$ then there is a subset $M$ of $N$ which is a neighborhood of $x$ and an index $\kappa$ with $\mathrm{St}(M, \mathcal{R}(\kappa)) \subset N$,

or the equivalent separation condition

(Y) If $F$ is a closed set not containing a given point $x$, then there exists a neighborhood $M$ of $x$ and an index $\mu$ such that $\mathrm{St}(M, \mathcal{R}(\mu)) \cap \mathrm{St}(F, \mathcal{R}(\mu)) = \emptyset$.

To such a space is assigned the system $\mathfrak{S}\langle\hat{\mathcal{R}}\rangle$. This class of spaces can be alternatively delineated in terms of a hypothetical system.

**Theorem 11.1.** *If $\mathfrak{S}$ is a representation system for a space $\mathfrak{X}$ satisfying the condition*

*(Z) For each neighborhood $N$ of a point $x$, there is a subset $M$ of $N$ which is a neighborhood of $x$ and an index $\kappa$ such that each term in the $\kappa$–tail of any cascade in $\mathfrak{S}$ which intersects $M$ is a subset of $N$,*

*then the framework $\hat{\mathcal{R}}\langle\mathfrak{S}\rangle$ satisfies condition (X). Inversely, if $\hat{\mathcal{R}}_0$ is a framework for a space $\mathfrak{X}$ satisfying condition (X), then $\mathfrak{S}\langle\hat{\mathcal{R}}_0\rangle$ is a representation system for $\mathfrak{X}$ satisfying condition (Z) with $\hat{\mathcal{R}}\langle\mathfrak{S}\langle\hat{\mathcal{R}}_0\rangle\rangle = \hat{\mathcal{R}}_0$.*

A representation system $\mathfrak{S}$ satisfying the condition (Z) is called a *localization system* for $\mathfrak{X}$ with $\hat{\mathcal{R}}\langle\mathfrak{S}\rangle$ the *associated localization structure.*

The fact that localization structures exist for each semi-uniformized space reveals that the spaces possessing a localization structure are precisely the regular topological spaces. A space having a localization structure is a normal space whenever the indexing relation is an order relation. Subspace and product systems formed from localization systems are localization systems.

Hewitt–Nachbin realcompactifications, Stone–Čech compactifications, and zero-dimensional compactifications have been constructed via the following general method: An infinite set $\mathbb{T}$ of continuous functions $f$ from a given $T_1$–topological $\mathfrak{X}$ to a space $\mathfrak{U}_f$ is specified for which the evaluation mapping $e$ of the space $\mathfrak{X}$ into the Tihonov product space $\mathfrak{Z}$ determined by the collection $(\mathfrak{U}_f : f \in \mathbb{T})$ is a homeomorphism. The desired space is the closure $\mathfrak{Y}^*$ of the subspace $\mathfrak{Y}$ of $\mathfrak{Z}$ determined by the set $Y = e(X)$. Now, each of the particular spaces $\mathfrak{U}_f$ can be assigned an Ascoli complete sequential localization system $\mathfrak{S}_f$ which is a determinant system satisfying condition (H); see page 242. In the first case, take each $\mathfrak{S}_f$ to be the system defined in Example 8.2 for the real line (in lieu of the rational line). In the second and third cases, take the $\mathfrak{S}_f$ to be the subspace systems of that system for the unit interval and the set $\{0, 1\}$, respectively. It ensues successively that the Moore product system $\mathfrak{S}_Z$ for the space $\mathfrak{Z}$, its subspace system $\mathfrak{S}_Y$, and the system $\mathfrak{S} = e^{-1}(\mathfrak{S}_Y)$ are determinant systems for their respective spaces. The **ACP**produces a completion $\mathfrak{Y}^+$ of $[\mathfrak{Y}, \mathfrak{S}]$ with respect to mappings $\langle\psi, g\rangle$ and $\mathfrak{Y}^+$ is a completion of $[\mathfrak{X}, \mathfrak{S}]$ with respect to the mappings $\langle\psi \circ e, g \circ e\rangle$. The recognition that the spaces $\mathfrak{Y}^*$ obtained in these situations are completions in the general sense is a consequence of the following fact.

