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ERROR LEVEL SATURATION FOR POPOFF'S GENERALIZED DERIVATIVE OPERATOR

Abstract

A saturation result with respect to data error level is presented for an approximate derivative operator of Kyrille Popoff.

1 INTRODUCTION

In 1938 K. Popoff [2], motivated by a geometric approximation of the normal line, introduced a notion of generalized derivative. Popoff's generalized derivative of a function f, which we shall denote f^p, is defined by

$$f^p(x) = \lim_{h \to 0} P_h f(x), \tag{1}$$

provided this limit exists, where

$$P_h f(x) = \frac{2}{h^2} \int_0^h f(x+t) - f(x)\, dt. \tag{2}$$

If f is differentiable at x, then

$$P_h f(x) - f'(x) = \frac{2}{h^2} \int_0^h \left[\frac{f(x+t) - f(x)}{t} - f'(x) \right] t\, dt.$$

One then sees from the definition of the derivative that for differentiable

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functions $f^p(x) = f'(x)$. On the other hand, there are examples of non-differentiable functions for which the Popoff derivative exists [2]. Popoff's definition therefore gives a generalization of the ordinary derivative.

Differentiation is a notoriously unstable process: uniformly close functions may have highly disparate derivatives. This instability is a significant challenge in scientific contexts which require approximation of the derivative when the observed function is contaminated with error. However, the Popoff approximation (2) is, for fixed h, stable with respect to uniform perturbations in f. This suggests the question of the attainable asymptotic order of approximation, with respect to error level ϵ, when approximating the unstable exact derivative f' with the stable approximate derivative $P_h f^\epsilon$, where f^ϵ is an ϵ-approximation of f. In this note we present a saturation result, with respect to error level in integrable data perturbations, for the Popoff approximation (2) of the derivative .

2 A SATURATION RESULT

We investigate the attainable order of approximation of $f'(x)$ by the Popoff approximation $P_h f^\epsilon(x)$, where f^ϵ is an integrable perturbation of f, as might arise, for example, in measured estimations of a given function f.

Lemma 1. *Suppose f has a bounded integrable second derivative on some open interval I containing x. If f^ϵ is a bounded integrable perturbation of f satisfying $|f(t) - f^\epsilon(t)| \leq \epsilon$ for $t \in I$, then*

$$|P_h f^\epsilon(x) - f'(x)| = O(|h|) + O(\epsilon/|h|)$$

for positive $|h|$ sufficiently small.

Proof. Suppose that $|f''(\theta)| \leq B$ for $\theta \in I$. Since

$$f(x+t) - f(x) = f'(x)t + \int_x^{x+t} f''(s)(x+t-s)\,ds, \tag{3}$$

we find that

$$|P_h f(x) - f'(x)| = \frac{2}{h^2}\left|\int_0^h \int_x^{x+t} f''(s)(x+t-s)\,ds\,dt\right| \leq \frac{B}{3}|h|.$$

Also,

$$|P_h f(x) - P_h f^\epsilon(x)| = \frac{2}{h^2}\left|\int_0^h f(x+t) - f^\epsilon(x+t) + f(x) - f^\epsilon(x)\,dt\right| \le \frac{4\epsilon}{|h|},$$

and the result follows. □

We see from this lemma that a choice of the parameter h in the Popoff approximation, in terms of the data error level ϵ, of the form $h = h(\epsilon)$, where $|h(\epsilon)| = const \times \sqrt{\epsilon}$ results in:

$$|P_h f^\epsilon(x) - f'(x)| = O(\sqrt{\epsilon}\,).$$

It is natural to ask if this order of approximation can be improved. The next Proposition, which may be termed a saturation result with respect to data error level, shows that real improvement is impossible, except in a trivial instance. This trivial case occurs when f is a linear function, for in this case, $P_h f(x) = f'(x)$ and hence

$$|P_h f^\epsilon(x) - f'(x)| = |P_h f^\epsilon(x) - P_h f(x)| \le \frac{4\epsilon}{|h|}.$$

Therefore, a choice of the parameter of the form $|h(\epsilon)| = const \times \epsilon^\nu$, with $0 < \nu < 1$, results in a rate

$$|P_h f^\epsilon(x) - f'(x)| = O(\epsilon^{1-\nu}),$$

which is arbitrarily close to the optimal order $O(\epsilon)$ for $\nu > 0$ and sufficiently small.

