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A CERTAIN 2-COLORING OF THE REALS

Abstract

There is a function $F : [\mathfrak{c}]^{<\omega} \to \{0,1\}$ such that if $A \subseteq [\mathfrak{c}]^{<\omega}$ is uncountable, then $\{F(a \cup b) : a, b \in A, a \neq b\} = \{0,1\}$. A corollary is that there is a function $f : \mathbb{R} \to \{0,1\}$ such that if $A \subseteq \mathbb{R}$ is uncountable, $2 \leq k < \omega$, then both 0 and 1 occur as the value of f at the sum of k distinct elements of A. This was originally proved by Hindman, Leader, and Strauss under CH, and they asked if it holds in general.

Here we solve a problem left open in the paper [2]. We prove that there is a coloring with two colors of the finite subsets of $\mathbb{R}$ such that if A is an uncountable subfamily of this set, then both colors occur as the color of $a \cup b$ for some $a, b \in A$, $a \neq b$. Consequently—and this is what Hindman, Leader, and Strauss were interested in—there is a 2-coloring of $\mathbb{R}$ such that if $A \subseteq \mathbb{R}$ is uncountable, then both colors occur as the color of $a + b$ for some $a, b \in A$, $a \neq b$. In fact, this holds for k-sums in place of 2-sums. In [2] this was proved under CH, and the authors raised the question if it holds without it. The statement is a generalization of Sierpiński's theorem, by which there is a coloring of the pairs of $\mathbb{R}$ with two colors, with no monocolored uncountable set ([5], see also e.g., in [1], Lemma 9.4.). The proof combines the main idea of Sierpiński's construction with some ideas in a current theory of Shelah, Todorcevic, and others producing very complicated colorings of pairs of sets (see e.g., [3], [4], [6]).

We just learned that the same result was independently proved by Dániel Soukup and William Weiss (Toronto).

Notation. Definitions. We use the notation and definitions of axiomatic set theory. In particular, ordinals are von Neumann ordinals, and each cardinal

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is identified with the least ordinal of that cardinality. Specifically, $2=\{0,1\}$ and $\mathfrak{c}$ denotes the least ordinal of cardinality continuum.

If S is a set, κ a cardinal, we define $[S]^\kappa=\{x\subseteq S:|x|=\kappa\}$, $[S]^{<\kappa}=\{x\subseteq S:|x|<\kappa\}$. For $n<\omega$, ${}^n2$ denotes the set of all $n\to 2$ functions. Similarly, ${}^{<\omega}2=\bigcup\{{}^n2:n<\omega\}$, ${}^\omega 2=\{f:\omega\to 2\}$, and ${}^{\le\omega}2={}^{<\omega}2\cup{}^\omega 2$. If $f,g\in{}^{\le\omega}2$, then $f\lhd g$ denotes that f is a proper initial segment of g, i.e., $f=g|n\neq g$ for some $n<\omega$. If $f\in{}^n2$, $x<2$, then $f\widehat{\ }x$ is that function $g\in{}^{n+1}2$, such that $f\lhd g$ and $g(n)=x$. If $n\le\omega$, then $<_{\text{lex}}$ is the lexicographic ordering on ${}^n2$, i.e., $f<_{\text{lex}}g$ iff there is $i<n$ with $f|i=g|i$, $f(i)<g(i)$.

Theorem 1. *There is a function $F:[\mathfrak{c}]^{<\omega}\to 2$ such that if $\{a_\alpha:\alpha<\omega_1\}$ are distinct finite subsets of $\mathfrak{c}$, $i<2$, then there are $\alpha<\beta$ such that $F(a_\alpha\cup a_\beta)=i$.*

Proof. Let $\{r_\alpha:\alpha<\mathfrak{c}\}\subseteq{}^\omega 2$ be distinct functions. For $\alpha\neq\beta$ set

$$\Delta(\alpha,\beta)=\min\{n:r_\alpha(n)\neq r_\beta(n)\}.$$

If $a\in[\mathfrak{c}]^{<\omega}$, $|a|\ge 2$, let

$$N=\max\left\{\Delta(\alpha,\beta):\alpha\neq\beta\in a\right\}.$$

Let $s\in{}^N2$ be lexicographically minimal such that there are $\beta_0,\beta_1\in a$ with $r_{\beta_0}|N=r_{\beta_1}|N=s$, $r_{\beta_i}(N)=i$ $(i<2)$. Define

$$F(a)=\begin{cases}0, & \text{if } \beta_0<\beta_1,\\ 1, & \text{if } \beta_1<\beta_0.\end{cases}$$

For the other sets a, i.e., when $|a|\le 1$, we define $F(a)$ arbitrarily.

