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Keith Neu, Science and Mathematics Division, Angelina College, 3500 South First Street, Lufkin, TX 75904, U.S.A. email: kneu@bpcc.edu

Filip Strobin, Institute of Mathematics, Łódź University of Thechnology, Wólczańska 215, 93-005 Łódź, Poland. email: filip.strobin@p.lodz.pl

NON-LINEAR IMAGES OF μ-SHADINGS, SHADINGS IN $\mathbb{R}^2$, AND QUOTIENT SETS OF μ-SHADINGS

Abstract

In the paper we prove that some natural modifications (for instance, images under some functions, Cartesian products, quotient sets) of certain types of μ-shadings (or shadings), are other examples of μ-shadings (or shadings).
The studies of shadings and μ-shadings were initiated by R. Mabry in 1990. Our work is a continuation of his and K. Neu's research in this field. In particular, we solve one problem posed by R. Mabry.

1 Introduction

In this paper we use the standard notations $S+t=\{s+t|s\in S\}$, $cS=\{cs|s\in S\}$, $A\times B=\{(a,b)|a\in A, b\in B\}$, $A\Delta B=(A\setminus B)\cup(B\setminus A)$ and $f(S)=\{f(s)|s\in S\}$. Moreover, the symbol $\biguplus$ is used to represent a disjoint union.

For any set $A\subseteq\mathbb{R}$, let $\tau(A)$ denote the set of all $t\in\mathbb{R}$ satisfying $A+t=A$ and let $\delta(A)$ denote the set of all $d\in\mathbb{R}$ satisfying $dA=A$. For a set $A\subseteq\mathbb{R}^2$, $\tau(A)$ and $\delta(A)$ are defined in the same way (the addition and the multiplication have a "complex" meaning, i.e., $(a,b)+(c,d)=(a+c,b+d)$ and $(a\cos\alpha, a\sin\alpha)(b\cos\beta, b\sin\beta)=(ab\cos(\alpha+\beta), ab\sin(\alpha+\beta))$. Additionally,

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let $\rho(A)$ represent the set of all $r \in \mathbb{R}$ such that set A is unchanged when the entire plane is rotated r radians counterclockwise about the origin. We will refer to the members of $\tau(A), \delta(A)$ and $\rho(A)$ as *translators, dilators*, and *rotators* of a set A. In Section 3, the notation Ae^{iB} and Ae^{it} will be used as an abbreviation for $\{(a\cos b, a\sin b)|a \in A, b \in B\}$ and $\{(a\cos t, a\sin t)|a \in A\}$, respectively. (in particular, we have $\rho(A) = \{t \in \mathbb{R} : e^{it}A = A\}$.)

Recall that a Banach measure is a finitely additive, isometry-invariant extension of the Lebesgue measure to $2^{\mathbb{R}}$. In [2], R. Mabry first demonstrated the existence of *shadings*, or sets $A \subseteq \mathbb{R}$ that give the expression $\dfrac{\mu(A\cap I)}{\lambda(I)}$ the same constant value for every bounded interval I and every Banach measure μ. This constant value can be made to be any number in $[0, 1]$. All of the shadings contructed in Mabry's paper are built using *Archimedean sets*. These are sets that satisfy $A + t = A$ for densely many $t \in \mathbb{R}$ (i.e., for which $\tau(A)$ is dense in $\mathbb{R}$). One of the important results proven in the paper states that for an Archimedean set A, the ratio $\dfrac{\mu(A\cap I)}{\lambda(I)}$ has the same constant value for every bounded interval I, for a given Banach measure μ. Note that unlike shadings, in the case of an Archimedean set A, the value of $\dfrac{\mu(A\cap I)}{\lambda(I)}$ might depend on the Banach measure chosen. Any set in which $\dfrac{\mu(A\cap I)}{\lambda(I)}$ has the same constant value for every bounded intervall I, for a given Banach measure μ, is called a *μ-shading*. Thus, Archimedean sets are examples of μ-shadings. For a μ-shading A, the ratio $\dfrac{\mu(A\cap I)}{\lambda(I)}$ is called the *μ-shade* of A and is denoted by $\mathrm{sh}_\mu A$. For a shading A, the ratio $\dfrac{\mu(A\cap I)}{\lambda(I)}$ is called the *shade* of A and is denoted by $\mathrm{sh}A$.

We will also refer to the following generalization of the notion of shadings. If $f : \mathbb{R} \to [0, 1]$ is continuous, then $A \subset \mathbb{R}$ is called an f-shading, if for each Banach measure μ and $x \in \mathbb{R}$, $\lim_{\lambda(I)\to 0} \frac{\mu(A\cap I)}{\lambda(I)} = f(x)$, where the limit is taken over all bounded intervals which contain x. Note that A is a shading with the shade $\mathrm{sh}A$ iff A is an f-shading for $f(x) = \mathrm{sh}A$, $x \in \mathbb{R}$ (cf. [2, Remark 5.12]).

An *almost isometry-invariant set* $A \subset \mathbb{R}$ is any set satisfying $|g(A) \triangle A| < \mathfrak{c}$ for any isometry g of $\mathbb{R}$. An *almost translation-invariant set* is similar to an almost isometry-invariant set, except that it is almost invariant under any translation, instead of any isometry. That is, an almost translation-invariant set A satisfies $|A \triangle (A + r)| < \mathfrak{c}$ for any $r \in \mathbb{R}$. It is important to mention that almost translation-invariant sets (and, in particular, almost isometry-invariant sets), like Archimedean sets, are μ-shadings for any Banach measure μ. This result follows from [3, Theorem 5.3] and the following fact:

Lemma 1.1. *[2, Lemma 4.5] Let $A \subset \mathbb{R}$ and $|A| < \mathfrak{c}$ (i.e., A is of a cardinality less than continuum). Then* $\mathrm{sh}A = 0$.

A *similarity-shading* A is a set in which $\dfrac{\mu(A \cap I)}{\lambda(I)}$ has the same value for every bounded interval I and for every improved Banach measure μ. (As in [3], we say that a Banach measure is *improved* if it satisfies $\mu(cE) = c\mu(E)$ for every $c > 0$, for every $E \subseteq \mathbb{R}$. The existence of such measures is demonstrated in [7, Corollary 11.5].) We will use the notation $\mathrm{sh}_{\sim}A$ to represent the number $\dfrac{\mu(A \cap I)}{\lambda(I)}$ for similarity-shading A. We will use the following:

Theorem 1.2. *[4, Theorem 2.3] Let $A \subset \mathbb{R}$ be a set satisfying $cA = A$ for densely many $c \in \mathbb{R}$ (i.e., for which $\delta(A)$ is dense). Then A is a μ-shading for any improved Banach measure.*

Theorem 1.3. *[3, Theorem 11.1] If μ is an improved Banach measure on $\mathbb{R}$ and $A \subset \mathbb{R}$ is a μ-shading, then $g(A)$ is also a μ-shading and* $\mathrm{sh}_\mu g(A) = \mathrm{sh}_\mu A$ *for every similarity transformation g on $\mathbb{R}$ (i.e., $g(x) = ax+b$ for some $a, b \in \mathbb{R}$ with $a \neq 0$).*

We will also have an occasion to use *t-Banach* measures, or measures that are exactly like Banach measures except that they are not necessarily reflection-invariant about the origin. (Such measures were discussed in [5].) Sets A in which $\dfrac{\mu(A \cap I)}{\lambda(I)}$ has the same value for every t-Banach measure μ and every bounded interval I are called *t-shadings*. This common value is denoted t-sh$A$. Most of the results in Section 2 involve almost isometry-invariant sets and almost translation-invariant sets that are also shadings, t-shadings, or similarity-shadings.

We will also consider Banach measures, improved Banach measures, shadings, μ-shadings, similarity shadings, etc. in $\mathbb{R}^2$. The definitions are very similar. The only differences are the following
– instead of intervals we take rectangles $I \times J$;
– improved Banach measures satisfy $\mu(cE) = c^2\mu(E)$ for $E \subset \mathbb{R}^2$ and $c > 0$.

Many of the results in this paper involve integrals. Integrals such as $\int_G f(T)d\mu_G(T)$, where G is an amenable group, f is a bounded function of G, and μ_G is a measure on G, have already been defined and utilized in [7].

In particular, given the measure space $(G, 2^G, \mu_G)$, and using H. L. Royden's integral definition in [6] as a motivation, S. Wagon defines $\int_G f(T)d\mu_G(T)$ to be the supremum of all integrals $\int_G \phi(T)d\mu_G(T)$, where $\phi(T) = \sum_{i=1}^{n} c_i\chi_{E_i}(T)$ is a simple function (i.e., any real function with a finite image) satisfying $0 \leq \phi(T) \leq f(T)$. For such a simple function ϕ, $\int_G \phi(T)d\mu_G(T)$ is defined to be $\sum_{i=1}^{n} c_i\mu(E_i)$. We would now like to alter the above definition somewhat, by integrating over a subset $E \subseteq \mathbb{R}$, rather than over a group G.

Definition 1.4. Let $E \subseteq \mathbb{R}$ be bounded and μ be a Banach measure. For any nonnegative simple function $\phi(x) = \sum_{i=1}^{n} c_i\chi_{E_i}(x)$, where $E = \biguplus_{i=1}^{n} E_i$ (recall that $\biguplus$ is used to represent a disjoint union), define $\int_E \phi(x)d\mu(x) = \sum_{i=1}^{n} c_i\mu(E_i)$. For any bounded function $f(x) \geq 0$, define $\int_E f(x)d\mu(x)$ to be the supremum of $\int_E \phi(x)d\mu(x)$ as ϕ ranges over all simple functions satisfying $0 \leq \phi(x) \leq f(x)$.

We will now verify some important integral properties, true for integrals defined on measure spaces in which the measure is countably additive.

