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TOWERS AND PERMITTED TRIGONOMETRIC THIN SETS

Abstract

In [3] we introduced the notion of perfect measure zero sets and proved that every perfect measure zero set is permitted for any of the four families of trigonometric thin sets $\mathcal{N}$, $\mathcal{A}$, $\mathcal{N}_0$, and $p\mathcal{D}$. Now we prove that the unions of less than $\mathfrak{t}$ perfect measure zero sets are permitted for the mentioned families. This strengthens a result of T. Bartoszyński and M. Scheepers [1] saying that every set of cardinality less than $\mathfrak{t}$ is $\mathcal{N}$-permitted.

1 Introduction

Let $\mathcal{F}$ be a family of sets of reals. Let $A, B \in \mathcal{F}$. We say that a set A is permitted for $\mathcal{F}$ if $A \cup B \in \mathcal{F}$ for every $B \in \mathcal{F}$. Let A be a set of reals. Then A is a pD-set (pseudo Dirichlet set) if there is an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that the sequence $\{\sin n_k \pi x\}_{k=0}^{\infty}$ converges quasinormally on A; i.e. there is a sequence of positive reals $\{\varepsilon_k\}_{k=0}^{\infty}$ converging to 0 such that $(\forall x \in A)(\forall^{\infty} k)\,|\sin n_k \pi x| < \varepsilon_k$. A is an N_0-set if there is an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty} |\sin n_k \pi x| < \infty$ for $x \in A$. A is an A-set if there is an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that $\{\sin n_k \pi x\}_{k=0}^{\infty}$ converges to 0 for $x \in A$. A is an N-set if there is a sequence of non-negative reals $\{\rho_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} \rho_n = \infty$ and the series $\sum_{n=0}^{\infty} |\rho_n \sin n\pi x|$ converges for $x \in A$. The families of all pD-sets, N_0-sets, A-sets, and N-sets are denoted by $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{A}$, and $\mathcal{N}$, respectively.

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A set A has perfect measure zero if for every sequence of positive reals $\{\varepsilon_n\}_{n=1}^{\infty}$ there is an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ and a sequence of finite families of intervals $\{\mathcal{I}_n\}_{n=1}^{\infty}$ such that $|\mathcal{I}_n| \le n$, $|I| < \varepsilon_n$ for every $I \in \mathcal{I}_n$ and $A \subseteq \bigcup_m \bigcap_{k \ge m} \bigcup \mathcal{I}_{n_k}$.

In [3] it was proved that every perfect measure zero set is permitted for any of the families $\mathcal{N}$, $\mathcal{A}$, $\mathcal{N}_0$, and $p\mathcal{D}$; every γ-set has perfect measure zero; every perfect measure zero set has strong measure zero; the subgroup of $\mathbb{R},+$ generated by a set having perfect measure zero has perfect measure zero; and every set of cardinality less than the additivity of Lebesgue measure has perfect measure zero.

A result of [1] says that every set of cardinality less than $\mathfrak{t}$ is permitted for $\mathcal{N}$. The cardinals $\mathfrak{t}$ and the additivity of Lebesgue measure are two the best known lower bounds for the minimal size of a set not being permitted for $\mathcal{N}$. It is worth mentioning that these two bounds are mutually independent. (See a discussion in [3].) Some more bounds of cardinal invariants of other families of trigonometric thin sets can be found in [2] and [1]. Using some ideas of [1] we prove the following result.

Main Theorem 1.1 *Let $\mathcal{F}$ be any of the families $\mathcal{N}$, $\mathcal{A}$, $\mathcal{N}_0$ and $p\mathcal{D}$. The unions of less than $\mathfrak{t}$ sets having perfect measure zero are permitted for $\mathcal{F}$.*

Let us recall that $\mathfrak{t}$ is the minimal size of a tower of subsets of ω, $\mathfrak{h}$ is the distributivity number of the Boolean algebra $\mathcal{P}(\omega)/\mathit{fin}$, and $\mathfrak{b}$ is the minimal size of an unbounded family of functions under the eventual dominance. It is well known that $\mathfrak{t} \le \mathfrak{h} \le \mathfrak{b}$.