Claim. *If $A,B\subseteq\mathfrak{c}$, $|A|=|B|=\aleph_1$, then there are $g\in{}^{<\omega}2$ and $\varepsilon<2$, such that $A'=\{\alpha\in A:g\widehat{\ }\varepsilon\lhd r_\alpha\}$ and $B'=\{\beta\in B:g\widehat{\ }(1-\varepsilon)\lhd r_\beta\}$ are both uncountable.*

Proof. For $s\in{}^{<\omega}2$ define $M(A,s)=\{\alpha\in A:s\lhd r_\alpha\}$ and similarly $M(B,s)=\{\beta\in B:s\lhd r_\beta\}$.

Set

$$A^*=\{\alpha\in A:\exists s\lhd r_\alpha,|M(A,s)|\le\aleph_0\}$$

and define B^* analogously for B. A^* is countable as the appropriate $\alpha\mapsto s$ mapping maps A^* to the countable ${}^{<\omega}2$ such that each preimage is countable. Similarly, B^* is countable.

Pick $\alpha\in A-A^*$, $\beta\in B-B^*$, $\alpha\neq\beta$. If $N=\Delta(\alpha,\beta)$, $g=r_\alpha|N=r_\beta|N$, $g\widehat{\ }\varepsilon\lhd r_\alpha$, $g\widehat{\ }(1-\varepsilon)\lhd r_\beta$, then

$$A'=\{\gamma\in A:r_\gamma|(N+1)=g\widehat{\ }\varepsilon\}$$

and

$$B' = \{\gamma \in B : r_\gamma|(N+1) = g\widehat{\ }(1-\varepsilon)\}$$

are uncountable by the choice of α, β. $\square$

In order to show that the function F defined above is good, assume that $\{a_\xi : \xi < \omega_1\} \subseteq [\mathfrak{c}]^{<\omega}$ are different. Using the Δ-system lemma, we can assume that $a_\xi = a \cup b_\xi$ where $a \cap b_\xi = b_\xi \cap b_\eta = \emptyset$ $(\xi < \eta)$, $|a| = \ell$, $|b_\xi| = k$. Here ℓ can be zero, but $k > 0$. Let $a = \{\gamma_i : i < \ell\}$, $b_\xi = \{\gamma^\xi_j : j < k\}$ be the increasing enumerations. By shrinking, we can achieve that for each $j < k$, $\{\gamma^\xi_j : \xi < \omega_1\}$ is of order type ω_1. With further shrinking, we can obtain that for each $j < k$, $\gamma^\xi_j < \gamma^\eta_j$ holds for $\xi < \eta$. (Another possibility is to use the Dushnik–Miller partition theorem $\omega_1 \to (\omega_1, (\omega)_k)^2$.) Still more shrinking and re-indexing gives that there is $M < \omega$, such that $r_{\gamma_i}|M = f_i$ $(i < \ell)$, $r_{\gamma^\xi_j}|M = g_j$ $(j < k)$ and the functions f_i, g_j are different.

We construct by recursion the uncountable sets U_j, V_j $(j \le k)$ as follows. $U_0 = V_0 = \omega_1$. Given U_j, V_j, we apply the Claim to $A = \{\gamma^\xi_j : \xi \in U_j\}$, $B = \{\gamma^\xi_j : \xi \in V_j\}$, and obtain the uncountable $U_{j+1} \subseteq U_j$, $V_{j+1} \subseteq V_j$, $N_j < \omega$, $g_j \in {}^{N_j}2$, $\varepsilon_j < 2$ such that

$$r_{\gamma^\xi_j}|(N_j+1) = g_j\widehat{\ }\varepsilon_j \quad (\xi \in U_{j+1})$$

and

$$r_{\gamma^\eta_j}|(N_j+1) = g_j\widehat{\ }(1-\varepsilon_j) \quad (\eta \in V_{j+1}).$$

Set $N = \max\{N_j : j < k\}$. Notice that $N > M$. Let g_j be the $<_{\text{lex}}$-minimal element of $\{g_j : N_j = N\}$.