Lemma 1.5. *Let $f \geq 0$ be a bounded function on a bounded set $E \subset \mathbb{R}$, and let (ϕ_k) be a sequence of a positive simple functions such that $\phi_k \nearrow f$ uniformly on E. Then*
$$\lim_{k\to\infty} \int_E \phi_k(x)d\mu(x) = \int_E f(x)d\mu(x).$$

Proof. If $\mu(E) = 0$, then $\int_E f(x)d\mu(x) = 0$ and the equality holds. Assume that $\mu(E) > 0$.
It is sufficient to show that for every simple function ϕ with $0 \leq \phi(x) \leq f(x)$ and for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ satisfying $\int_E \phi_k(x)d\mu(x) \geq \int_E \phi(x)d\mu(x) - \varepsilon$. Choose k large enough so that $f(x) - \phi_k(x) < \frac{\varepsilon}{\mu(E)}$ for all $x \in E$. Let $E' = \{x \in E | \phi(x) < \phi_k(x)\}$ and let $E'' = \{x \in E | \phi(x) \geq \phi_k(x)\}$. Clearly, $\int_{E'} \phi_k(x)d\mu(x) \geq \int_{E'} \phi(x)d\mu(x)$, so it is enough to prove $\int_{E''} \phi_k(x)d\mu(x) \geq \int_{E''} \phi(x)d\mu(x) - \varepsilon$. If $x \in E''$, then $0 \leq \phi(x) - \phi_k(x) \leq \frac{\varepsilon}{\mu(E)}$. This implies $\int_{E''}(\phi(x) - \phi_k(x))d\mu(x) \leq \varepsilon$. But $\int_{E''}(\phi(x) - \phi_k(x))d\mu(x) = \int_{E''} \phi(x)d\mu(x) - \int_{E''} \phi_k(x)d\mu(x)$, which implies the result. $\square$

Proposition 1.6. *Let f, g be two nonnegative bounded functions defined on a bounded set $E \subset \mathbb{R}$, $c > 0$ and $t \in \mathbb{R}$.*

(i) $\int_E (f(x) + g(x))d\mu(x) = \int_E f(x)d\mu(x) + \int_E g(x)d\mu(x)$;

(ii) $\int_E cf(x)d\mu(x) = c\int_E f(x)d\mu(x)$;

(iii) $\int_E f(x)d\mu(x) = \int_{E-t} f(x+t)d\mu(x)$.

PROOF. It is obvious that for every nonnegative bounded function on a bounded set E, there exists a sequence (ϕ_k) of nonnegative simple functions with $\phi_k \nearrow f$ uniformly to f. Hence the result follows from Lemma 1.5 and an obvious fact that it is true for simple functions. $\square$

Proposition 1.7. *If $f \geq 0$ and E are Lebesgue measurable and bounded, then $\int_E f(x)d\mu(x) = \int_E f(x)d\lambda(x)$. That is, the integral in Definition 1.4 becomes the Lebesgue integral. If f is continuous and bounded, and E is a bounded interval, then $\int_E f(x)d\mu(x) = \int_E f(x)dx$. That is, the integral in Definition 1.4 becomes the Riemann integral.*

PROOF. We only need to prove the first statement. It follows from Lemma 1.5 and an obvious fact that if $f \geq 0$ and E are Lebesgue measurable (and bounded), then there exists a sequence (ϕ_k) of nonnegative simple Lebesgue measurable functions with $\phi_k \nearrow f$ uniformly. $\square$

2 Continuous, non-linear images of almost isometry-invariant sets

Lemma 2.1. *Let μ be a measure on the family of all bounded subsets of $\mathbb{R}$, which agrees with the Lebesgue measure for open and bounded sets. Then μ is an extension of the Lebesgue measure for bounded sets.*

PROOF. It is easy to see that for every bounded Lebesgue measurable set $E \subset \mathbb{R}$, $\mu(E) \leq \lambda(E)$. This easily gives the thesis. $\square$

Lemma 2.2. *Let μ be a measure on the family of all bounded subsets of $\mathbb{R}$, which is an extension of the Lebesgue measure (for bounded sets). Let (I_n) be a sequence of intervals of a given length, with $\biguplus_{i=1}^{\infty} I_i = \mathbb{R}$. Define $\nu(E) = \sum_{i=1}^{\infty} \mu(E \cap I_i)$, $E \subset \mathbb{R}$. Then*

(i) if μ is isometry-invariant, then ν is a Banach measure;

(ii) if μ is translation-invariant, then ν is a t-Banach measure;

(iii) if μ is isometry-invariant and $\mu(cE) = c\mu(E)$ for bounded E and $c > 0$, then ν is an improved Banach measure.

Moreover, the measure ν does not depend on the choice of a sequence (I_n).

Proof. We will show only (i), since proofs of rest parts are very similar. Let $E = A \uplus B$. Since $\mu(E \cap I_i) = \mu(A \cap I_i) + \mu(B \cap I_i)$ for all i, $\nu(E) = \nu(A) + \nu(B)$. Hence ν is finitely additive. Now let T be an isometry on $\mathbb{R}$. We have $\nu(T(E)) = \sum_{i=1}^{\infty} \mu(T(E) \cap I_i) = \sum_{i=1}^{\infty} \mu(E \cap T^{-1}(I_i)) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu(E \cap T^{-1}(I_i) \cap I_j) = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \mu(E \cap T^{-1}(I_i) \cap I_j) = \sum_{j=1}^{\infty} \mu(E \cap I_j) = \nu(E)$. Hence ν is translation-invariant. Now let E be any Lebesgue measurable subset of $\mathbb{R}$. Using the countable additivity of the Lebesgue measure we have $\nu(E) = \sum_{i=1}^{\infty} \mu(E \cap I_i) = \sum_{i=1}^{\infty} \lambda(E \cap I_i) = \lambda(E)$. Hence ν is an extension of the Lebesgue measure. □

The idea behind the construction of the measure in the following lemma, like the one mentioned in [5, Lemma 4.2], is due in part to J. Roberts and R. Mabry.

Lemma 2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for some $s < t$, $f_{|[s,t]}$ is continuous and strictly increasing, and $f(\mathbb{R} \setminus [s,t]) \cap I = \emptyset$, where $I = [f(s), f(t)]$ and μ is a (fixed) Banach measure. Then there exists a t-Banach measure ν such that for any bounded set $E \subset \mathbb{R}$, $\nu(E) = \frac{1}{\lambda(I)} \int_S \mu(f(E+x) \cap I) d\mu(x)$, where S is the support of the function $x \to \mu(f(E+x) \cap I)$.*

Proof. It is easy to see that we may assume that f is an increasing, continuous bijection, and I is any nonempty closed and bounded interval.
We will prove that for bounded sets $E \subset \mathbb{R}$, $\psi(E) \equiv \frac{1}{\lambda(I)} \int_S \mu(f(E+x) \cap I) d\mu(x)$ is additive, translation-invariant, and an extension of the Lebesgue measure. The conclusion of this lemma would then follow from Lemma 2.2(ii).
Let $a = inf(E), b = sup(E)$. Also, assume $I = [c,d]$. Then $S \subseteq [f^{-1}(c) - b, f^{-1}(d) - a]$. This proves that S is bounded and so the integral is defined.
Let A, B be two disjont subsets of $\mathbb{R}$. Since for every x,

$$f((A \uplus B) + x) \cap I = (f(A+x) \cap I) \uplus (f(B+x) \cap I),$$

we get

$$\begin{aligned} \psi(A \uplus B) &= \frac{1}{\lambda(I)} \int_S \mu(f((A \uplus B) + x) \cap I) d\mu(x) \\ &= \frac{1}{\lambda(I)} \int_S \left(\mu(f(A+x) \cap I) + \mu(f(A+x) \cap I)\right) d\mu(x) = \psi(A) + \psi(B). \end{aligned}$$

The last equality follows from Proposition 1.6(i) and the fact that the supports of $\mu(f(A+x) \cap I)$ and $\mu(f(B+x) \cap I)$ are included in S. This proves additivity.

To prove translation-invariance, note that

$$\psi(E+t) = \frac{1}{\lambda(I)} \int_{S-t} \mu(f(E+x+t) \cap I) d\mu(x).$$

That is, changing E to $E+t$ changes the support of the integrand function from S to $S-t$. But by Proposition 1.6(iii), $\int_S \mu(f(E+x) \cap I) d\mu(x) = \int_{S-t} \mu(f(E+x+t) \cap I) d\mu(x)$, which implies $\psi(E+t) = \psi(E)$. Now we will show that for any bounded interval J of positive measure, $\psi(J) = \lambda(J)$. Let $s = f^{-1}(c)$ and $t = f^{-1}(d)$. Choose $u \in \mathbb{R}$ so that $c < f(u) \le d$, let $K = [s,u]$, and define $g(x) = \mu(f(K+x) \cap [c,d]) = \mu(f([x+s, x+u]) \cap [c,d])$. Since f is continuous, $f([x+s,x+u])$ is an interval for all x, which implies $f([x+s,x+u]) \cap [c,d]$ is also an interval, a single point, or the empty set for any x. This means we can write $g(x) = \lambda(f([x+s,x+u]) \cap [c,d])$. We have

$$g(x) = \begin{cases} f(x+u) - c, & -u+s \le x \le 0 \\ f(x+u) - f(x+s), & 0 \le x \le t-u \\ d - f(x+s), & t-u \le x \le t-s \\ 0, & \text{otherwise.} \end{cases}$$

Since $f(x)$ is continuous, so is $g(x)$. This means $g(x)$ is Riemann integrable, so by Proposition 1.7 and easy computations, we get

$$\begin{aligned} \psi(K) = \frac{1}{\lambda(I)} \int_S g(x) d\mu(x) &= \frac{1}{\lambda(I)} \int_{-u+s}^{t-s} g(x) dx \\ &= \frac{1}{\lambda(I)} (d-c)(u-s) = u - s = \lambda(K). \end{aligned}$$

Because ψ is translation-invariant and we can choose u to be as close to s as we like, we have proven that $\psi(J) = \lambda(J)$ whenever J is an interval satisfying $\lambda(J) < t-s$. Because ψ is also additive, $\psi(J) = \lambda(J)$ for all bounded J. Now since functions defined as g above (for intervals) are continuous, and from Proposition 1.7 (and the monotone convergence theorem), we also have that if (I_n) is a sequence of pairwise disjoint intervals with $\bigcup_{n\in\mathbb{N}} I_n$ bounded, then $\psi(\bigcup_{n\in\mathbb{N}} I_n) = \sum_{n\in\mathbb{N}} \psi(I_n) = \sum_{n\in\mathbb{N}} \lambda(I_n) = \lambda(\bigcup_{n\in\mathbb{N}} I_n)$. Hence, by Lemma 2.1, ψ is an extension of the Lebesgue measure for bounded sets. $\square$

The first few theorems below involve almost isometry-invariant sets and almost translation-invariant sets that also happen to be shadings. From [5, Theorem 2.2], we know that if an almost isometry-invariant set A is a shading, then sh$A = 0$ or sh$A = 1$. From [5, Theorem 2.3], we also know that if an almost translation-invariant set A is a shading, then sh$A = 0, \frac{1}{2}$, or 1. Let's see what happens when one of these shadings is mapped by a continuous, increasing function.