**Theorem 11.2.** *If $\mathfrak{Z}$ is a Kolmogorov space with an Ascoli complete localization system $\mathfrak{S}_Z$ satisfying condition (H) then the completion $\mathfrak{Y}^+$ of each subspace $\mathfrak{Y}$ of $\mathfrak{Z}$ (relative to the subspace system $\mathfrak{S}_Y$) derived via the* **ACP***is homeomorphic to the subspace $\mathfrak{Y}^*$ of $\mathfrak{Z}$ determined by the set $Y^* = \mathrm{Cl}_Z(Y)$.*

Concerning the statements in the preceding section, there is

**Theorem 11.3.** *For a space having a localization structure with a well-directed index scheme, the conditions (L2) through (L4) are equivalent.*

Each uniform function on a subset of a space having an assigned framework to a space having a localization structure is a continuous function. In regard to extensions of continuous functions the classical Sierpiński–Zygmund and Lavrentiev theorems have the following generalized forms.

**Theorem 11.4.** *If $\mathfrak{X}$ and $\mathfrak{Y}$ are spaces having localization structures of identical type with $\mathfrak{Y}$ a Hausdorff space satisfying* **BCP***, then each function $f : P \to Y$ continuous on a subset $P$ of $X$ can be extended to a function $g : Q \to Y$ continuous on a $\mathcal{G}_\Delta$-set $Q$ where $P \subset Q \subset \mathrm{Cl}_X(P)$.*

**Theorem 11.5.** *If* **BCP** *is valid for Hausdorff spaces $\mathfrak{X}$ and $\mathfrak{Y}$ having localization structures of identical type then each homeomorphism between subsets $P$ of $X$ and $R$ of $Y$ can be extended to a homeomorphism between $\mathcal{G}_\Delta$–sets $Q$ and $S$ with $P \subset Q \subset \mathrm{Cl}_X(P)$ and $R \subset S \subset \mathrm{Cl}_Y(R)$.*

For generalizations of aspects of the descriptive theory of sets and functions to spaces possessing a localization structure, consult [27, 28, 29, 30, 38].

## 12 Pseudometrized spaces

A space $\mathfrak{X}$ is said to be *pseudometrized* (respectively, *metrized*) by a specific pseudometric (respectively, metric) $d$ on $X$ if it is basically equivalent to the space whose domains are the $d$–open balls. The *primary sequential framework* assigned to a pseudometrized space is that whose $n^{th}$ term consists of all open balls with radius at most $(2n)^{-1}$. This constitutes a localization structure for $\mathfrak{X}$. We assemble some characterizations of principal completeness for this framework.

**Theorem 12.1.** *The following conditions are equivalent relative to the primary sequential framework for a pseudometrized space.*

(M1) *All principal sequences of open balls converge,*

(M2) *Every Cauchy filter converges,*

(M3) *Each fundamental sequence of points converges,*

(M4) *The intersection of any family of closed sets, which has the finite intersection property and contains arbitrarily small sets, is nonempty,*

(M5) *Every regular sequence of pairwise intersecting open balls converges,*

(M6) *Each diminishing sequence of pairwise intersecting closed balls converges to some point belonging to all of its terms,*

(M7) *The intersection of any descending, diminishing sequence of closed balls is nonempty,*

(M8) *Every descending, diminishing sequence of open balls converges,*

(M9) *Each diminishing sequence of open balls with closure* $\mathrm{Cl}[S(n+1)] \subset S(n)$ *for all* $n \in \mathbb{N}$ *converges to some point belonging to all of its terms,*

(M10) *Every totally bounded infinite set has at least one limit point.*

*For all* $i \neq 2$ *or* 4, *condition* (Mi) *is equivalent to the condition* (Mi*)*: There exists a dense set* $Q$ *for which the condition* (Mi) *holds with the proviso that the centers of the hypothetical balls (or points) belong to* $Q$.
*For a metrized space there are the additional equivalences*

(M11) *All totally bounded closed sets are compact,*

(M12) *Each contraction mapping from a nonempty closed set to itself has a fixed point.*

See further ([25, 34, 42, 47, 50, 55].