The next result shows that under suitable conditions the saturation order of the Popoff operator is $\circ(\sqrt{\epsilon})$ with the linear functions comprising the associated saturation class.

Theorem 2. *Suppose that f has a continuous bounded second derivative on I and that for some choice $h(\epsilon) \to 0$ as $\epsilon \to 0$,*

$$|P_h f^\epsilon(x) - f'(x)| = \circ(\sqrt{\epsilon}\,)$$

for each $x \in I$ and all bounded integrable perturbations f^ϵ satisfying $|f(t) - f^\epsilon(t)| \le \epsilon$ for all $t \in I$. Then f is a linear function on I.

Proof. For any given $x \in I$, we have by (3),

$$P_h f(x) = f'(x) + \frac{2}{h^2}\int_0^h \int_x^{x+t} f''(s)(x+t-s)\,ds\,dt.$$

Suppose that $f''(x) > 0$. Then for some positive η, $f''(s) > \eta$, for all $s \in (x-h, x+h)$, where h is positive and sufficiently small. We then have

$$P_h f(x) - f'(x) > \frac{2\eta}{h^2}\int_0^h \int_x^{x+t} (x+t-s)\,ds\,dt = \frac{\eta}{3}h.$$

Now let $\nu(t) = H(t-x)$, where H is the Heaviside function:

$$H(s) = \begin{cases} 0, & s < 0, \\ \frac{1}{2}, & s = 0, \\ 1, & s > 0. \end{cases}$$

Then

$$P_h \nu(x) = \frac{2}{h^2}\int_0^h \nu(x+t) - \nu(x)\,dt = \frac{2}{h^2}\int_0^h H(t) - \frac{1}{2}\,dt = \frac{1}{h}.$$

Therefore, if $f^\epsilon = f + \epsilon\nu$, then $|f(t) - f^\epsilon(t)| \le \epsilon$ for all $t \in I$ and

$$P_h f^\epsilon(x) - f'(x) = P_h f(x) - f'(x) + \epsilon P_h \nu(x) > \frac{\eta}{3}h + \frac{\epsilon}{h}.$$

But by assumption, $P_h f^\epsilon(x) - f'(x) = \circ(\sqrt{\epsilon}\,)$ as $\epsilon \to 0$, and hence we find that

$$\frac{\eta}{3}\frac{h(\epsilon)}{\sqrt{\epsilon}} + \frac{\sqrt{\epsilon}}{h(\epsilon)} \to 0$$

as $\epsilon \to 0$, which is impossible. Therefore $f''(x)$ is not positive. In the same way, one finds that $f''(x)$ is not negative and hence for each $x \in I$ it follows that $f''(x) = 0$. Therefore f is linear on I. $\square$

A similar analysis is carried out for the Lanczos generalized derivative in [1]. Kindred results are suggested for other approximations. For example, if

$$C_h f(x) = \frac{1}{h^2}\int_0^h f(x+t) - f(x-t)\ dt,$$

then $C_h f(x) \to f'(x)$ as $h \to 0$, if $f'(x)$ exists. Furthermore, if f has a continuous third derivative in a neighborhood of x, then

$$|C_h f^\epsilon(x) - f'(x)| = O(h^2) + O(\epsilon/|h|),$$

for any integrable f^ϵ with $|f(t) - f^\epsilon(t)| \leq \epsilon$ in a neighborhood of x. A pairing of the parameter h with the error level ϵ of the form $h(\epsilon) = const \times \epsilon^{1/3}$ then results in:

$$|C_{h(\epsilon)} f^\epsilon(x) - f'(x)| = O(\epsilon^{2/3}).$$

Arguing as above shows that this order cannot generally be improved to $\circ(\epsilon^{2/3})$ unless f is quadratic in a neighborhood of x.

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