2 The Proof of the Case $\mathcal{N}$

Let E be an N-set, i.e. there is a sequence $\{\rho_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \rho_n = \infty$ and $E = \{x \in \mathbb{R} : \sum_{n=1}^{\infty} \rho_n |\sin n\pi x| < \infty\}$. Let us denote $s_n = \sum_{k=1}^{n} \rho_k$. There is a surjective monotone function $\iota : \omega \to \omega \setminus \{0\}$ such that

$$\sum_{n=1}^{\infty} \frac{\rho_n}{s_n^{1+1/\iota^2(n)}} < \infty.$$

Let $\rho'_n = \rho_n / s_n$ and $\delta_n = s_n^{-1/\iota^2(n)}$. Then $\sum_{n=1}^{\infty} \rho'_n = \infty$ and $\sum_{n=1}^{\infty} \rho'_n \delta_n < \infty$. There is an increasing function $f \in {}^{\omega}\omega$ such that $\sum_{n=f(k)}^{f(k+1)-1} \rho'_n \ge 1$ for all $k \in \omega$. Let us denote $J_k = [f(k), f(k+1))$ and

$$\varepsilon_m = \min\left\{ \frac{\delta_n}{n s_n} : (\exists\, k)\, \iota(k) = m \;\&\; n \in J_k \right\}. \tag{2.1}$$

Lemma 2.1 *Suppose $Z \subseteq \omega$ is a finite set, $0 < \delta \leq 1$ and $x_1, \dots, x_n \in \mathbb{R}$. There is $Z' \subseteq Z$ such that $|Z'| \geq \delta^n |Z|$ and*

$$(\forall\, i, j \in Z')(\forall\, l = 1, \dots, n)\, |\sin(i-j)\pi x_l| < 2\pi\delta.$$

PROOF. First let $n = 1$. Find m such that $1/(2m) < \delta \leq 1/m$ and divide the interval $[0,1]$ into m subintervals of length $1/m$. There is a set $Z' \subseteq Z$ such that $|Z'| \geq |Z|/m \geq \delta|Z|$ and for all $i \in Z'$, $\{ix_1\}$ are in the same subinterval. (Let us recall that $\{y\}$ denotes the fractional part of a real y.) Hence $|\sin(i-j)\pi x_1| \leq \pi/m < 2\pi\delta$ for all $i, j \in Z'$. Now in case $n > 1$ apply the previous result n times. □

Lemma 2.2 *There is a system of functions $f_\alpha \in {}^\omega\mathbb{R}$ for $\alpha < \mathfrak{b}$ such that*

(1) $(\forall^\infty n)\, 0 \leq f_\alpha(n) \leq s_n \delta_n^{\iota(n)}$,

(2) $(\forall\, k \in \omega)\, \lim_{n\to\infty} f_\alpha(n)\delta_n^{k\iota(n)} = \infty$,

(3) *If $\beta < \alpha < \mathfrak{b}$, then $(\forall^\infty n)\, f_\alpha(n) \leq f_\beta(n)\delta_n^{\iota(n)}$.*

PROOF. Let us set $f_0(n) = s_n \delta_n^{\iota(n)}$ and similarly in the non-limit steps let us set $f_{\alpha+1}(n) = f_\alpha(n)\delta_n^{\iota(n)}$. Then we get

$$\lim_{n\to\infty} f_0(n)\delta_n^{k\iota(n)} = \lim_{n\to\infty} s_n^{1-(k+1)/\iota(n)} \geq \lim_{n\to\infty} s_n^{1/2} = \infty$$

and condition (2) can be easily verified also for the non-limit step. Let $\alpha < \mathfrak{b}$ be limit. For each $\beta < \alpha$ there is $g_\beta \in {}^\omega\omega$ with $\lim_{n\to\infty} g_\beta(n) = \infty$ such that $\lim_{n\to\infty} f_\beta(n)\delta_n^{g_\beta(n)\iota(n)} = \infty$ (by (2) in the induction hypothesis). Let $g \in {}^\omega\omega$ be such that $\lim_{n\to\infty} g(n) = \infty$ and $(\forall\, \beta < \alpha)(\forall^\infty n)\, g(n) \leq g_\beta(n)$. Then for any $\beta < \alpha$, $\lim_{n\to\infty} f_\beta(n)\delta_n^{g(n)\iota(n)} = \infty$. We can find $h \in {}^\omega\omega$ with $\lim_{n\to\infty} h(n) = \infty$ such that $(\forall\, \beta < \alpha)(\forall^\infty n)\, h(n) \leq f_\beta(n)\delta_n^{g(n)\iota(n)}$. Let us set $f_\alpha(n) = h(n)\delta_n^{-g(n)\iota(n)}$. Then