Theorem 2.4. *Let f be a continuous and increasing bijection. Let A be an almost isometry-invariant shading of shade 0. Then* $\text{sh}f(A) = 0$.

Proof. First observe that if ϕ is a t-Banach measure, then $\psi(E) = \frac{1}{2}(\phi(E) + \phi(-E))$ is a Banach measure. Since $\text{sh}A = 0$, setting $E = A \cap I$ for any bounded interval I in the equation above gives $0 = \frac{1}{2}(\phi(A \cap I) + \phi(-A \cap -I))$, which implies $\phi(A \cap I) = 0$. This means t-sh$A = 0$. Now let $I = [c, d]$ be any bounded interval and let μ be any Banach measure. By Lemma 2.3, there exists a t-Banach measure ν satisfying

$$\nu(E) = \frac{1}{\lambda(I)} \int_S \mu(f(E + x) \cap I) d\mu(x), \tag{1}$$

where S is the support of $\mu(f(E + x) \cap I)$. As in the proof of Lemma 2.3, let $s = f^{-1}(c)$ and $t = f^{-1}(d)$. Let $J = [f^{-1}(c-1), t]$ and set $E = A \cap J$. Then (1) gives us $0 = \frac{1}{\lambda(I)} \int_S \mu(f((A \cap J) + x) \cap I) d\mu(x) = \frac{1}{\lambda(I)} \int_S \mu(f((A + x) \cap (J + x)) \cap I) d\mu(x) = \frac{1}{\lambda(I)} \int_S \mu(f(A + x) \cap f(J + x) \cap I) d\mu(x) = \frac{1}{\lambda(I)} \int_S \mu(f(A) \cap f(J + x) \cap I) d\mu(x)$. (The last equality follows from the fact that A is almost isometry invariant and Lemma 1.1.) It is not difficult to show from Definition 1.4 that if $h(x)$ is any nonnegative function, H is any bounded set, and if $0 = \int_H h(x) d\mu(x)$, then for any $\varepsilon > 0$, $\mu(\{x \in H | h(x) > \varepsilon\}) = 0$. This means, in particular, that if $L \subseteq H$ is any interval, then there exists an $x \in L$ such that $h(x) \leq \varepsilon$. Note that in the integral $\frac{1}{\lambda(I)} \int_S \mu(f(A) \cap f(J + x) \cap I) d\mu(x)$, $f(J + x) \cap I = I$ for all $x \in [0, s - f^{-1}(c-1)] = [0, f^{-1}(c) - f^{-1}(c-1)]$. Since f is increasing, so is f^{-1}, which implies $[0, f^{-1}(c) - f^{-1}(c-1)]$ is an interval of positive measure. Thus there exists an x in $[0, f^{-1}(c) - f^{-1}(c-1)]$ such that $\mu(f(A) \cap f(J + x) \cap I) = \mu(f(A) \cap I) \leq \varepsilon$. For this fixed interval I we can choose ε as small as we like, so we conclude that $\mu(f(A) \cap I) = 0$. Since I and μ were arbitrary, $\text{sh}f(A) = 0$. $\square$

Since for every shading $A \subset \mathbb{R}$, sh(A^c)=1-sh(A), the above result implies the following:

Corollary 2.5. *Let f be a continuous and increasing bijection. Let A be an almost isometry-invariant shading of shade 1. Then* $\text{sh}f(A) = 1$.

Before we move on to the case of an almost translation-invariant shading of shade $\frac{1}{2}$, we sate a few more propositions. Two of them follow from Theorem 1.2 (see Introduction).

Proposition 2.6. *Let A be a set in which $\delta(A)$ is dense. If $f(x) = x^{\frac{p}{q}}$ for p and q odd integers, then $f(A)$ is a μ-shading for any improved Banach measure μ.*

PROOF. Let $r = \dfrac{p}{q}$. For any $c \in \delta(A)$, we have $c^r A^r = (cA)^r = A^r$. (here for any $B \subset \mathbb{R}$, we define $B^r = \{b^r : b \in B\}$.) Hence $(\delta(A))^r \subset \delta(A^r)$. In particular, $\delta(A^r)$ is dense and the thesis follows from Theorem 1.2. $\square$

Proposition 2.7. *Let A be a set in which $\delta(A)$ is dense. Let $A^+ = A \cap (0, \infty)$. Then $\ln(A^+)$ is Archimedean, and therefore a μ-shading for any Banach μ.*

PROOF. Let $\delta^+(A) = \delta(A) \cap (0, \infty)$. Since $\ln \delta^+(A)$ is dense, it is enough to show $\ln A^+ + \ln \delta^+(A) = \ln A^+$. Let $c \in \delta^+(A)$. Then $\ln A^+ + \ln c = \ln cA^+ = \ln A^+$. $\square$

Proposition 2.8. *Let A be an Archimedean set. Then $e^A \cup (-e^A)$ is a μ-shading for any improved Banach μ.*

PROOF. By Theorem 1.2 and the fact that $-(e^A \cup (-e^A)) = e^A \cup (-e^A)$, it is enough to show $ce^A = e^A$ for densely many $c \in (0, \infty)$. Since $\tau(A)$ is dense in $\mathbb{R}$, $e^{\tau(A)}$ is dense in $(0, \infty)$. But for any $t \in \tau(A)$, $e^t e^A = e^{t+A} = e^A$. This means $e^{\tau(A)} e^A = e^A$. $\square$

We are now ready for a result involving an almost translation-invariant $\frac{1}{2}$-shading.

Theorem 2.9. *Let $f : \mathbb{R} \to \mathbb{R}$ be an odd continuous and increasing bijection. Let A be an almost translation-invariant $\frac{1}{2}$-shading. If $f(A)$ is a μ-shading for some Banach μ, then $\mathrm{sh}_\mu f(A) = \frac{1}{2}$.*

PROOF. Define $A_1 = \mathbb{R} \setminus (A \cup (-A)), A_2 = A \setminus (A \cap (-A)), A_3 = (-A) \setminus (A \cap (-A)), A_4 = A \cap (-A)$. All four of these pairwise disjoint sets are almost translation-invariant, and by [5, Corollary 2.4], $\mathrm{sh}(A \cap (-A)) = 0$. This implies $\mathrm{sh} A_2 = \mathrm{sh} A_3 = \frac{1}{2}$ and so $\mathrm{sh} A_1 = \mathrm{sh} A_4 = 0$. Using Theorem 2.4, we can then say that $\mathrm{sh} f(A_1) = \mathrm{sh} f(A_4) = 0$. Using the reflection-invariance of μ, we have $\mathrm{sh}_\mu(f(A)) = \mathrm{sh}_\mu(-f(A)) = \mathrm{sh}_\mu f(-A) = \mathrm{sh}_\mu f(A_3 \uplus A_4) = \mathrm{sh}_\mu f(A_3) + \mathrm{sh}_\mu f(A_4) = \mathrm{sh}_\mu f(A_3) = \mathrm{sh}_\mu f(A_3) + \mathrm{sh}_\mu f(A_1) = \mathrm{sh}_\mu f(A_3 \uplus A_1) = \mathrm{sh}_\mu f(A^c) = \mathrm{sh}_\mu (f(A))^c = 1 - \mathrm{sh}_\mu f(A)$. This forces $\mathrm{sh}_\mu f(A) = \frac{1}{2}$. $\square$

Example 2.10. There exists an almost translation-invariant $\frac{1}{2}$-shading S satisfying $\mathrm{sh}_\mu f(S) = \frac{1}{2}$ for any improved Banach measure μ, for any function of the form $f(x) = x^{\frac{p}{q}}$, where p and q are odd integers. Let H be a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$, let $\{h_\alpha\}_{\alpha < \mathfrak{c}}$ be an injective well-ordering of H, let $\mathbb{Q}_e$

represent the set of all rationals of the form $\left(\frac{r}{s}\right)2^n$, where n is an even integer, and let $\mathbb{Q}_o$ represent the set of all rationals of the form $\left(\frac{r}{s}\right)2^n$, where n is an odd integer. Here r and s are both positive odd integers. Define $S = \{\sum_{i=1}^{n} q_i h_{\alpha_i} + q h_\beta : h_{\alpha_i}, h_\beta \in H, q_i \in \mathbb{Q}, q \in \mathbb{Q}_o \text{ or } q \in -\mathbb{Q}_e, \alpha_i < \beta\}$. Clearly, for every $t \in \mathbb{R}$, $S\Delta(S+t) \subset sp\{h_\alpha : \alpha \leq \alpha_t\}$, where $\alpha_t < \mathfrak{c}$ is such that $t \in sp\{h_\alpha : \alpha \leq \alpha_t\}$. Hence, by that fact that S and $-S$ are pairwise disjoint and $\mathbb{R} = S \cup (-S) \cup \{0\}$, S is an almost translation-invariant $\frac{1}{2}$-shading. Also, if r, s are positive odd integers, then $\left(\frac{r}{s}\right)S = S$ and $\left(-2\frac{r}{s}\right)S = S$. This implies $cS = S$ for densely many $c \in \mathbb{R}$. By Proposition 2.6, $f(S)$ is a μ-shading for any improved Banach measure μ, for any function of the form $f(x) = x^{\frac{p}{q}}$, where p, q are odd integers. The above theorem then implies $\mathrm{sh}_\mu f(S) = \frac{1}{2}$; in fact, we can write $\mathrm{sh}_\sim f(S) = \frac{1}{2}$. (See Introduction.)