To verify that Hausdorff's completion of a metrized space $\mathfrak{X}$ is a completion in the present sense, we assign to $\mathfrak{X}$ the system $\mathfrak{S}$ of descending sequences of open balls whose $n^{th}$ term has radius at most $(2n)^{-1}$ and apply the **ACP**to obtain the completion $\mathfrak{X}^+ = (X^+, \mathcal{D}^+)$ of $[\mathfrak{X}, \mathfrak{S}]$ with respect to $\langle \varphi, h \rangle$. The function

$$d^+(w^+, x^+) = \sup\{d(S(n), T(n)) : \hat{S} \in w^+, \hat{T} \in x^+, \text{ and } n \in \mathbb{N}\},$$

defined for all points $w^+, x^+ \in X^+$, is a pseudometric on $X^+$. However, as seen from the tri-rational completion of the rational line, $\mathfrak{X}^+$ need not be basically equivalent to the space $\mathfrak{X}^* = (X^+, \mathcal{D}^*)$ with $\mathcal{D}^*$ the family of $d^+$–open balls. Let $\varphi^* : \mathcal{B} \to \mathcal{D}^*$ associate with the open ball $B(x, r)$ in $\mathcal{B}$ the open ball $B^*(h(x), r)$ in $\mathcal{D}^*$. The space $\mathfrak{X}^*$ is a completion of $[\mathfrak{X}, \mathfrak{S}]$ with respect to $\langle \varphi^*, h \rangle$ that satisfies the condition (M8*) relative to the primary sequential framework for $\mathfrak{X}^*$ with $Q = h(X)$ and consequently satisfies the condition (M5).

Denote by $Y$ the quotient set obtained by identifying points $w^+, x^+ \in X^+$ with $d^+(w^+, x^+) = 0$, denote by $q : X^+ \to Y$ the quotient mapping, and denote by $d_Y$ the quotient metric. Then the composition $i = q \circ h : X \to Y$ is an isometry. Denote by $\mathfrak{Y} = (Y, \mathcal{E})$ the space whose domains are the open balls determined by $d_Y$. Let $\chi : \mathcal{D}^* \to \mathcal{E}$ associate with each open ball $B^*(x^+, r)$ in $\mathcal{D}^*$ the open ball $B_Y(q(x^+), r)$ in $\mathcal{E}$ and set $\psi = \chi \circ \varphi^*$. The space $\mathfrak{Y}$ is a completion of $[\mathfrak{X}, \mathfrak{S}]$ with respect to $\langle \psi, i \rangle$ satisfying condition (M5) with respect to the primary sequential framework for $\mathfrak{Y}$ which is isometric to Hausdorff's completion. Summarizing, we have

**Theorem 12.2.** *Each metrized space has a metrized completion satisfying* **BCP** *in which it is isometrically embedded.*

Deduced in consequence is

**Theorem 12.3.** *A metrized space $\mathfrak{X}$ is principally complete relative to its primary sequential framework if and only if, for every isometric mapping of $\mathfrak{X}$ into a metrized space $\mathfrak{Y}$, the image of $X$ is a closed set in $\mathfrak{Y}$.*