$$\lim_{n\to\infty} f_\alpha(n)\delta_n^{k\iota(n)} = \lim_{n\to\infty} h(n)\delta_n^{(k-g(n))\iota(n)} \geq \lim_{n\to\infty} h(n) = \infty$$

for all $k \in \omega$ and

$$(\forall^\infty n)\, f_\alpha(n) = h(n)\delta_n^{-g(n)\iota(n)} \leq f_{\beta+1}(n) \leq f_\beta(n)\delta_n^{\iota(n)}$$

for all $\beta < \alpha$. □

Lemma 2.3 *Let $\kappa < \mathfrak{h}$, let A_α, $\alpha < \kappa$, be perfectly measure zero sets, let a be an infinite subset of ω and let $\{\varepsilon_n\}_{n=1}^\infty$ be a given sequence of positive* *reals. There is a sequence of integers $\{n_k\}_{k=0}^{\infty}$ elements of a and a system of sequences of finite systems of intervals $\{\mathcal{I}_n^{\alpha}\}_{n=1}^{\infty}$, $\alpha<\kappa$, such that $|\mathcal{I}_n^{\alpha}|\le n$, $(\forall\, I\in\mathcal{I}_n^{\alpha})\,|I|<\varepsilon_n$ and $A_\alpha\subseteq\bigcup_m\bigcap_{k\ge m}\bigcup\mathcal{I}_{n_k}^{\alpha}$ for each α. (The sequence $\{n_k\}_{k=0}^{\infty}$ is independent of α.)*

Proof. Let $\mathfrak{I}$ be the family of all infinite sequences of finite families of intervals $\{\mathcal{I}_n\}_{n=1}^{\infty}$ such that $|\mathcal{I}_n|\le n$ and $(\forall\, I\in\mathcal{I}_n)\,|I|<\varepsilon_n$. The sets

$$D_\alpha=\left\{b\in[\omega]^\omega:(\exists\,\{\mathcal{I}_n\}_{n=1}^{\infty}\in\mathfrak{I})\,A_\alpha\subseteq\bigcup_m\bigcap_{n\in b\setminus m}\bigcup\mathcal{I}_n\right\}$$

are open dense in $[\omega]^\omega$. (See the proof of Theorem 1.2 (iii) of [3].) Choose $b\subseteq a$, $b\in\bigcap_{\alpha<\kappa}D_\alpha$ and let $\{n_k\}_{k=0}^{\infty}$ be the increasing enumeration of the set b. □

The last lemma enables us to choose n_k uniformly for a given family of perfect measure zero sets and so for the case of N-sets in the Main Theorem it is enough to prove the following proposition. Notice that in the proposition $\iota(n_k)$ can be replaced by n_k. (But it is easier handling terms $\iota(n_k)$ in the proof.)

Proposition 2.4 *Let $\{n_k\}_{k=0}^{\infty}$ be a given increasing sequence of integers and $\{\varepsilon_n\}_{n=1}^{\infty}$ be the sequence of positive reals defined by (2.1) for a given N-set E. Whenever $\nu<\mathfrak{t}$ and $\{\mathcal{I}_n^{\alpha}\}_{n=1}^{\infty}$, $\alpha\le\nu$, are sequences of finite families of intervals such that $|\mathcal{I}_n|\le n$ and $(\forall\, I\in\mathcal{I}_n)\,|I|\le\varepsilon_n$, then $E\cup\bigcup_{\alpha\le\nu}\bigcup_m\bigcap_{k\ge m}\bigcup\mathcal{I}_{\iota(n_k)}^{\alpha}$ is an N-set.*

Proof. Let $\mathcal{I}_n^{\alpha}=\{[a_{n,l}^{\alpha},a_{n,l}^{\alpha}+\varepsilon_n]\}_{l=1}^{n}$ for $n\ge 1$ and $\alpha\le\nu$ and let us set $P_\alpha=\bigcup_m\bigcap_{k\ge m}\bigcup\mathcal{I}_{\iota(n_k)}^{\alpha}$. By induction we build $\{\varphi_\alpha:\alpha\le\nu\}$ and $\{a_\alpha:\alpha\le\nu\}\subseteq[\omega]^\omega$ such that

