Real Analysis Exchange
Vol. 23(2), 1997–1998, pp. 783–786

Artur Bartoszewicz, Technical University of Łódź, Institute of Mathematics, Al. Politechniki 11, 90-924 Łódź, Poland

ON DENSITY POINTS OF SUBSETS OF METRIC SPACE WITH RESPECT TO THE MEASURE GIVEN BY RADON-NIKODYM DERIVATIVE

We start from the well known definition of density point of measurable set with respect to the Lebesgue measure on the real line.

Definition 1. *0 is a density point of the set E if*

$$\lim_{h \to 0} \frac{\lambda(E \cap (-h,h))}{2h} = 1.$$

This is equivalent to saying that the sequence of characteristic functions of the sets $(nE) \cap (-1,1)$ tends in (Lebesgue) measure to the characteristic function of the interval $(-1,1)$ (see [1]).

But (using the Riesz theorem) we have that 0 is a density point of E with respect to the Lebesgue measure if and only if any subsequence of $\chi_{(nE \cap (-1,1))}$ contains a subsequence which converges to $\chi_{(-1,1)}$ almost everywhere.

To describe convergence almost everywhere we need only the σ-ideal of sets of Lebesgue measure zero. So it follows that the notion of a density point of any subset of real line with respect to the Lebesgue measure can be described without measure. The only one thing that we need is the σ-ideal of null sets.

Now we will assume that μ is an arbitrary measure, finite on the balls, on the Borel subsets of some metric space X. The natural generalization leads to the following definition:

Definition 2. *Let x_0 be a point of support μ. We say that x_0 is the density point of the measurable set A if*

$$\lim_{r \to 0} \frac{\mu(A \cap B_r)}{\mu(B_r)} = 1$$

where B_r denotes the ball $B(x_0, r)$.

Key Words: density point, metric space, the Radon-Nikodym measure
Mathematical Reviews subject classification: Primary 26A21; Secondary 04A15
Received by the editors November 4, 1997

A natural problem is: what can we say about density points of sets with respect to another measure ν, which has the same as μ σ- ideal of null sets.

We will show that, in general, the density point of A with respect to μ need not be the density point of A with respect to ν, but we will find some sufficient conditions for ν under which density points for μ are also density points for ν.

In our considerations we will use the notion of Radon-Nikodym derivative of measure ν with respect to an another measure μ, (i.e. the integrable function f such that for any measurable set A, $\nu(A) = \int_A f d\mu$).

We start from the following:

Lemma 1. *Let $D_1, D_2, \ldots, D_k$ be a family of disjoint measurable sets, such that for some $r_0 > 0$, $B(x_0, r_0) = B_{r_0} \subset \bigcup_{i=1}^k D_i$. Let $d_1, d_2, \ldots, d_k$ be positive numbers. Denote by f the linear combination of characteristic functions of the sets D_i with coefficients d_i, i.e. $f = d_1\chi_{D_1} + d_2\chi_{D_2} + \ldots + d_k\chi_{D_k}$. Let us define the new measure ν for any measurable set B as the integral*

$$\nu(B) = \int_B f d\mu = \sum_{i=1}^k d_i \mu(B \cap D_i).$$

Then, if x_0 is a density point of the set A with respect to μ, it is also such a point with respect to ν.

PROOF. We want to show that

$$\lim_{r\to 0} \frac{\nu(A \cap B_r)}{\nu(B_r)} = 1.$$

We know that for any positive ε there exists a positive δ such that for any $r < \delta$ we have

$$1 - \frac{\mu(A \cap B_r)}{\mu(B_r)} = \frac{\mu(B_r) - \mu(A \cap B_r)}{\mu(B_r)} < \varepsilon.$$

So:

$$1 - \frac{\nu(A \cap B_r)}{\nu(B_r)} = 1 - \frac{\sum_{i=1}^k d_i \mu(A \cap B_r \cap D_i)}{\sum_{i=1}^k d_i \mu(B_r \cap D_i)} =$$

$$\frac{\sum_{i=1}^k d_i(\mu(B_r \cap D_i) - \mu(A \cap B_r \cap D_i))}{\sum_{i=1}^k d_i \mu(B_r \cap D_i)} \leq \frac{M}{m} \cdot \frac{\mu(B_r) - \mu(A \cap B_r)}{\mu(B_r)} < \frac{M}{m} \cdot \varepsilon,$$

where m and M are the minimum and the maximum values of d_i respectively. $\square$

Remark 1. *Observe that the last inequality depends only on the maximum and minimum values of d_i. We will use this fact subsequently.*

Remark 2. *It is quite easy to show that Lemma 1 is not true if $d_i = 0$ for some i, $1 \leq i \leq k$. For example, take A as a subset of $\langle 0, 1\rangle$ with density point 0 (with respect to the Lebesgue measure λ) and such that, for any $r > 0$, $\lambda(\langle 0, r\rangle \setminus A) > 0$. Evidently 0 is not a density point of A with respect to ν given by the formula $\nu(B) = \int_B f d\lambda$, where $f = 0 \cdot \chi_A + 1 \cdot \chi_{\langle 0,1\rangle \setminus A}$. Moreover, 0 is the dispersion point of A with respect to ν and 0 belongs to the support of ν.*

Now we are in the position to prove the following theorem.

