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ON THE DIFFERENCES OF DARBOUX UPPER SEMICONTINUOUS FUNCTIONS

Abstract

In this article I prove that the difference of Darboux upper semicontinuous functions possesses the Darboux property. Moreover I show that each Darboux function which is the difference of two (bounded) upper semicontinuous functions can be written as the difference of two (bounded) Darboux upper semicontinuous functions.

In 1984, J. G. Ceder and T. L. Pearson asked, "Which functions can be written as the difference of two Darboux upper semicontinuous functions?" [5, p. 186]. As each upper semicontinuous function is the sum of two Darboux upper semicontinuous functions [7, Corollary], one could expect that the same holds for the differences of Darboux upper semicontinuous functions. However, it was pointed out in [7] that it is not true.

Since upper semicontinuous functions are Baire one, their difference is Baire class one, also. In 1921, W. Sierpiński constructed a bounded Baire one function which cannot be written as the difference of two upper semicontinuous functions [13]. As in [6, p. 132], we will denote the class of all differences of upper semicontinuous functions by $\widehat{\mathcal{B}}_1$. An easy line of reasoning yields that the difference of two Darboux upper semicontinuous functions possesses the Darboux property(Proposition 3). So we get a conjecture that the class of all Darboux functions in $\widehat{\mathcal{B}}_1$ is the answer to the above mentioned question. This conjecture is proved as Theorem 4.

The real line is denoted by $\mathbb{R}$ and the set of positive integers by $\mathbb{N}$. The word *function* always denotes a mapping from $\mathbb{R}$ into $\mathbb{R}$. The *oscillation of a function f on a non-empty set $A \subset \mathbb{R}$* will be denoted by $\omega(f, A)$ $\big($i.e., $\omega(f,A) = \sup\{|f(x) - f(y)| : x, y \in A\}\big)$. Similarly, the *oscillation of a*

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function f *at a point* $x \in \mathbb{R}$ will be denoted by $\omega(f,x)$ $\big($ i.e., $\omega(f,x) = \lim_{r\to 0^+} \omega\big(f,[x-r,x+r]\big)\big)$.

There are many conditions which are equivalent to the Darboux property of a Baire one function [1, Theorem 1.1, p. 9]. We will use two of them. The first is due to H. Sen & J. L. Massera, and the other one to I. Maximoff.

Theorem 1 *For a Baire one function* f *the following are equivalent:*

1) f *is Darboux*

2) *for each* $x \in \mathbb{R}$

$$\max\Big\{\liminf_{t\to x^-} f(t), \liminf_{t\to x^+} f(t)\Big\} \le f(x) \le \min\Big\{\limsup_{t\to x^-} f(t), \limsup_{t\to x^+} f(t)\Big\}$$

3) *for each* $x \in \mathbb{R}$ *there is a perfect set* P *such that* x *is a point of bilateral accumulation of* P *and* $f|P$ *is continuous at* x.

Proposition 2 *There is a bounded Darboux Baire one function which does not belong to* $\widehat{\mathcal{B}}_1$.

PROOF. Clearly $\widehat{\mathcal{B}}_1$ is closed with respect to addition. Suppose that each bounded Darboux Baire one function is in $\widehat{\mathcal{B}}_1$. Using that each bounded Baire one function is the sum of two Darboux Baire one functions [8, Corollary 9] $\big($see also [2, Theorem B]$\big)$, we get that each bounded Baire one function is in $\widehat{\mathcal{B}}_1$, contrary to the Sierpiński's result [13]. $\square$

Remark 1 *By Maximoff's theorem* [12] *and the above proposition, we get that there is a bounded Lebesgue function which does not belong to* $\widehat{\mathcal{B}}_1$.