(1) $\varphi_\alpha:\bigcup_{k\in a_\alpha}J_{n_k}\to[\omega]^{<\omega}$,

(2) $(\forall\, k\in a_\alpha)(\forall\, n\in J_{n_k})(\forall\, i,j\in\varphi_\alpha(n))(\forall\, l=1,\dots,\iota(n_k))$
$|\sin(i-j)n\pi a_{\iota(n_k),l}^{\alpha}|<2\pi\delta_n$,

(3) $(\forall\, n\in\operatorname{dom}(\varphi_\alpha))\ \max\varphi_\alpha(n)\le s_n$,

(4) $(\forall^\infty n\in\operatorname{dom}(\varphi_\alpha))\,|\varphi_\alpha(n)|\ge f_\alpha(n)$,

(5) If $\beta\le\alpha$, then $a_\alpha\subseteq^* a_\beta$ and $(\forall^\infty n\in\operatorname{dom}(\varphi_\alpha))\,\varphi_\alpha(n)\subseteq\varphi_\beta(n)$.

For $\alpha=0$ set $a_0=\omega$ and for $n\in J_{n_k}$. Using Lemma 2.1 find $\varphi_0(n)\subseteq\{i:i\le s_n\}$ so that $|\varphi_0(n)|\ge s_n\delta_n^{\iota(n_k)}\ge s_n\delta_n^{\iota(n)}\ge f_0(n)$ and

$$(\forall\, i,j\in\varphi_0(n))\,|\sin(i-j)n\pi a_{\iota(n_k),l}^{0}|<2\pi\delta_n$$

(whenever $\delta_n \leq 1$). Similarly using Lemma 2.1 we can find $\varphi_{\alpha+1}(n) \subseteq \varphi_\alpha(n)$ so that (2) and (4) hold and we can set $a_{\alpha+1} = a_\alpha$. Let $\alpha \leq \nu$ be limit. For $k \in \omega$ let

$$\begin{aligned} U_k = \big\{ &F \in {}^{J_{n_k}}([\omega]^{<\omega}) : (\forall\, n \in \operatorname{dom}(F))\ \max F(n) \leq s_n \\ &\ \&\ |F(n)| \geq f_\alpha(n)\ \&\ (\forall\, i, j \in F(n))(\forall\, l = 1, \dots, \iota(n_k)) \\ &\ |\sin(i-j)\pi n a^\alpha_{\iota(n_k),l}| < 2\pi\delta_n \big\}. \end{aligned}$$

The set $U = \bigcup_{k\in\omega} U_k$ is countably infinite. For $\beta < \alpha$ let

$$X_\beta = \big\{ F \in U : (\forall\, n \in \operatorname{dom}(F))\, F(n) \subseteq \varphi_\beta(n) \big\}.$$

For all $n \in J_{n_k}$, $\iota(n) \geq \iota(n_k)$. Hence for all but finitely many k for all $n \in J_{n_k}$, $f_\alpha(n) \leq f_\beta(n)\delta_n^{\iota(n)} \leq f_\beta(n)\delta_n^{\iota(n_k)}$ and using Lemma 2.1 we easily see that X_β is infinite. Moreover, whenever $\beta < \gamma < \alpha$, then $X_\gamma \subseteq^* X_\beta$. Since $\alpha < \mathfrak{t}$, there is $X \subseteq U$ such that $X \subseteq^* X_\beta$ for all $\beta < \alpha$. Since each U_k is finite we can choose X so that $|X \cap U_k| \leq 1$ for each $k \in \omega$. Let us define $\varphi_\alpha = \bigcup X$ and $a_\alpha = \{k : X \cap U_k \neq \emptyset\}$.

Let $\varphi = \varphi_\nu$ and let $a \subseteq \omega$ be the set of all $k \in a_\nu$ such that for each $n \in J_{n_k}$ there are $i_n, j_n \in \varphi(n)$, $j_n < i_n$, and for each such n let us put $\lambda_n = i_n - j_n$; $\lambda_n \leq s_n$. The set a contains all but finitely many $k \in a_\nu$. We prove that the series

$$\sum_{k\in a} \sum_{n\in J_{n_k}} \rho'_n |\sin \lambda_n n\pi x|$$

converges for $x \in E \cup \bigcup_{\alpha\leq\nu} P_\alpha$. This finishes the proof since