We are now ready to consider almost isometry-invariant sets that are also similarity-shadings. Unlike almost isometry-invariant shadings, almost isometry-invariant similarity-shadings can take on any μ-shade in $[0,1]$. For example, for any $k \geq 1$ and any $j \in \{0, ..., 2^k - 1\}$, define

$$S_j^k = \left\{ \sum_{i=1}^{n} q_i h_{\alpha_i} + q h_\beta : h_{\alpha_i}, h_\beta \in H, q_i \in \mathbb{Q}, q \in T_j^k \cup (-T_{p_j}^k), \alpha_i < \beta \right\},$$

where H is the Hamel basis from the above example, $p_j = (j + 2^{k-1}) \mod 2^k$ and each T_i^k is given by $T_i^k = \left\{2^n\left(\frac{r}{s}\right) | r, s \text{ are positive odd integers}, n \in 2^k\mathbb{Z} + i\right\}$. Then it can be shown (in analogous way as in Example 2.10) that each S_j^k has a μ-shade $\frac{1}{2^k}$ for any improved Banach μ.

Before we get to the main theorems, we must state and prove a lemma that can be used to create an improved Banach measure from an arbitrary Banach measure. If G is the multiplicative group of positive real numbers, then G is commutative, and so by [7, Theorem 10.4 b)], G is amenable. This implies the existence of the measure μ_G on G.

Lemma 2.11. *Let μ be any Banach measure. Then there exists an improved Banach measure ν satisfying $\nu(E) = \int_G \frac{\mu(x^{-1}E)}{x^{-1}} d\mu_G(x)$ for any bounded set E, where G is the multiplicative group of positive real numbers.*

Proof. We will prove that $\psi(E) \equiv \int_G \frac{\mu(x^{-1}E)}{x^{-1}} d\mu_G(x)$ is finitely-additive, isometry-invariant, and an extension of the Lebesgue measure for bounded sets E. Lemma 2.2 would then imply the existence of a Banach ν. To verify that the integral is defined, we must still prove $\frac{\mu(x^{-1}E)}{x^{-1}}$ is bounded. Since

E is bounded, say $E \subseteq I$ for some bounded interval I, we have $\dfrac{\mu(x^{-1}E)}{x^{-1}} \leq \dfrac{\lambda(x^{-1}I)}{x^{-1}} = x^{-1}\dfrac{\lambda(I)}{x^{-1}} = \lambda(I) < \infty$. Thus $\int_G \dfrac{\mu(x^{-1}E)}{x^{-1}} d\mu_G(x)$ is defined for all bounded E. Finite additivity follows easily from the additivity of μ and the additivity of the integral. (See p. 147 of [7].) Since $\dfrac{\mu(x^{-1}(E+t))}{x^{-1}} = \dfrac{\mu(x^{-1}E + x^{-1}t)}{x^{-1}} = \dfrac{\mu(x^{-1}E)}{x^{-1}}$, ψ is translation-invariant. Since $\mu(x^{-1}(-E)) = \mu(-x^{-1}E) = \mu(x^{-1}E)$, ψ is reflection-invariant. Finally, for any Lebesgue measurable set $A \subset \mathbb{R}$, $\dfrac{\mu(x^{-1}A)}{x^{-1}} = \dfrac{\lambda(x^{-1}A)}{x^{-1}} = x^{-1}\dfrac{\lambda(A)}{x^{-1}} = \lambda(A)$, so ψ is an extension of the Lebesgue measure. We will show that for any bounded $E \subset \mathbb{R}$ and $c > 0$, $\psi(cE) = c\psi(E)$. We note that ψ is a *left-invariant mean*, meaning x can be replaced with $c^{-1}x$ in the integral without changing its value. (See [7, p. 147].) Thus for every bounded $E \subset \mathbb{R}$ and $c > 0$, $\dfrac{\psi(cE)}{c} = \int_G \dfrac{\mu(x^{-1}(cE))}{cx^{-1}} d\mu_G(x) = \int_G \dfrac{\mu((c^{-1}x)^{-1}E)}{(c^{-1}x)^{-1}} d\mu_G(x) = \int_G \dfrac{\mu(x^{-1}(E))}{x^{-1}} d\mu_G(x) = \psi(E)$. Hence, by Lemma 2.2, there exists an improved Banach measure ν which satisfies the thesis. □

In [3, Theorem 10.1], it is proven that there exists a set A, a constant $c > 0$, and a Banach measure ν such that $\nu(cA) \neq c\nu(A)$. That is, it is demonstrated that not every Banach measure is an improved Banach measure. The following corollary to Lemma 2.11 gives a sufficient condition for $\nu(cA) \neq c\nu(A)$ for some $c \in \mathbb{R}^+$ and for some Banach measure ν, for a given set A.

Corollary 2.12. *Let A be a bounded set such that $\mu(A)$ has the same value for every improved Banach measure μ, but $\mu(A) \neq \nu(A)$ for some (non-improved) Banach measure ν. Then there exists $c \in \mathbb{R}^+$ such that $\nu(cA) \neq c\nu(A)$.*

PROOF. By contradiction. Assume $\nu(x^{-1}A) = x^{-1}\nu(A)$ for every $x > 0$. Let k be the common value of $\mu(A)$, where μ is an arbitrary improved Banach measure. By Lemma 2.11, there exists an improved Banach measure ζ satisfying $\zeta(E) = \int_G \dfrac{\nu(x^{-1}E)}{x^{-1}} d\mu_G(x)$ for bounded E. Setting $E = A$ gives us

$$k = \int_G \frac{\nu(x^{-1}A)}{x^{-1}} d\mu_G(x) = \int_G \nu(A) d\mu_G(x) = \nu(A),$$

a contradiction. □

Proposition 2.13. *Let A be an almost isometry-invariant similarity-shading with $\delta(A)$ dense, and let $f(x) = x^{\frac{p}{q}}$ for odd integers p and q. Then* $\mathrm{sh}_{\sim} f(A) = \mathrm{sh}_{\sim} S$.

PROOF. Let μ be an improved Banach measure and let I be any bounded interval. From Lemma 2.3, there exists a t-Banach measure ν satisfying for bounded E

$$\nu(E) = \frac{1}{\lambda(I)} \int_S \mu(f(E+x) \cap I) d\mu(x), \tag{2}$$

where S is the support of $\mu(f(E+x) \cap I)$. This means $\eta(E) = \frac{1}{2}(\nu(E)+\nu(-E))$ is a Banach measure. Using Lemma 2.11, there exists an improved Banach ϕ such that

$$\phi(E) = \int_G \frac{\eta(y^{-1}E)}{y^{-1}} d\mu_G(y) \tag{3}$$

for every bounded E. Let $y \in \mathbb{R}^+$. Then $y^{-1}A$ is an almost isometry-invariant set with $\delta(y^{-1}A)$ dense. Setting $E = y^{-1}A \cap [0, y^{-1}]$ in equation (2), we have

$$\begin{aligned} \nu(y^{-1}A \cap [0, y^{-1}]) &= \frac{1}{\lambda(I)} \int_S \mu(f(((y^{-1}A) \cap [0, y^{-1}]) + x) \cap I) d\mu(x) \\ &= \frac{1}{\lambda(I)} \int_S \mu(f((y^{-1}A) \cap [x, x+y^{-1}]) \cap I) d\mu(x). \end{aligned}$$

By Proposition 2.6, $f(y^{-1}A)$ is a μ-shading since μ is an improved Banach measure, so

$$\begin{aligned} \mu(f((y^{-1}A) \cap [x, x+y^{-1}]) \cap I) &= \mu(f(y^{-1}A) \cap f([x, x+y^{-1}]) \cap I) \\ &= (\mathrm{sh}_\mu f(y^{-1}A))(\lambda(f([x, x+y^{-1}]) \cap I)). \end{aligned}$$

This implies by the proof of Lemma 2.3 that $\nu((y^{-1}A) \cap [0, y^{-1}]) = y^{-1}(\mathrm{sh}_\mu f(y^{-1}A))$. Since $\nu(-((y^{-1}A) \cap [0, y^{-1}])) = \nu((y^{-1}A) \cap (-[0, y^{-1}])) = \nu((y^{-1}A) \cap [0, y^{-1}])$ (these equalities follow from the fact that $y^{-1}A$ is almost isometry-invariant), we get $\eta((y^{-1}A) \cap [0, y^{-1}]) = y^{-1}(\mathrm{sh}_\mu f(y^{-1}A))$. But

$$\begin{aligned} \mathrm{sh}_\mu f(y^{-1}A) &= \mathrm{sh}_\mu((y^{-1})^{\frac{p}{q}} f(A)) \\ &= \mathrm{sh}_\mu f(A) \end{aligned}$$

by Theorem 1.3, so $\eta((y^{-1}A) \cap [0, y^{-1}]) = y^{-1}(\mathrm{sh}_\mu f(A))$. Now set $E = A \cap [0,1]$ in equation (3). Since ϕ is an improved Banach measure, we have $\mathrm{sh}_{\sim} A = \int_G \mathrm{sh}_\mu f(A) d\mu_G(y) = \mathrm{sh}_\mu f(A)$. But μ was an arbitrary improved Banach measure, so $\mathrm{sh}_{\sim} f(A) = \mathrm{sh}_{\sim} A$. □

The following theorem shows that it is possible for the continuous image of an almost isometry-invariant set (almost never a shading) to be a shading.

Theorem 2.14. *Let A be an almost isometry-invariant similarity-shading with $\delta(A)$ dense. If*

$$h(x) = \begin{cases} 0, & x \le 0 \\ \ln x, & x > 0 \end{cases},$$

then $\mathrm{sh}(h(A)) = \mathrm{sh}_{\sim} A$.