The pseudometrizable spaces form a subclass of the class of gaugeable spaces. Let $\mathfrak{X} = (X, \mathcal{D})$ be a space, $\mathbb{T}$ an infinite set, and $\Xi = (\mathfrak{X}_t : t \in \mathbb{T})$ a collection of spaces $\mathfrak{X}_t = (X, \mathcal{D}_t)$ pseudometrized by (not necessarily different) pseudometrics $d_t$ on $X$ with respective open balls $B_t(x, r)$. We say that $\mathfrak{X}$ is a *gaugeable space relative to* $\Xi$ if it is basically equivalent to the space whose domains are the *generalized open balls* $B(x, r, \lambda) = \cap\{B_t(x, r) : t \in \lambda\}$ with $x \in X$, $r \in \mathbb{R}^+$, and $\lambda$ a nonempty finite subset of $\mathbb{T}$. A space $\mathfrak{X}$ with such a specified collection is called a *gauged space*. The *primary Moore framework for a gauged space* $\hat{\mathcal{R}}$ is that where $\mathcal{R}(\lambda)$ consists of the generalized open balls $B(x, r, \mu)$ for all $x \in X$, $\mu \succ \lambda$, and $r \leq (2|\lambda|)^{-1}$. This framework is a localization structure for $\mathfrak{X}$.

**Theorem 12.4.** *For the primary Moore framework of a gaugeable space, the conditions (L1) through (L5) are equivalent.*

## 13 The Induction Principle

The symbol $X$ below denotes a set containing at least two elements which is ordered by a relation $<$. Elements smaller (respectively, larger) than a given element are called its *predecessors* (respectively, *successors*).

Let $\Phi$ be a propositional function of elements of $X$ and let $E = \{u \in X : \Phi(x)$ is true for all predecessors of $u\}$. The *Induction Principle* is the statement

(O1) If $E$ is nonempty and every element in $E$ has a successor in $E$, then $\Phi(x)$ is true for all $x \in X$.

This may be rephrased in the alternate logical form

(O1′) If $E$ is nonempty and $\Phi(x)$ is false for some element of $X$, then there exists an element in $E$ which has no successor in $E$.

Bolzano deduced the latter statement for the real line assuming the validity of his convergence principle. In that setting the equivalent form (O1) is referred to as *Bolzano's Induction Principle* or the *Real Induction Principle* (see [10, 14, 31, 43, 46 p.174]).

We list several related statements. *Dedekind's Principle,* the *Supremum Principle*, and the *Infimum Principle* are, respectively,

(O2) If $(U, V)$ is a partition of $X$ with $U < V$ then either $U$ has a final element or $V$ has an initial element.

(O3) Every nonempty subset of $X$ having an upper bound in $X$ has a supremum in $X$.

(O4) Every nonempty subset of $X$ having a lower bound in $X$ has an infimum in $X$.

**Theorem 13.1.** *The conditions* (O1) *through* (O4) *are equivalent for the ordered set $X$.*

A subset of an ordered set satisfying any one of these conditions with respect to its relativized ordering is called an *inductive set.*

An *ideal* of an ordered set is a nonempty subset which contains all predecessors of each of its elements; it is a *proper ideal* if it is a proper subset of that set. The *principal ideals* of an ordered set $M$ are the sets of the form $\{x \in M : x \leq w\}$ with $w \in M$. Let $\mathcal{J}(\mathcal{M})$ be the family of proper ideals of an ordered set $M$, with the exclusion of all non-principal ideals having a supremum in $M$. The family $\mathcal{J}(\mathcal{M})$ is an inductive set relative to ordering by inclusion whose subfamily of principal ideals is order isomorphic to $M$. Ergo

**Theorem 13.2.** *Each ordered set is order isomorphic to a subset of an inductive set.*

If $M$ is a subset of an inductive set $U$ then the set $\sigma_U(M) = \{\sup(I) : I \in \mathcal{J}(\mathcal{M})\}$ of suprema relative to $U$ is an inductive set containing $M$ and there results