$$\sum_{k\in a} \sum_{n\in J_{n_k}} \rho'_n = \infty.$$

First notice that by (2) for all but finitely many $k \in a$ and for all $n \in J_{n_k}$

$$|\sin \lambda_n n\pi a^\alpha_{\iota(n_k),l}| \leq 2\pi\delta_n \quad \text{for } l = 1, \dots, \iota(n_k)$$

and since

$$|\sin \lambda_n n\pi\varepsilon_{\iota(n_k)}| \leq \lambda_n n\varepsilon_{\iota(n_k)} \leq s_n n\varepsilon_{\iota(n_k)} \leq \delta_n$$

we obtain that

$$|\sin \lambda_n n\pi x| \leq 3\pi\delta_n \quad \text{for } x \in \bigcup \mathcal{I}^\alpha_{\iota(n_k)} \text{ and } n \in J_{n_k} \tag{2.2}$$

holds for all but finitely many $k \in a$.

Let $x \in P_\alpha$, $\alpha \le \nu$. There is m such that $x \in \bigcup \mathcal{I}^\alpha_{\iota(n_k)}$ and (2.2) holds for all $k \in a \setminus m$. Then

$$\sum_{k \in a\setminus m} \sum_{n \in J_{n_k}} \rho'_n |\sin \lambda_n n\pi x| \le \sum_{k \in a\setminus m} \sum_{n \in J_{n_k}} 3\rho'_n \delta_n < \infty.$$

If $x \in E$, then

$$\sum_{k \in a} \sum_{n \in J_{n_k}} \rho'_n |\sin \lambda_n n\pi x| \le \sum_{k \in a} \sum_{n \in J_{n_k}} \rho'_n \lambda_n |\sin n\pi x| \le \sum_{n=1}^{\infty} \rho_n |\sin n\pi x| < \infty.$$

□

3 The Proof of the Cases $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{A}$

All these proofs are the same and we will outline the case of N_0-sets. Let $E = \{x : \sum_{k=0}^{\infty} |\sin m_k \pi x| < \infty\}$ with $\{m_k\}_{k=0}^{\infty}$ strictly increasing. Let $\delta_n = 1/n^2$ and $\varepsilon = \delta_n / m_k$ for $k = \sum_{j=1}^{n} (1/\delta_j)^{j^2}$. The particular steps of the proof are analogous to the case of N-sets.

Lemma 3.1 *There is a system of functions $f_\alpha \in {}^\omega\mathbb{R}$ for $\alpha < \mathfrak{b}$ such that*

(1) $(\forall^\infty n)\, 0 \le f_\alpha(n) \le (1/\delta_n)^{n^2} \delta_n^n$,

(2) $(\forall\, k \in \omega)\, \lim_{n\to\infty} f_\alpha(n) \delta_n^{kn} = \infty$,

(3) *If $\beta < \alpha < \mathfrak{b}$, then $(\forall^\infty n)\, f_\alpha(n) \le f_\beta(n) \delta_n^n$.*

PROOF. Same as the proof of Lemma 2.2. □

Again it is enough to prove the following proposition. (Notice that the same proposition holds for pD-sets and A-sets. It is enough to consider either quasinormal or pointwise convergence in the definition of the set E instead of absolute convergence of a series.)

Proposition 3.2 *Let $\{n_k\}_{k=0}^{\infty}$ be a given increasing sequence of integers and $\nu < \mathfrak{t}$. If $\{\mathcal{I}_n^\alpha\}_{n=1}^{\infty}$, $\alpha \le \nu$, are sequences of finite families of intervals such that $|\mathcal{I}_n| \le n$ and $(\forall\, I \in \mathcal{I}_n)\, |I| \le \varepsilon_n$, then $E \cup \bigcup_{\alpha \le \nu} \bigcup_m \bigcap_{k \ge m} \bigcup \mathcal{I}^\alpha_{n_k}$ is an N_0-set.*

PROOF. Let $\mathcal{I}_n^\alpha = \{[a^\alpha_{n,l}, a^\alpha_{n,l} + \varepsilon_n]\}_{l=1}^{n}$ and put

$$P_\alpha = \bigcup_m \bigcap_{k \ge m} \bigcup \mathcal{I}^\alpha_{n_k}.$$

In the same way as in the proof of Proposition 2.4 we build $\{\varphi_\alpha : \alpha \le \nu\}$ and $\{a_\alpha : \alpha \le \nu\} \subseteq [\omega]^\omega$ by induction so that