PROOF. Let μ be any Banach measure. By Lemma 2.3 (used for $s = e$, $t = e^2$), there exists a t-Banach measure ν such that $\nu(E) = \dfrac{1}{\lambda(I)} \displaystyle\int_S \mu(h(E + x) \cap I) d\mu(x)$ for bounded E, where S is the support of the integrand function. Define η and ϕ as in the proof of Theorem 2.13. Let $y \in \mathbb{R}^+$. Then $y^{-1}A$ is almost isometry-invariant with $\delta(y^{-1}A)$ dense. By Proposition 2.7, $h(y^{-1}A)$ is a μ-shading. To show $\eta((y^{-1}A) \cap [0, y^{-1}]) = y^{-1}(\mathrm{sh}_\mu h(A))$, we follow the exact same steps as in the proof of Theorem 2.13, except we must show $\mathrm{sh}_\mu h(y^{-1}A) = \mathrm{sh}_\mu h(A)$ for this particular function. We have $\mathrm{sh}_\mu h(y^{-1}A) = \mathrm{sh}_\mu \ln(y^{-1}A^+) = \mathrm{sh}_\mu(\ln y^{-1} + \ln A^+) = \mathrm{sh}_\mu \ln A^+ = \mathrm{sh}_\mu h(A)$. Setting $E = A \cap [0, 1]$ in equation (3) as before gives us $\mathrm{sh}_{\sim} A = \mathrm{sh}_\mu h(A)$, but this time μ is not restricted to be an improved Banach measure. We conclude that $\mathrm{sh}(h(A)) = \mathrm{sh}_{\sim} A$. $\square$

3 Shadings in $\mathbb{R}^2$

The first several results in this section involve the Cartesian product of subsets of the real line. In most of the results we prove something involving, say $A \times B$, but it should be understood that since $B \times A$ is the set $A \times B$ reflected with respect to $y = x$ in the plane (an isometry), that result is also in most cases true for $B \times A$. Many of these Cartesian product results were inspired by the proof of [7, Corollary 1.6c)]. Related results can be also found in [1].

The following two Lemmas are analogous to Lemmas 2.1 and 2.2. We skip the proof.

Lemma 3.1. *Let μ be a measure on the family of all bounded subsets of $\mathbb{R}^2$, such that for any open and bounded set $U \subset \mathbb{R}^2$, $\mu(U) = \lambda(U)$. Then μ is an extension of the Lebesgue measure for bounded sets.*

Lemma 3.2. *Let μ be a measure on the family of all bounded subsets of $\mathbb{R}^2$, which is an isometry invariant extension of the Lebesgue measure for bounded sets. Let (I_n) and (J_n) be sequences of nonempty intervals of a given length* *with* $\biguplus_{n=1}^{\infty} I_n = \biguplus_{n=1}^{\infty} J_n = \mathbb{R}$. *Define* $\nu(E) \equiv \sum_{n,m=1}^{\infty} \mu(E \cap (I_n \times J_m))$, $E \subset \mathbb{R}^2$. *Then* ν *is a Banach measure.*

Lemma 3.3. *For any Archimedean set* $A \subseteq \mathbb{R}$, *any bounded* $S \subset \mathbb{R}$, *and any Banach measure* μ *on* $\mathbb{R}^2$, $\dfrac{\mu((A \cap I) \times S)}{\lambda(I)} = \mu((A \cap [0,1]) \times S)$ *for any bounded interval* I.

Proof. The proof is similar to the proofs of [2, Theorem 6.1] and [3, Theorem 5.3]. It utilizes the horizontal translation invariance of μ. □

Let $I_n = (n, n+1]$, $n \in \mathbb{N}$. For any $S \subset \mathbb{R}$ and any Banach measure μ on $\mathbb{R}^2$ with $\mu(S \times [0,1]) > 0$, define $\mu_S : 2^{\mathbb{R}} \to [0, \infty]$ in the following way:

$$\mu_S(E) = \frac{1}{\mu(S \times [0,1])} \sum_{n \in \mathbb{Z}} \mu(S \times (E \cap I_n)).$$

Clearly, for any $E \subset \mathbb{R}$, we have $\mu_S(E) = \sum_{n \in \mathbb{Z}} \frac{\mu((E \cap I_n) \times S)}{\mu([0,1] \times S)}$, and if E is bounded, then

$$\mu_S(E) = \frac{\mu(S \times E)}{\mu(S \times [0,1])}.$$

Lemma 3.4. *For any bounded set* $S \subset \mathbb{R}$ *and any Banach measure* μ *on* $\mathbb{R}^2$ *with* $\mu([0,1] \times S) \neq 0$, μ_S *is a Banach measure. If additionally* S *is an interval and* μ *is an improved Banach measure, then* μ_S *is also improved.*

Proof. Additivity of μ_S follows from that of μ, as does translation and reflection invariance. Now assume $E = I$, a bounded interval. If we set $A = \mathbb{R}$ in Lemma 3.3, we get $\dfrac{\mu(S \times I)}{\lambda(I)} = \mu(S \times [0,1])$, or $\dfrac{\mu(S \times I)}{\mu(S \times [0,1]} = \lambda(I)$, provided $\lambda(I) \neq 0$. If $\lambda(I) = 0$, then S bounded implies $S \subseteq J$ for some bounded interval J so that $\dfrac{\mu(S \times I)}{\mu(S \times [0,1])} \leq \dfrac{\mu(J \times I)}{\mu(S \times [0,1])} = \dfrac{(\mu(J))(\mu(I))}{\mu(S \times [0,1])} = 0$. Now let (I_n) be a sequence of pairwise disjont intervals with $\bigcup_{n \in \mathbb{N}} I_n$ bounded, and let J be a bounded interval with $S \subset J$. For any $k \in \mathbb{N}$,

$$\lambda\left(\bigcup_{n=1}^{k} I_n\right) = \mu_S\left(\bigcup_{n=1}^{k} I_n\right) \leq \mu_S\left(\bigcup_{n \in \mathbb{N}} I_n\right) = \lambda\left(\bigcup_{n=1}^{k} I_n\right) + \mu_S\left(\bigcup_{n=k+1}^{\infty} I_n\right) \leq$$

$$\leq \lambda\left(\bigcup_{n=1}^{k} I_n\right) + \frac{\mu\left(J \times \left(\bigcup_{n=k+1}^{\infty} I_n\right)\right)}{\mu(S \times [0,1])} = \lambda\left(\bigcup_{n=1}^{k} I_n\right) + \frac{\lambda\left(\bigcup_{n=k+1}^{\infty} I_n\right) \lambda(J)}{\mu(S \times [0,1])}.$$

By taking the limit $k \to \infty$, we get $\mu_S\left(\bigcup_{n\in\mathbb{N}} I_n\right) = \lambda\left(\bigcup_{n\in\mathbb{N}} I_n\right)$. By Lemmas 2.1 and 2.2, we have that μ_S is finitely additive, isometry-invariant, and an extension of the Lebesgue measure.
If S is an interval and μ is improved, then for any bounded E and $c > 0$ we get by Lemma 3.3 (for $A = \mathbb{R}$),

$$\frac{\mu\left(S \times (cE)\right)}{\mu(S \times [0,1])} = \frac{c^2\mu\left(\frac{S}{c} \times E\right)}{\mu\left(\frac{S}{c} \times [0,1]\right)} \frac{\mu\left(\frac{S}{c} \times [0,1]\right)}{\mu(S \times [0,1])} = c\frac{\mu(S \times E)}{\mu(S \times [0,1])}.$$

This ends the proof. $\square$

Lemma 3.5. *Assume that $A \subset \mathbb{R}$ is a ν-shading for every Banach measure ν. Then for every Banach measure μ on $\mathbb{R}^2$ with $\mathrm{sh}_{\mu_{[0,1]}}(A) > 0$, and every bounded interval I, $\mu_{A\cap I} = \mu_{A\cap[0,1]}$.*

PROOF. At first note that $\mu((A\cap I)\times[0,1]) = \mu_{[0,1]}(A\cap I) = \mathrm{sh}_{\mu_{[0,1]}}(A)\lambda(I) > 0$. For every bounded set E with $\mu_{[0,1]}(E) > 0$, we have

$$\begin{aligned}
\mu_{A\cap I}(E) =& \frac{\mu((A\cap I)\times E)}{\mu((A\cap I)\times[0,1])} = \frac{\mu(E\times(A\cap I))}{\mu_{[0,1]}(A\cap I)} = \frac{\mu_E(A\cap I)\mu_{[0,1]}(E)}{\mu_{[0,1]}(A\cap I)} \\
=& \frac{\mu_E(A\cap I)}{\lambda(I)} \frac{\lambda(I)}{\mu_{[0,1]}(A\cap I)} \mu_{[0,1]}(E) \\
=& \frac{\mu_E(A\cap[0,1])}{1} \frac{1}{\mu_{[0,1]}(A\cap[0,1])} \mu_{[0,1]}(E) \\
=& \frac{\mu((A\cap[0,1])\times E)}{\mu((A\cap[0,1])\times[0,1])} = \mu_{A\cap[0,1]}(E).
\end{aligned}$$

If $\mu_{[0,1]}(E) = 0$, then

$$\mu_{A\cap I}(E) = \frac{\mu((A\cap I)\times E)}{\mu((A\cap I)\times[0,1])} \le \frac{\mu(I\times E)}{\mu((A\cap I)\times[0,1])} = 0.$$

The last equality follows from the fact that for each $a \in \mathbb{R}$, $\mu([a,a+1]\times E) = 0$. $\square$

Theorem 3.6. *Let μ be a Banach measure on $\mathbb{R}^2$ and $A, B \subset \mathbb{R}$. Assume that one of the conditions holds:*

(i) B is a shading and A is a $\mu_{[0,1]}$-shading;

(ii) A, B are η-shadings for every Banach measure η.

Then $A \times B$ is a μ-shading.
If $\mathrm{sh}_{\mu_{[0,1]}} A = 0$, then $\mathrm{sh}_\mu(A \times B) = 0$
If $\mathrm{sh}_{\mu_{[0,1]}} A > 0$, then $\mathrm{sh}_\mu(A \times B) = \mathrm{sh}_{\mu_{[0,1]}}(A)\mathrm{sh}_{\mu_{A\cap[0,1]}}(B)$.
In particular, if B is a shading, we have $\mathrm{sh}_\mu(A \times B) = \mathrm{sh}_{\mu_{[0,1]}}(A)\mathrm{sh}(B)$.