**Theorem 13.3.** *If $X$ and $Y$ are inductive sets, $P \subset X$, and $R \subset Y$ then each order isomorphism between $P$ and $R$ can be extended to an order isomorphism between $\sigma_X(P)$ and $\sigma_Y(R)$.*

Assume now that $\mathfrak{X} = (X, \mathcal{D})$ is an *ordered space*; i.e., $X$ is an ordered set having at least two elements and $\mathfrak{X}$ is basically equivalent to a space whose domains are the nonempty intervals $(a, b)$, as well as the intervals $[\theta, b)$ when $X$ has an initial element $\theta$, and $(a, \omega]$ when $X$ has a final element $\omega$. The intervals $[a, b]$ with $a < b$ are termed *segments.* The *Covering Principle* is

(O5) Each family of domains covering a segment of $X$ has a finite subcovering.

**Theorem 13.4.** *The conditions* (O1) *through* (O5) *are equivalent for the space* $\mathfrak{X}$.

The *Connectedness Principle* and *Intermediate Value Principle* are, respectively, as follows.

(O6) Given two nonempty families of domains, the sets in each family being disjoint from those in the other family, the union of all the domains is not an interval.

(O7) For each continuous function $f : X \to X$, if $a, b, y \in X$ and $f(a) < y < f(b)$, then there is a point $c \in X$ between $a$ and $b$ such that $f(c) = y$.

**Theorem 13.5.** *When $X$ is ordinally dense, conditions* (O1) *through* (O7) *are equivalent for the space $\mathfrak{X}$.*

A family of sets is a *monotone family* if, for each pair of its sets, one is a subset of the other. The *Monotone Intersection Principle* and *Descending Interval Principle* are

(O8) The intersection of each monotone family of segments is nonempty.

(O9) Every descending sequence of segments has a nonempty intersection.

A point of $X$ is designated a limit point for a subset $S \subset X$ if every domain containing that point contains infinitely many points of $S$. The *Limit Point Principle* is

(O10) For each bounded infinite set, there is at least one limit point.

The *Subsequential Convergence Principle* and *Monotone Convergence Principle* are

(O11) Every bounded sequence of points has a convergent subsequence.

(O12) Each bounded monotone sequence of points converges.

**Theorem 13.6.** *If $X$ has a denumerable subset ordinally dense in $X$ then conditions* (O1) *through* (O12) *are equivalent for the space $\mathfrak{X}$.*

It is noted that the concept of a limit point was apparently introduced by Weierstrass who proved that every bounded infinite set of points in a Euclidean space has at least one limit point (see [16 pp. 58, 77]). The designation *Bolzano–Weierstrass Theorem* recognizes that the linear case of Weierstrass' theorem is a consequence of Bolzano's Induction Principle.

## 14 Completeness of ordered spaces

Assume that $\mathfrak{X} = (X, \mathcal{D})$ is an ordered space having neither initial nor final elements and containing a set $\mathbb{E}$ of regular cardinality $\aleph_\alpha$ having the property that between any two different elements of $X$ there are $\aleph_\alpha$ elements of $\mathbb{E}$. Denote by $e^* = (e(\xi) : \xi \in \mathbb{W}_\alpha)$ a fixed enumeration of $\mathbb{E}$.

The *primary Moore framework*, $\hat{\mathcal{R}}$ for $\mathfrak{X}$ (relative to $e^*$) is defined by assigning the Moore index scheme for $\mathbb{T} = \mathbb{W}_\alpha$ and taking $\mathcal{R}(\lambda)$ to be the family of domains with endpoints in $\mathbb{E}$ which contain at most one of the elements $e(\xi)$ with $\xi \in \lambda$ and has none of these elements as an endpoint, for each index $\lambda$. This framework constitutes a localization structure for $\mathfrak{X}$.