Theorem 1. *Let f be an integrable function (with respect to μ) on the metric space X, fulfilling the inequality $0 < m \leq f(x) \leq M$ for some radius r_0, and for $x \in B_{r_0} = B(x_0, r_0)$. Define $\nu(B) = \int_B f d\mu$. If x_0 is a density point of a set A with respect to μ, then it is also a density point of this set with respect to ν. (Of course, the assumed condition of f is satisfied μ-almost everywhere.)*

Proof. Let f_n be a nondecreasing sequence of simple functions uniformly tending to f a.e. Since $f(x) \geq m > 0$ we can assume that $f_n(x) \geq m$ for each $x \in B_{r_0}$ and n. For if a nondecreasing sequence $\{g_n\}$ of functions uniformly tends to f almost everywhere, then for each n and for each $x \in B_{r_0}$ we put $f_n(x) = \max\{m, g_n(x)\}$.

Hence
$$\lim_{r \to 0} \frac{\int_{B_r} f_n d\mu - \int_{B_r \cap A} f_n d\mu}{\int_{B_r} f_n d\mu} = 0.$$

We have:
$$1 - \frac{\int_{B_r \cap A} f d\mu}{\int_{B_r} f d\mu} \leq I_1 + I_2$$

where
$$I_1 = 1 - \frac{\int_{B_r \cap A} f_n d\mu}{\int_{B_r} f_n d\mu}$$

and
$$I_2 = \Big| \frac{\int_{B_r \cap A} f_n d\mu}{\int_{B_r} f_n d\mu} - \frac{\int_{B_r \cap A} f d\mu}{\int_{B_r} f d\mu} \Big|.$$

The difference I_1 is as small as we want, for sufficiently small r. Moreover (see Remark 1), the inequality $I_1 < \frac{M}{m} \cdot \varepsilon$ does not depend of n!

For I_2 we have:

$$
\begin{aligned}
I_2 = & \frac{\mid \int_{A\cap B_r} f_n d\mu \int_{B_r} f d\mu - \int_{A\cap B_r} f d\mu \int_{B_r} f_n d\mu \mid}{\int_{B_r} f_n d\mu \cdot \int_{B_r} f d\mu} \leq \\
\leq & \frac{\mid \int_{A\cap B_r} f_n d\mu \int_{B_r} f d\mu - \int_{A\cap B_r} f d\mu \int_{B_r} f d\mu \mid}{\int_{B_r} f_n d\mu \cdot \int_{B_r} f d\mu} + \\
& + \frac{\mid \int_{A\cap B_r} f d\mu \int_{B_r} f d\mu - \int_{A\cap B_r} f d\mu \int_{B_r} f_n d\mu \mid}{\int_{B_r} f_n d\mu \cdot \int_{B_r} f d\mu} \\
= & \frac{\int_{B_r} f d\mu}{\int_{B_r} f_n d\mu} \cdot \frac{\mid \int_{A\cap B_r} f_n d\mu - \int_{A\cap B_r} f d\mu \mid}{\int_{B_r} f d\mu} + \\
& + \frac{\int_{A\cap B_r} f d\mu}{\int_{B_r} f_n d\mu} \cdot \frac{\mid \int_{B_r} f d\mu - \int_{B_r} f_n d\mu \mid}{\int_{B_r} f d\mu} \\
< & \frac{2M \cdot \varepsilon}{m^2}
\end{aligned}
$$

if n is so large that $\mid f - f_n \mid < \varepsilon$ on B_{r_0}. □

Remark 3. *It is not difficult to see, that without the assumptions about the double boundedness of f Theorem 1 can be false.*

Corollary 1. *Let μ, ν be the finite on balls Borel measures on the metric space X. Assume that μ and ν are equivalent, i.e. the σ- ideals of null sets with respect to μ and ν are the same. Let the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$ have the property that for any ball $B(x_0, r)$ there exist positive numbers m, M such that $m \leq \frac{d\nu}{d\mu}(x) \leq M$ for x belonging to $B(x_0, r)$. Then any measurable set A has the same sets of density points with respect to μ and ν.*

PROOF. By our Theorem we have, that if x_0 is a density point of A with respect to μ, it is also a density point with respect to ν. But, as is well known, the Radon-Nikodym derivative $\frac{d\mu}{d\nu} = \frac{1}{\frac{d\nu}{d\mu}}$ has the same property of double boundedness on balls as $\frac{d\nu}{d\mu}$ and hence density points with respect to ν are also density points with respect to μ. □

References

[1] W. Poreda, E.Wagner-Bojakowska, W.Wilczyński, *A category analogue of the density topology* Fund. Math. **125** (1985), 167–173.

[2] W. Rudin, *Real and Complex Analysis*,McGraw-Hill, 1987.