Proof. Assume first that $\mathrm{sh}_{\mu_{[0,1]}} A = 0$. For every bounded intervals I, J, we have

$$\mu\left((A \times B) \cap (I \times J)\right) = \mu\left((A \cap I) \times (B \cap J)\right) \leq \mu\left((A \cap I) \times J\right) = 0.$$

Assume now that $\mathrm{sh}_{\mu_{[0,1]}} A > 0$. For every bounded intervals I, J, we have

$$\begin{aligned}\frac{\mu\left((A \times B) \cap (I \times J)\right)}{\lambda(I)\lambda(J)} &= \frac{\mu\left((A \cap I) \times (B \cap J)\right)}{\lambda(I)\lambda(J)} \\ &= \frac{\mu\left((A \cap I) \times (B \cap J)\right)}{\mu\left((A \cap I) \times [0,1]\right)\lambda(J)} \frac{\mu\left((A \cap I) \times [0,1]\right)}{\lambda(I)} \\ &= \frac{\mu_{A\cap I}(B \cap J)}{\lambda(J)} \frac{\mu_{[0,1]}(A \cap I)}{\lambda(I)} = \frac{\mu_{A\cap I}(B \cap J)}{\lambda(J)} \mathrm{sh}_{\mu_{[0,1]}}(A).\end{aligned}$$

If (i) is satisfied, then the last formula is equal $\mathrm{sh}(B)\mathrm{sh}_{\mu_{[0,1]}}(A)$, and if (ii) is satisfied, then by Lemma 3.5, we have

$$\frac{\mu_{A\cap I}(B \cap J)}{\lambda(J)} \mathrm{sh}_{\mu_{[0,1]}}(A) = \frac{\mu_{A\cap[0,1]}(B \cap J)}{\lambda(J)} \mathrm{sh}_{\mu_{[0,1]}}(A) = \mathrm{sh}_{\mu_{[0,1]}}(A)\mathrm{sh}_{\mu_{A\cap[0,1]}}(B).$$

$\square$

Directly from the above Theorem and the second part of Lemma 3.4, we get the following corollaries. The first one answers a conjecture from the Conclusion section of [2].

Corollary 3.7. *Let $A, B \subseteq \mathbb{R}$ be shadings. Then $\mathrm{sh}(A \times B) = (\mathrm{sh}A)(\mathrm{sh}B)$.*

Corollary 3.8. *Let $A \subset \mathbb{R}$ be a similarity shading and $B \subset \mathbb{R}$ be a shading. Then $A \times B$ is a similarity shading and $\mathrm{sh}_\sim(A \times B) = \mathrm{sh}_\sim(A)\mathrm{sh}(B)$.*

Now we strengthen Corollary 3.7 to f-shadings on $\mathbb{R}^2$ (see Introduction).

Theorem 3.9. *Let $f, g : \mathbb{R} \to [0,1]$ be continuous functions. Let F be an f-shading on $\mathbb{R}$ and let G be a g-shading on $\mathbb{R}$. Then $\mathrm{sh}(F \times G)(x,y) = f(x)g(y)$.*

Proof. We may assume $f(x)$ and $g(y)$ are not both zero. If they are, then it is enough to prove the result for, say $F \times \mathbb{R}$ at (x,y), since $F \times G \subseteq F \times \mathbb{R}$. But in that case $g(y) = 1$. So assume $f(x)$ and $g(y)$ are not both zero. WLOG, assume $f(x) \neq 0$. Let I, J be two bounded intervals such that $(x, y) \in I \times J$. We have

$$\frac{\mu((F\cap I)\times(G\cap J))}{\lambda(I)\lambda(J)} = \frac{\mu((F\cap I)\times(G\cap J))\mu((F\cap I)\times[0,1])}{\mu((F\cap I)\times[0,1])\lambda(I)\lambda(J)}$$

$$= \frac{\mu_{F\cap I}(G\cap J)\mu_{[0,1]}(F\cap I)}{\lambda(J)\lambda(I)} = \frac{\mu_{F\cap[x-\frac{1}{2},x+\frac{1}{2}]}(G\cap J)\mu_{[0,1]}(F\cap I)}{\lambda(J)\lambda(I)}.$$

(The last eqality and the facts that $\mu((F\cap I)\times[0,1]) > 0$ and $\mu((F\cap[x-\frac{1}{2}, x+\frac{1}{2}])\times[0,1]) > 0$ follow from [2, Remark 5.12] which states that if $E \subset \mathbb{R}$ is Lebesgue measurable and A is an f-shading, then for every Banach μ, $\mu(A\cap E) = \int_E f\ d\lambda$.) Now if we take the limit from both sides, we get the thesis. $\square$

A measure similar to the one given in Lemma 3.10 was mentioned by J. Roberts and R. Mabry.

Lemma 3.10. *Let η be a Banach measure on $\mathbb{R}$ and let G represent the group of isometries on $\mathbb{R}^2$. Then there exists a Banach measure μ on $\mathbb{R}^2$ satisfying $\mu(E) = \int_G(\int_S \eta(l_x\cap T^{-1}(E))d\eta(x))d\mu_G(T)$ for every bounded set E. Here l_x represents the vertical line through x on the x-axis and S represents the support of the integrand function $\eta(l_x\cap T^{-1}(E))$.*

PROOF. Define $\psi(E) = \int_G(\int_S \eta(l_x\cap T^{-1}(E))d\eta(x))d\mu_G(T)$ for all bounded $E \subset \mathbb{R}$. Additivity is easily verified using the additivity of η and that of the two integrals. Isometry-invariance follows from the fact that ψ is a left-invariant mean. If R is a rectangle, then $\eta(l_x\cap T^{-1}(R))$ is continuous, meaning by Proposition 1.7, $\int_S \eta(l_x\cap T^{-1}(R))d\eta(x)$ reduces to a Riemann integral. It is easy to see that a Riemann sum for $\int_S \eta(l_x\cap T^{-1}(R))d\eta(x)$ approximates the area of $T^{-1}(R)$ and so $\int_S \eta(l_x\cap T^{-1}(R))d\eta(x) = \lambda(T^{-1}(R))$. Since the mapping $\eta(l_x\cap T^{-1}(R))$ is continuous, again by Proposition 1.7, if (R_n) is a sequence of pairwise disjoint rectangles with $\bigcup_{n\in\mathbb{N}} R_n$ bounded, then

$$\int_S \eta\left(l_x\cap T^{-1}\left(\bigcup_{n\in\mathbb{N}} R_n\right)\right)d\eta(x) = \sum_{n\in\mathbb{N}}\int_S \eta\left(l_x\cap T^{-1}(R_n)\right)d\eta(x)$$

$$= \sum_{n\in\mathbb{N}}\lambda\left(T^{-1}(R_n)\right) = \lambda\left(T^{-1}\left(\bigcup_{n\in\mathbb{N}} R_n\right)\right).$$

By Lemma 3.1, ψ is an extension of the Lebesgue measure for bounded E. Use Lemma 3.2 to get μ. $\square$

Lemma 3.11. *Let η be an improved Banach measure on $\mathbb{R}$, and let μ be a Banach measure on $\mathbb{R}^2$ as in Lemma 3.10, chosen for η. If $A \subset \mathbb{R}$ is an η-shading, then for every bounded interval I, $\eta(A \cap I) = \mu_{[0,1]}(A \cap I)$.*

Proof. It is enough to show that $\int_S \eta(l_x \cap T^{-1}(E))d\eta(x) = \lambda(I)\mathrm{sh}_\eta A$ whenever $E = (A \cap I) \times [0,1]$. To do this, we will slice the set $A \times \mathbb{R}$ with lines from every direction and show that we (almost) always get an η-shading of η-shade sh$A$. First consider a nonvertical line, say $y = mx + c$. This line will intersect the y-axis at $(0, c)$. We will think of this point as the "origin" of the line and the right direction as the positive direction. Now choose $a \in A$ on the x-axis. It's corresponding point on the line is $(a, ma + c)$. The distance between $(0, c)$ and $(a, ma + c)$ is $\sqrt{m^2a^2 + a^2} = |a|\sqrt{m^2 + 1}$. Hence the intersection of the line and $A \times \mathbb{R}$ is $\sqrt{m^2 + 1}A$. But by Theorem 1.3, $\mathrm{sh}_\eta(\sqrt{m^2 + 1}A) = \mathrm{sh}_\eta A$. Now consider slicing the set with vertical slices. (This is equivalent to assuming T^{-1} is an isometry that does not change the orientation of $A \times \mathbb{R}$.) Let P represents the projection of $T^{-1}((A \cap I) \times [0,1])$ into the x-axis. We have $\eta(l_x \cap T^{-1}((A \cap I) \times [0,1])) = 0$ whenever $x \notin P$ and $\eta(l_x \cap T^{-1}((A \cap I) \times [0,1])) = 1$ whenever $x \in P$. Hence $\int_S \eta(l_x \cap T^{-1}((A \cap I) \times [0,1]))d\eta(x) = \int_P 1 d\eta(x) = \lambda(I)\mathrm{sh}_\eta A$ in this case. □

Theorem 3.12. *Let $A \subset \mathbb{R}$ be a shading, η be an improved Banach measure on $\mathbb{R}$, $B \subset \mathbb{R}$ be an η-shading and μ be a Banach measure defined as in Lemma 3.10. Then $A \times B$ is a μ-shading with $\mathrm{sh}_\mu(A \times B) = \mathrm{sh}(A)\mathrm{sh}_\eta(B)$.*

Proof. By Lemma 3.11, B is an $\mu_{[0,1]}$-shading with $\mathrm{sh}_{\mu_{[0,1]}}(B) = \mathrm{sh}_\eta(B)$. Hence, by Theorem 3.6(i), $A \times B$ is a μ-shading with $\mathrm{sh}_\mu(A \times B) = \mathrm{sh}(A)\mathrm{sh}_\eta(B)$. □

An *Archimedean* subset of $\mathbb{R}^2$ is any set A satisfying $A + \overrightarrow{t} = A$ for densely many vectors $\overrightarrow{t} = < t_1, t_2 >$, i.e., for which $\tau(A)$ is dense in $\mathbb{R}^2$.