Proposition 3 *Let* f_1 *and* f_2 *be Darboux upper semicontinuous functions. Then* $f_1 - f_2$ *has the Darboux property.*

PROOF. Clearly $f_1 - f_2$ is in Baire class one. To prove it is Darboux, we will use the Sen & Massera's condition. Let $x \in \mathbb{R}$. Since f_2 is Darboux, there is a sequence $x_n \nearrow x$ such that $f_2(x_n) \to f_2(x)$. So

$$\begin{aligned}\liminf_{t\to x^-}(f_1-f_2)(t) &\le \liminf_{n\to\infty}(f_1-f_2)(x_n)\\ &= \liminf_{n\to\infty} f_1(x_n) - \lim_{n\to\infty} f_2(x_n) \le (f_1-f_2)(x).\end{aligned}$$

Similarly we can prove the other necessary inequalities. $\square$

Theorem 4 *Let* f_1 *and* f_2 *be upper semicontinuous functions such that* $f_1 - f_2$ *is Darboux. There is a function* α *such that* $f_1+\alpha$ *and* $f_2+\alpha$ *are Darboux and upper semicontinuous, and* $0 \le \alpha \le \sup\big\{\omega(f_i,x) : x \in \mathbb{R},\ i \in \{1,2\}\big\}$ *on* $\mathbb{R}$.

PROOF. For $i \in \{1,2\}$ put

$$A_i = \big\{x \in \mathbb{R} : \limsup_{t\to x^-} f_i(t) < f_i(x) \text{ or } \limsup_{t\to x^+} f_i(t) < f_i(x)\big\}.$$

By Lemma 4 of [4], the cardinality of $A_1 \cup A_2$ is less than that of continuum. But $A_1 \cup A_2$ is a Borel set, so it is countable. (See also Lemma A1 of [11].) For $i \in \{1,2\}$ arrange the elements of A_i in a sequence, $\{a_{in} : n < r_i\}$, where r_i is a finite number or the infinity.

For each i and n use Maximoff's condition and find a perfect set P_{in} such that a_{in} is a point of bilateral accumulation of P_{in} and $(f_1 - f_2)|P_{in}$ is continuous at a_{in}; clearly we may assume that for each $x \in P_{in}$

$$\max\big\{|(f_1 - f_2)(x) - (f_1 - f_2)(a_{in})|, f_1(x) - f_1(a_{in}), f_2(x) - f_2(a_{in})\big\} \le 1/n.$$

Then for each i, each n and each $k > n$ the sets $K_{ink1} = P_{in} \cap (a_{in} - 1/k, a_{in})$ and $K_{ink2} = P_{in} \cap (a_{in}, a_{in} + 1/k)$ are F_σ and have the cardinality of continuum. So by Lemma 2 of [14], we can find a family $\big\{Q_{inkj} : i,j \in \{1,2\},\, n < r_i,\, k > n\big\}$, consisting of pairwise disjoint non-empty nowhere dense perfect sets, $Q_{inkj} \subset K_{inkj} \setminus (A_1 \cup A_2)$ for each i, n, k, and j. Using that Borel measurable functions are also Marczewski measurable, we may assume that $f_1|Q_{inkj}$ and $f_2|Q_{inkj}$ are continuous (also see [9]). We may also assume that $\omega(f_s, Q_{inkj}) \le 1/k$ for $s \in \{1,2\}$. Let $S = \sup\big\{\omega(f_i, x) : x \in \mathbb{R},\, i \in \{1,2\}\big\}$. For each i, n, k, and j set $w_{inkj} = \min\big\{\max\big\{f_i(a_{in}) - \sup f_i(Q_{inkj}), 0\big\}, S\big\}$, and construct a Darboux upper semicontinuous function α_{inkj} such that $\alpha_{inkj} = 0$ outside of Q_{inkj}, $\sup\big\{\alpha_{inkj}(x) : x \in \mathbb{R}\big\} = w_{inkj}$, and $0 \le \alpha_{inkj} \le w_{inkj}$ on $\mathbb{R}$ [3].

Let $\alpha = \sum_{i=1}^{2} \sum_{n<r_i} \sum_{k>n} \sum_{j=1}^{2} \alpha_{inkj}$. It is obvious that $0 \le \alpha \le S$ on $\mathbb{R}$. To complete the proof fix an $s \in \{1,2\}$ and an $x \in \mathbb{R}$.