We now have that if $\xi \in U_k$, $\eta \in V_k$, then $F(a_\xi \cup a_\eta) = \varepsilon_j$ iff $\gamma^\xi_j < \gamma^\eta_j$ iff $\xi < \eta$. As we can choose $\xi \in U_k$, $\eta \in V_k$ such that either of $\xi < \eta$ or $\eta < \xi$ hold, both 0 and 1 are attained as $F(a_\xi \cup a_\eta)$ for some ξ, η. $\square$

Corollary 2. *There is a function $f : \mathbb{R} \to \{0,1\}$ such that if $A \subseteq \mathbb{R}$, $|A| = \aleph_1$, $2 \le k < \omega$, then both 0 and 1 occur as $f(a_0 + a_1 + \cdots + a_{k-1})$ for some distinct $a_0, a_1, \ldots, a_{k-1} \in A$.*

Proof. Fix a Hamel basis $B = \{b_\alpha : \alpha < \mathfrak{c}\}$ over $\mathbb{Q}$ for $\mathbb{R}$. Each $x \in \mathbb{R}$, can uniquely be written as

$$x = \sum_{\alpha < \omega_1} \lambda_\alpha b_\alpha$$

where each λ_α is rational and $\operatorname{supp}(x) = \{\alpha : \lambda_\alpha \ne 0\}$ is finite.

We define $f(x) = F(\mathrm{supp}(x))$. We show that f is as required.

Assume first that $k = 2$. Let $\{x_\xi : \xi < \omega_1\}$ be distinct reals. Set $a_\xi = \mathrm{supp}(x_\xi) \in [\mathfrak{c}]^{<\omega}$. By repeatedly shrinking the system, we can assume that every a_ξ has the same number of elements, k, and the sets $\{a_\xi : \xi < \omega_1\}$ form a Δ-system, i.e., $a_\xi \cap a_\eta = a$ $(\xi \neq \eta)$. Let $a_\xi = \{\gamma_i^\xi : i < k\}$ be the increasing enumeration of a_ξ and λ_i^ξ be the corresponding coefficients, that is,

$$x_\xi = \sum_{i<k} \lambda_i^\xi b_{\gamma_i^\xi}.$$

By further shrinking the system we can assume that $\lambda_i^\xi = \lambda_i$ and that there is a set I such that $a = \{\gamma_i^\xi : i \in I\}$, that is, the elements of a occupy the same positions in the a_ξ's.

If now $\xi < \eta$, then

$$\mathrm{supp}(x_\xi + x_\eta) = a_\xi \cup a_\eta$$

as

$$x_\xi + x_\eta = \sum_{i\in I} 2\lambda_i b_{\gamma_i^\xi} + \sum_{i\notin I} \lambda_i b_{\gamma_i^\xi} + \sum_{i\notin I} \lambda_i b_{\gamma_i^\eta},$$

where the b_τ's are different on the right hand side.

We can therefore apply the Theorem and obtain $\xi_0 < \eta_0$ and $\xi_1 < \eta_1$ such that $f(x_{\xi_0} + x_{\eta_0}) = 0$ and $f(x_{\xi_1} + x_{\eta_1}) = 1$.

We now consider the case $k \geq 3$. Assume that $\{x_\xi : \xi < \omega_1\}$ are distinct reals and $i < 2$. Define

$$y_\xi = \frac{1}{2}(x_0 + \cdots + x_{k-3}) + x_{k-2+\xi}$$

and apply the previous argument to $\{y_\xi : \xi < \omega_1\}$. It gives $\xi < \eta$ such that the value of f is i at

$$y_\xi + y_\eta = x_0 + x_1 + \cdots + x_{k-3} + x_{k-2+\xi} + x_{k-2+\eta},$$

the sum of k distinct elements of $\{x_\xi : \xi < \omega_1\}$. $\square$

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