Theorem 3.13. *Every Archimedean subset of $\mathbb{R}^2$ is a μ-shading for every Banach measure μ of $\mathbb{R}^2$.*

Proof. The proof is nearly identical to that of the proofs of [2, Theorem 6.1] and [3, Theorem 5.3] with squares and rectangles used instead of intervals. □

Proposition 3.14. *If A, B are Archimedean subsets of $\mathbb{R}$, then $A \times B$ is an Archimedean subset of $\mathbb{R}^2$.*

Proof. It is clear that $\tau(A \times B) \supseteq \tau(A) \times \tau(B)$. Moreover, the set $\tau(A) \times \tau(B)$ is dense since $\tau(A)$ and $\tau(B)$ are dense. □

We will now consider another way to generate shadings in $\mathbb{R}^2$ from shadings in $\mathbb{R}$. It involves polar rectangles, so we first need to prove the following.

Lemma 3.15. *Let $K \geq 0$, and let $A \subseteq \mathbb{R}^2$ be a set satisfying $\mu(A \cap R) = K\lambda(R)$ for every polar rectangle R. Then $\mu(A \cap I) = K\lambda(I)$ for every rectangle $I \subset \mathbb{R}^2$. That is, A is a μ-shading in $\mathbb{R}^2$ with* $\mathrm{sh}_\mu A = K$.

PROOF. We will prove the result for squares, since any rectangle can be approximated as closely as we like with finitely many squares. Let I be a square with a center C. Let I_i be a square with center C inside I that satisfies $\lambda(I_i) = \lambda(I) - \varepsilon$, and let I_o be a square with center C containing I that satisfies $\lambda(I_o) = \lambda(I) + \varepsilon$. Cover I_i with finitely many disjoint polar rectangles small enough so that they are all strictly inside I. Call the union of these polar rectangles R_1. Now cover I with finitely many disjoint polar rectangles small enough so that they are all strictly inside I_o. Call the union of these polar rectangles R_2. We can then say $K\lambda(R_1) \leq \mu(A \cap I) \leq K\lambda(R_2)$. Since $\lambda(R_1) \geq \lambda(I) - \varepsilon$ and $\lambda(R_2) \leq \lambda(I) + \varepsilon$, $K\lambda(I) - K\varepsilon \leq \mu(A \cap I) \leq K\lambda(I) + K\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\mu(A \cap I) = K\lambda(I)$. This means A is a μ-shading of shade K. $\square$

Recall that if $A \subset \mathbb{R}^2$, then $\delta(A) = \{t \in \mathbb{R}^2 : tA = A\}$, where the multiplication has a "complex meaning".

Theorem 3.16. *Let $A \subseteq \mathbb{R}^2$ and assume $\delta(A)$ is dense in $\mathbb{R}^2$. Then A is a μ-shading for any improved Banach μ on $\mathbb{R}^2$.*

PROOF. Let μ be an improved Banach measure. We first show that for every polar rectangle R and every $d \in \mathbb{R}^2$,

$$\mu(A \cap dR) = |d|^2 \mu(A \cap R). \tag{4}$$

Let $\epsilon > 0$. There exists $d' \in \delta(A)$ such that $||d'|^2 - |d|^2| \leq \epsilon$ and $\lambda(dR \Delta d'R) < \epsilon$ (Δ means the symmetric difference). By rotation-invariance of μ and the fact that μ is improved, we have

$$\mu(A \cap d'R) = \mu(d'A \cap d'R) = \mu(d'(A \cap R)) = |d'|^2 \mu(A \cap R).$$

Moreover, by

$$A \cap dR \subset (A \cap d'R) \cup ((A \cap dR) \setminus (A \cap d'R)) \subset (A \cap d'R) \cup (dR \setminus d'R),$$

we have

$$\mu(A \cap dR) \leq \mu(A \cap d'R) + \epsilon.$$

In the same way we can show that

$$\mu(A\cap d'R)\le \mu(A\cap dR)+\epsilon.$$

Summing up, we get

$$|\mu(A\cap dR)-|d|^2\mu(A\cap R)|\le$$

$$|\mu(A\cap dR)-\mu(A\cap d'R)|+||d'|^2\mu(A\cap R)-|d|^2\mu(A\cap R)|\le \epsilon(1+\mu(A\cap R)).$$

Since $\epsilon>0$ was taken arbitrarily, we get (4).
If $\alpha\in[0,2\pi]$, put $R_\alpha=\{se^{i\beta}: s\in[0,1],\ \beta\in[0,\alpha]\}=[0,1]e^{i[0,\alpha]}$. Now we show that for every $\alpha\in[0,2\pi]$,

$$\mu(A\cap R_\alpha)=\frac{\alpha}{2\pi}\mu(C\cap A), \tag{5}$$

where C is the unit circle. Let $\epsilon>0$ and $t\in(0,2\pi)$ be such that

$$\left(\frac{\alpha}{2\pi}-\epsilon\right)\le\left(\frac{\alpha-t}{2\pi+t}\right) \text{ and } \left(\frac{\alpha}{2\pi}+\epsilon\right)\ge\left(\frac{\alpha+t}{2\pi-t}\right). \tag{6}$$

Now fix $n,m\in\mathbb{N}$ with

$$nt\le 2\pi<(n+1)t \text{ and } mt\le\alpha<(m+1)t. \tag{7}$$

We have by (4),

$$n\mu(A\cap R_t)\le\mu(A\cap C)\le(n+1)\mu(A\cap R_t), \tag{8}$$

and

$$m\mu(A\cap R_t)\le\mu(A\cap R_\alpha)\le(m+1)\mu(A\cap R_t). \tag{9}$$

By (6)–(9), we have

$$\left(\frac{\alpha}{2\pi}-\epsilon\right)\mu(A\cap C)\le\left(\frac{\alpha-t}{2\pi+t}\right)\mu(A\cap C)\le\frac{m}{n+1}\mu(A\cap C)\le\mu(A\cap R_\alpha)$$

and

$$\left(\frac{\alpha}{2\pi}+\epsilon\right)\mu(A\cap C)\ge\left(\frac{\alpha+t}{2\pi-t}\right)\mu(A\cap C)\ge\frac{m+1}{n}\mu(A\cap C)\ge\mu(A\cap R_\alpha).$$

Since $\epsilon>0$ was taken arbitrarily, we get (5).
Lemma 3.15, (4) and (5) imply the thesis. Indeed, if R is any polar rectangle,

then for some reals $r_2 > r_1 \geq 0$, $d \in [0, 2\pi]$ and $\alpha \in [0, 2\pi]$, $R = e^{id}(r_2 R_\alpha \setminus r_1 R_\alpha)$. Hence

$$\mu(A \cap R) = \mu(A \cap (e^{id}(r_2 R_\alpha \setminus r_1 R_\alpha))) = \mu(A \cap r_2 R_\alpha) - \mu(A \cap r_1 R_\alpha) =$$

$$= (r_2^2 - r_1^2)\mu(A \cap R_\alpha) = (r_2^2 - r_1^2)\frac{\alpha}{2\pi}\mu(A \cap C) = \lambda(R)\frac{\mu(A \cap C)}{\pi},$$

which shows that A is a μ-shading with $\mathrm{sh}A = \frac{\mu(A \cap C)}{\pi}$. □

Recall that $\rho(A) = \{t \in \mathbb{R} : Ae^{it} = A\}$.

Corollary 3.17. *Let $A \subseteq \mathbb{R}^2$. If $\delta(A) \cap (\mathbb{R} \times \{0\})$ and $\rho(A)$ are both dense in $\mathbb{R}$, then A is a μ-shading for any improved Banach μ.*

PROOF. Let $B = \delta(A) \cap (\mathbb{R} \times \{0\})$. Clearly, the denseness of B and $\rho(A)$ implies the denseness of $Be^{i\rho(A)}$ in $\mathbb{R}^2$. It is also clear that $Be^{i\rho(A)} \subset \delta(A)$. □

We can use Theorems 3.13 and 3.16 to state three results similar to Propositions 2.6 – 2.8. Proofs of the newer propositions are similar to the original proofs and will be omitted. Note that we consider the functions which appear in the following results as complex functions (we identify here $\mathbb{C}$ with $\mathbb{R}^2$).

Proposition 3.18. *Let $A \subseteq \mathbb{R}^2$ and assume $\delta(A)$ is dense. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(z) = z^t$ (t any real number $t \neq 0$). Then $f(A)$ is a μ-shading for any improved Banach μ.*

Proposition 3.19. *Let $A \subseteq \mathbb{R}^2$ and assume $\delta(A)$ is dense. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(z) = \log z$ (f is the multi-valued logarithmic function). Then $f(A)$ is a μ-shading for any Banach μ.*

Proposition 3.20. *Let $A \subseteq \mathbb{R}^2$ and assume $\tau(A)$ is dense. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(z) = e^z$. Then $f(A)$ is a μ-shading for any improved Banach μ.*

The following lemma will be used for the next shading result. It is an easy generalization to $\mathbb{R}^2$ of the measure used in the proof of [3, Theorem 4.1].

Lemma 3.21. *Let $c > 0$ and let μ be any Banach measure on $\mathbb{R}^2$. Then $\nu(E) = \dfrac{\mu(cE)}{c^2}$ is a Banach measure on $\mathbb{R}^2$.*

PROOF. Additivity is obvious as is the fact that ν is an extension of the Lebesgue measure. Any isometry of $\mathbb{R}^2$ can be written as a composition of translations and reflections about lines through the origin, so it is enough to show that ν is invariant with respect to each of these two isometries separately.