I. First we will prove that $f_s + \alpha$ is upper semicontinuous at x. Take a sequence $x_m \to x$ with $\lim\limits_{m\to\infty} (f_s+\alpha)(x_m) = \limsup_{t\to x}(f_s+\alpha)(t)$. We consider several cases.

If $\alpha(x_m) = 0$ for infinitely many m, then

$$\lim_{m\to\infty} (f_s + \alpha)(x_m) = \lim_{m\to\infty} f_s(x_m) \le f_s(x) \le (f_s + \alpha)(x).$$

So we may assume that $x_m \in Q_{i_m n_m k_m j_m}$ for each m. Since $i_m, j_m \in \{1,2\}$, we may assume there are $i, j \in \{1,2\}$ such that $i_m = i$ and $j_m = j$ for each m. If $n_m + k_m$ does not tend to infinity, then there are n and k such that $x_m \in Q_{inkj}$ for infinitely many m. Hence $x \in Q_{inkj}$, and since $f_s|Q_{inkj}$ is continuous and α_{inkj} is upper semicontinuous; so

$$\lim_{m\to\infty} (f_s + \alpha)(x_m) = \lim_{m\to\infty} f_s(x_m) + \lim_{m\to\infty} \alpha(x_m) \le (f_s + \alpha)(x).$$

Now let $n_m + k_m \to \infty$. If n_m does not tend to infinity, then $k_m \to \infty$ and there is an n such that $x_m \in Q_{ink_mj}$ for infinitely many m. Hence $x = a_{in}$, and

- $i = s$ implies

$$\begin{aligned}\lim_{m\to\infty}(f_s+\alpha)(x_m) &\le \limsup_{m\to\infty}\big(f_s(x_m)+w_{snk_mj}\big)\\ &\le \limsup_{m\to\infty}\ \max\big\{f_s(a_{sn}), f_s(x_m)\big\} = f_s(x) = (f_s+\alpha)(x).\end{aligned}$$

- $i = 3 - s$ implies

$$\begin{aligned}\lim_{m\to\infty}(f_s+\alpha)(x_m) &\le \limsup_{m\to\infty}(f_s - f_{3-s})(x_m) + \limsup_{m\to\infty}(f_{3-s}+\alpha)(x_m)\\ &\le (f_s - f_{3-s})(a_{in}) + (f_{3-s}+\alpha)(a_{in}) = (f_s+\alpha)(x).\end{aligned}$$

(We used the previous case and the fact that $(f_s - f_{3-s})|P_{in}$ is continuous at a_{in}.)

Finally let $n_m \to \infty$. Then $|x_m - a_{in_m}| \le 1/n_m \to 0$, so $a_{in_m} \to x$ and

- $i = s$ implies

$$\begin{aligned}&\lim_{m\to\infty}(f_s+\alpha)(x_m) \le \limsup_{m\to\infty}\big(f_s(x_m)+w_{sn_mk_mj}\big)\\ &\le \limsup_{m\to\infty}\ \max\big\{f_s(a_{sn_m}), f_s(x_m)\big\} \le f_s(x) \le (f_s+\alpha)(x).\end{aligned}$$

- $i = 3 - s$ implies

$$\begin{aligned}&\limsup_{m\to\infty}\big((f_{3-s}+\alpha)(x_m) - f_{3-s}(a_{in_m})\big)\\ &\quad\le \limsup_{m\to\infty}\ \max\big\{f_{3-s}(x_m) - \sup f_{3-s}(Q_{in_mk_mj}), f_{3-s}(x_m) - f_{3-s}(a_{in_m})\big\}\\ &\quad\le \limsup_{m\to\infty} 1/n_m = 0\end{aligned}$$

and

$$\begin{aligned}\lim_{m\to\infty}(f_s+\alpha)(x_m) &\le \limsup_{m\to\infty}\big((f_s - f_{3-s})(x_m) - (f_s - f_{3-s})(a_{in_m})\big)\\ &\quad + \limsup_{m\to\infty}\big((f_{3-s}+\alpha)(x_m) - f_{3-s}(a_{in_m})\big)\\ &\quad + \limsup_{m\to\infty} f_s(a_{in_m})\\ &\le \limsup_{m\to\infty} 1/n_m + 0 + f_s(x) = f_s(x) \le (f_s+\alpha)(x).\end{aligned}$$