(1) $\varphi_\alpha : \{n_k : k \in a_\alpha\} \to [\omega]^{<\omega}$,

(2) $(\forall n \in \mathrm{dom}(\varphi_\alpha))(\forall i, j \in \varphi_\alpha(n))(\forall l = 1, \dots, n)\, |\sin(m_i - m_j)\pi a^\alpha_{n,l}| < 2\pi\delta_n$,

(3) $(\forall n \in \mathrm{dom}(\varphi_\alpha))(\forall i \in \varphi_\alpha(n))\ \sum_{j=1}^{n-1}(1/\delta_j)^{j^2} \le i < \sum_{j=1}^{n}(1/\delta_j)^{j^2}$,

(4) $(\forall^\infty n \in \mathrm{dom}(\varphi_\alpha))\, |\varphi_\alpha(n)| \ge f_\alpha(n)$,

(5) If $\beta \le \alpha$, then $a_\alpha \subseteq^* a_\beta$ and $(\forall^\infty n \in \mathrm{dom}(\varphi_\alpha))\, \varphi_\alpha(n) \subseteq \varphi_\beta(n)$.

For $\alpha \le \nu$ limit let U_n be the set of all pairs (n, F) such that $F \in [\omega]^{<\omega}$, $|F| \ge f_\alpha(n)$, $(\forall i \in F)\ \sum_{j=1}^{n-1}(1/\delta_j)^{j^2} \le j < \sum_{j=1}^{n}(1/\delta_j)^{j^2}$, and $(\forall i, j \in F)(\forall l = 1, \dots, n)\, |\sin(m_i - m_j)\pi a^\alpha_{n,l}| < 2\pi\delta_n$. The set $U = \bigcup_{n\in\omega} U_n$ is countably infinite and for $\beta < \alpha$ the sets $X_\beta = \{(n, F) : F \subseteq \varphi_\beta(n)\}$ are infinite and decreasing with respect to $\subseteq^*$. There is an infinite set $X \subseteq U$ such that $X \subseteq^* X_\beta$ for all $\beta < \alpha$ and $|X \cap U_n| \le 1$ for each n. Let us set $\varphi_\alpha = X$ and $a_\alpha = \{k : U_{n_k} \cap X \ne \emptyset\}$.

Let $\varphi = \varphi_\nu$ and let a be the set of all $k \in a_\nu$ for which there are two different members $i_k, j_k \in \varphi(n_k)$, $j_k < i_k$, and for each such k let us set $\lambda_k = m_{i_k} - m_{j_k}$. Notice that both sequences $\{i_k\}_{k=0}^\infty$ and $\{j_k\}_{k=0}^\infty$ are strictly increasing. Given $\alpha \le \nu$ for all but finitely many $k \in a$, $n_k \in \mathrm{dom}(\varphi_\alpha)$. Hence for all but finitely many $k \in a$

$$|\sin\lambda_k\pi a^\alpha_{n_k,l}| \le 2\pi\delta_{n_k} \quad \text{for } l = 1, \dots, n_k$$

and

$$|\sin\lambda_k\pi\varepsilon_{n_k}| \le (m_{i_k} - m_{j_k})\pi\varepsilon_{n_k} \le m_{i_k}\pi\varepsilon_{n_k} \le \pi\delta_{n_k}.$$

Consequently

$$|\sin\lambda_k\pi x| \le 3\pi\delta_{n_k} \quad \text{for } x \in \bigcup \mathcal{I}^\alpha_{n_k} \tag{3.1}$$

for all but finitely many $k \in a$. We prove that $\sum_{k\in a}|\sin\lambda_k\pi x|$ converges for $x \in E \cup \bigcup_{\alpha\le\nu} P_\alpha$.

Let $x \in P_\alpha$. There is m such that $x \in \bigcup \mathcal{I}^\alpha_{n_k}$ for all $k \ge m$ and such that (3.1) holds for all $k \in a\setminus m$. Then $\sum_{k\in a\setminus m}|\sin\lambda_k\pi x| \le \sum_{k\in a\setminus m} 3\pi\delta_{n_k} < \infty$. For $x \in E$, $\sum_{k\in a}|\sin\lambda_k\pi x| \le \sum_{k\in a}|\sin m_{i_k}\pi x| + \sum_{k\in a}|\sin m_{j_k}\pi x| < \infty$. □

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