Let $\overrightarrow{t}$ be a vector in $\mathbb{R}^2$. Then $\nu(E+\overrightarrow{t}) = \dfrac{\mu(c(E+\overrightarrow{t}))}{c^2} = \dfrac{\mu(cE+c\overrightarrow{t})}{c^2} = \dfrac{\mu(cE)}{c^2} = \nu(E)$. This proves translation invariance. The set of isometries representing reflections about lines through the origin is precisely the group of orthogonal 2×2 matrices with real number coefficients with determinant ± 1. Let B be one such matrix. We have $\nu(BE) = \dfrac{\mu(c(BE))}{c^2} = \dfrac{\mu(B(cE))}{c^2} = \dfrac{\mu(cE)}{c^2} = \nu(E)$. Here we use some basic matrix properties as well as the isometry invariance of μ. $\square$

Lemma 3.22. *Let μ be any Banach measure on $\mathbb{R}^2$. Then there exists a Banach measure ν on $\mathbb{R}$ satisfying $\nu(E) = 2\mu([0,1]e^{iE})$ for every set $E \subset [0,2\pi]$.*

Proof. Let
$$\nu(E) = 2\sum_{k\in\mathbb{Z}} \mu\left([0,1]e^{i(E\cap([0,2\pi)+2k\pi))}\right)$$
for any $E\subset\mathbb{R}$. Additivity follows from that of μ. We show that ν is translation invariant. If $E\subset\mathbb{R}$, $t\in[0,2\pi]$ and $k\in\mathbb{Z}$,
$$\begin{aligned}\mu\left([0,1]e^{i((E+t)\cap([2k\pi,2(k+1)\pi))}\right) =& \mu\left([0,1]e^{i((E\cap([2k\pi-t,2(k+1)\pi-t))+t)}\right)\\ =&\mu\left([0,1]e^{i(E\cap([2k\pi-t,2(k+1)\pi-t))}\right)\\ =&\mu\left([0,1]e^{i(E\cap([2k\pi-t,2k\pi))}\right)\\ &+\mu\left([0,1]e^{i(E\cap([2k\pi,2(k+1)\pi-t))}\right).\end{aligned}$$

From this we can easily deduce that ν is translation invariant. Reflection-invariance follows from reflection-invariance about the x-axis of μ. (For each point $re^{i\theta}$ in the set $[0,1]e^{i(E\cap[2k\pi,2(k+1)\pi))}$, replacing θ with $-\theta$ reflects that point with respect to the x-axis.) If I is an interval, then $\nu(I) = 2(\frac{1}{2}\lambda(I)) = \lambda(I)$. Proceeding as in the proof of Lemma 3.4, we show that ν is an extension of the Lebesgue measure. $\square$

The next result says that if we take a shading in $\mathbb{R}$ and wrap 2π of it around the unit circle, then form a set in the plane by sketching rays that begin at the origin and intersect points on the shading, we get a shading in $\mathbb{R}^2$.

Theorem 3.23. *Let A be a shading in $\mathbb{R}$. Then* $\mathrm{sh}([0,\infty)e^{i[0,2\pi]\cap A}) = \mathrm{sh}A$.

PROOF. We show that if $d > 0$ and $\alpha \in [0, 2\pi]$, then

$$\mu(B \cap dR_\alpha) = d^2 \mathrm{sh}(A) \frac{\alpha}{2}, \tag{10}$$

where $R_\alpha = [0,1]e^{i[0,\alpha]}$ and $B = [0,\infty)e^{i[0,2\pi] \cap A}$.
By Lemma 3.21, there exists a Banach measure ψ such that for bounded E,

$$\psi(E) = \frac{\mu(dE)}{d^2}. \tag{11}$$

By Lemma 3.22, there exists a Banach ν on $\mathbb{R}$ satisfying

$$\nu(E) = 2\psi([0,1]e^{iE}) \tag{12}$$

for every set $E \subset [0, 2\pi]$. Setting $E = A \cap [0, \alpha]$ in (12) gives us $\alpha(\mathrm{sh}A) = 2\psi([0,1]e^{i[0,\alpha]\cap A})$, which implies

$$\frac{\alpha}{2}(\mathrm{sh}A) = \frac{\mu([0,d]e^{i[0,\alpha]\cap A})}{d^2} = \frac{\mu(B \cap dR_\alpha)}{d^2}.$$

This gives (10).
Now if R is any polar rectangle, then $R = (r_2(R_\alpha \setminus R_\beta)) \setminus (r_1(R_\alpha \setminus R_\beta))$ for some $2\pi \geq \alpha \geq \beta \geq 0$ and $r_2 \geq r_1 \geq 0$. By (10), we have

$$\begin{aligned}
\mu(A \cap R) =& \mu(A \cap (r_2(R_\alpha \setminus R_\beta)) \setminus (r_1(R_\alpha \setminus R_\beta))) \\
=& \mu(A \cap (r_2(R_\alpha \setminus R_\beta))) - \mu(A \cap (r_1(R_\alpha \setminus R_\beta))) \\
=& \mu(A \cap r_2 R_\alpha) - \mu(A \cap r_2 R_\beta) - \mu(A \cap r_1 R_\alpha) + \mu(A \cap r_1 R_\beta) \\
=& \mathrm{sh}A(r_2^2 - r_1^2)\left(\frac{\alpha}{2} - \frac{\beta}{2}\right) = \lambda(R)\mathrm{sh}A.
\end{aligned}$$

□

4 Quotient sets of μ-shadings

In [4, Theorem 3.2] it was proven that if μ is a Banach measure and A is an Archimedean set satisfying $\mathrm{sh}_\mu A > \frac{1}{k+1}$, then $\mathrm{sh}_\mu(A - A) \geq \frac{1}{k}$. If we restrict μ to be an improved Banach measure, then we can prove a similar result involving quotients rather than differences. The main result follows the lemma.

Lemma 4.1. *Let μ be an improved Banach measure on $\mathbb{R}$, and let $H \subset \mathbb{R}$ be a set in which $\delta(H)$ is dense. Also assume $\mathrm{sh}_\mu H > \dfrac{k-1}{k}$, where $k \geq 2$ is an integer. Then there exist $h_1, h_2, \cdots, h_k \in \mathbb{R}$ such that $\mathrm{sh}_\mu \left(\cap_{i=1}^{k} (h_i H)\right) > 0$ and for every $j = 2, ..., k$, $h_j \in \bigcap_{i=1}^{j-1} h_i H$.*

Proof. Set any $h_1, h_2, \cdots, h_k \in \mathbb{R}^2 \setminus \{0\}$. Then $\bigcap_{i=1}^{k} h_i H$ is a μ-shading (since $\delta(H) \subset \delta(\bigcap_{i=1}^{k} h_i H)$) and we have

$$\begin{aligned} \mathrm{sh}_\mu \left(\cap_{i=1}^{k} (h_i H)\right) &= 1 - \mathrm{sh}_\mu \left(\cup_{i=1}^{k} (h_i H)^c\right) \\ &\geq 1 - \sum_{i=1}^{k} \mathrm{sh}_\mu (h_i H)^c \\ &= 1 - k\left(1 - \mathrm{sh}_\mu H\right) \\ &= k\mathrm{sh}_\mu H - (k-1). \end{aligned}$$

Thus $\mathrm{sh}_\mu \left(\cap_{i=1}^{k} (h_i H)\right) > 0$ if $\mathrm{sh}_\mu H > \dfrac{k-1}{k}$. Hence for $1 \leq j \leq k$ we also have $\mathrm{sh}_\mu \left(\cap_{i=1}^{j} (h_i H)\right) > 0$. The h_i can therefore be chosen recursively so that for every $j = 2, ..., k$, $h_j \in \left(\bigcap_{i=1}^{j-1} h_i H\right) \setminus \{h_1, ..., h_{j-1}\}$. □

Theorem 4.2. *Let μ be an improved Banach measure and let A be a set in which $\delta(A)$ is dense. Also assume $0 \notin A$ and $\mathrm{sh}_\mu A > \dfrac{1}{k+1}$ for an integer $k \geq 1$. Then $\mathrm{sh}_\mu \left(\dfrac{A}{A}\right) \geq \dfrac{1}{k}$.*

Proof. Assume to the contrary that $\mathrm{sh}_\mu \left(\dfrac{A}{A}\right) < \dfrac{1}{k}$ and let $H = \left(\dfrac{A}{A}\right)^c$. Clearly $cH = H$ for densely many $c \in \mathbb{R}$, and $\mathrm{sh}_\mu H > \dfrac{k-1}{k}$. Choose distinct $h_1, h_2, \cdots, h_k$ as per the Lemma and then take $h_{k+1} \in \cap_{i=1}^{k} (h_i H) \setminus \{h_1, h_2, \cdots, h_k\}$, this being possible since the latter intersection has positive μ-shade. It follows that the sets $h_1 A, h_2 A, \cdots, h_k A, h_{k+1} A$ are pairwise disjoint. (To see this, note that if $x \in h_i A \cap h_j A$ for $i \neq j$, then $\dfrac{h_i}{h_j} \in \dfrac{A}{A}$, which is impossible.) But the sum of μ-shades of disjoint μ-shadings cannot exceed unity, so $1 \geq \sum_{i=1}^{k+1} \mathrm{sh}_\mu (h_i A) = (k+1)\mathrm{sh}_\mu A$, which implies that $\mathrm{sh}_\mu A \leq \dfrac{1}{k+1}$, a contradiction. □

The next result is also analogous to a difference set result, but applies to quotient sets. It is similar to Corollary 3.8 in [5] in structure.

Theorem 4.3. *Let A be a shading and let B be a μ-shading satisfying* $\mathrm{sh}A + \mathrm{sh}_\mu B > 1$. *If $0 \notin B$ but $0 \in A$, then $\frac{A}{B} = \mathbb{R}$. If $0 \notin A$ but $0 \in B$, then $\frac{B}{A} = \mathbb{R}$.*

PROOF. We will prove $\frac{A}{B} = \mathbb{R}$. The other equality follows similarly. Assume there exists an $r \neq 0$ such that $r \notin \frac{A}{B}$. Then $rB \cap A = \emptyset$, which implies $B \cap \frac{1}{r}A = \emptyset$. By [3, Theorem 4.1], $\frac{1}{r}A$ is a shading satisfying $\mathrm{sh}\left(\frac{1}{r}A\right) = \mathrm{sh}A$. But $B \cap \frac{1}{r}A = \emptyset$ implies $1 \geq \mathrm{sh}_\mu(B) + \mathrm{sh}\left(\frac{1}{r}A\right) = \mathrm{sh}_\mu B + \mathrm{sh}A$, a contradiction. □

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