II. To prove that $f_s + \alpha$ is Darboux we will use the Sen & Massera's condition. We consider several cases.

• If $x \notin A_s$ and $\alpha(x) = 0$, then

$$\limsup_{t \to x^-}(f_s + \alpha)(t) \geq \limsup_{t \to x^-} f_s(t) = f_s(x) = (f_s + \alpha)(x);$$

similarly $\limsup_{t\to x^+}(f_s + \alpha)(t) = (f_s + \alpha)(x)$.

• If $x \notin A_s$ and $\alpha(x) \neq 0$, then $x \in Q_{inkj}$ for some i, n, k, and j. Since α_{inkj} is Darboux and upper semicontinuous, and $\alpha_{inkj} = 0$ on $\mathbb{R} \setminus Q_{inkj}$, so x is a point of bilateral accumulation of Q_{inkj}, and there are $z_1, z_2, \dots \in Q_{inkj}$ with $z_m \nearrow x$ and $\lim_{m\to\infty} \alpha_{inkj}(z_m) = \alpha_{inkj}(x)$. Using that $f_s|Q_{inkj}$ is continuous we get $\limsup_{t\to x^-}(f_s+\alpha)(t) \geq \lim_{m\to\infty}(f_s+\alpha)(z_m) = (f_s+\alpha)(x)$; similarly $\limsup_{t\to x^+}(f_s + \alpha)(t) = (f_s + \alpha)(x)$.

• If $x \in A_s$, then $x = a_{sn}$ for some $n < r_s$. For $m > n$ let $z_m \in Q_{snm1}$ be such that $\alpha_{snm1}(z_m) \geq w_{snm1} - 1/m$. We have

$$\begin{aligned}
&\limsup_{t\to x^-}(f_s + \alpha)(t) \geq \lim_{m\to\infty}(f_s + \alpha)(z_m) \\
&\geq \lim_{m\to\infty} \min\bigl\{f_s(a_{sn}) - \omega(f_s, Q_{snm1}), f_s(z_m) + S\bigr\} - 1/m \\
&\geq \min\bigl\{f_s(a_{sn}), \liminf_{t\to x} f_s(t) + S\bigr\} = f_s(a_{sn}) = (f_s + \alpha)(x);
\end{aligned}$$

similarly $\limsup_{t\to x^+}(f_s + \alpha)(t) = (f_s + \alpha)(x)$. □

Remark 2 *Theorem* 4 *implies, in particular, that each function which is the difference of two bounded upper semicontinuous functions can be written as the difference of two bounded Darboux upper semicontinuous functions. It should be however emphasized that there is an ambivalent set whose characteristic function* <u>*cannot*</u> *be expressed that way.* (*Recall that a Baire one function whose range is discrete belongs to* $\widehat{\mathcal{B}}_1$*; see, e.g.,* [6, 2.D.15].) *Indeed, denote by* $\widehat{\mathfrak{B}}_1$ *the class of all differences of bounded upper semicontinuous functions and suppose that each characteristic function of an ambivalent set belongs to* $\widehat{\mathfrak{B}}_1$*. It is clear that* $\widehat{\mathfrak{B}}_1$ *is closed with respect to addition. We get that each Baire one function with finite range is in* $\widehat{\mathfrak{B}}_1$*. Hence* $\widehat{\mathfrak{B}}_1$ *is dense in the class of all bounded Baire one functions. But this contradicts Corollary* 3.22 *of* [10].

Remark 3 *Using a slightly more complicated method it is possible to prove that the function* α *in Theorem* 4 *can be chosen so that it is continuous whenever both* f_1 *and* f_2 *are. I do not know, however, whether each a.e. continuous function in* $\widehat{\mathcal{B}}_1$ *is the difference of two a.e. continuous upper semicontinuous functions.*

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