**Theorem 14.1.** *For the primary Moore framework, conditions* (L1) *through* (L5)*,* (O1) *through* (O7)*, and the following are equivalent:*

(O13) *Every descending cascade of segments has a nonempty intersection.*

(O14) *For any set $Q$ ordinally dense in $X$, each descending cascade of segments whose endpoints belong to $Q$ has a nonempty intersection.*

(O15) *Every descending regular cascade has a nonempty intersection.*

(O16) *Each descending regular cascade converges.*

The space $\mathfrak{X}$ is called a *Dedekind continuum of order $\alpha$* if any one of these conditions is satisfied.

If $\mathfrak{X}_0 = (X_0, \mathcal{D}_0)$ is the particular space with $X_0 = \mathbb{E}$, the family $\mathcal{D}_0$ is comprised of the intervals $(a, b)$, and $\mathfrak{S}_0$ is the system of descending cascades regular with respect to the primary framework for $\mathfrak{X}_0$ then application of the **ACP**yields

**Theorem 14.2.** *The space $\mathfrak{X}_0$ can be order isomorphically embedded in a Dedekind continuum of order $\alpha$ which is a principal completion of* $[\mathfrak{X}_0, \mathfrak{S}_0]$.

Assume now that $\mathfrak{X}$ and $\mathfrak{Y}$ are ordered spaces as specified above with assigned primary Moore frameworks of identical type. Then Theorem 13.3 yields the following counterpart of Lavrentiev's Theorem which is fundamental in the divination of certain analogies (see [35, 36, 37]).

**Theorem 14.3.** *If* **BCP** *is valid for the spaces $\mathfrak{X}$ and $\mathfrak{Y}$ then each order isomorphism between subsets $P$ of $X$ and $Q$ of $R$ can be extended to an order isomorphism between $\mathcal{G}_\Delta$–sets $Q$ and $S$ where $P \subset Q \subset \mathrm{Cl}_X(P)$ and $R \subset S \subset \mathrm{Cl}_Y(R)$.*

**Unification Query**: Is there a general extension theorem having both Theorem 11.5 and Theorem 14.3 as special cases for the real line?

Suppose now that the set $\mathbb{E}$ of regular cardinality $\aleph_\alpha$ is an $\eta_\alpha$-set (cf. [21]). The primary $\omega_\alpha$*-sequential framework* $\hat{\mathcal{R}}$ for $\mathfrak{X}$ (relative to $e^*$) is comprised of the families $\mathcal{R}(\lambda)$ of nonempty intervals $(a, b)$ having endpoints in $\mathbb{E}$ which contain at most one of the elements $e(\xi)$ with $\xi \leq \lambda$ and has none of these elements as an endpoint. This framework is a localization structure for $\mathfrak{X}$.

**Theorem 14.4.** *For the primary $\omega_\alpha$-sequential framework, the conditions* (L1) *through* (L5) *and* (O13) *through* (O16) *are equivalent.*

The space $\mathfrak{X}$ is termed a *homogeneous continuum of order* $\alpha$ whenever any one of these conditions is satisfied.

If $\mathfrak{X} = (X, \mathcal{D})$ is the particular space with $X = \mathbb{E}$, the family $\mathcal{D}$ consists of the nonempty intervals $(a, b)$ with endpoints in $\mathbb{E}$ , and $\mathfrak{S}$ is the system of descending $\omega_\alpha$-sequences regular with respect to the primary $\omega_\alpha$-sequential framework for $\mathfrak{X}$, then we derive from the **ACP**

**Theorem 14.5.** *The space $\mathfrak{X}$ can be order isomorphically embedded in a homogeneous continuum of order $\alpha$ which is a principal completion of* $[\mathfrak{X}, \mathfrak{S}]$ *and is an $\eta_\alpha$-set.*

Finally, we have

**Theorem 14.6.** *In the case that $\alpha = 0$, the space $\mathfrak{X}$ is principally complete relative to the primary $\omega_0$-sequential framework if and only if it is principally complete relative to the primary Moore framework.*

For further results see [15, 40].

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