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THE WIDE DENJOY INTEGRAL AS THE LIMIT OF A SEQUENCE OF STEPFUNCTIONS IN A SUITABLE CONVERGENCE

Abstract

In this paper we shall prove that a function $f : [a, b] \to \overline{\mathbb{R}}$ that is $\mathcal{D}$–integrable on $[a, b]$ can be defined as the limit of a $\mathcal{D}$-controlled convergent sequence of stepfunctions (see the second part of Theorem 2). In the last section we show that Ridder's α- and β-integrals can also be defined as the limit of some controlled convergent sequences of stepfunctions (see Theorem 4).

1 Introduction

E. J. McShane in [11], and F. Riesz and B. Sz-Nagy in [14] developed the Lebesgue integration on an interval $I \subset \mathbb{R}^n$ using the monotone convergence of stepfunctions. In [10], Lee and Chew showed that a function $f : [a, b] \to \overline{\mathbb{R}}$ that is $\mathcal{D}^*$–integrable on $[a, b]$ can be defined as the limit of a controlled convergent sequence of stepfunctions. However, their proof is not complete (that this result is indeed true is shown in [5]).

In a recent paper [7], Kurzweil and Jarník proved an analogue result of Lee and Chew for the multidimensional case. For the one-dimensional case, we shall prove that a function $f : [a, b] \to \overline{\mathbb{R}}$ that is $\mathcal{D}$–integrable on $[a, b]$ can be defined as the limit of a $\mathcal{D}$-controlled convergent sequence of stepfunctions (see the second part of Theorem 2). In the last section we show that Ridder's α- and β-integrals can be defined as the limit of controlled convergent sequences of stepfunctions (see Theorem 4).

The results in this paper are heavily based on Lemma 3, and its proof uses a technique that seems to be new (see Remark 3).

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We denote by $|X|$ the outer measure of the set X. Let $m(A)$ denote the Lebesgue measure of A, whenever $A \subset \mathbb{R}$ is Lebesgue measurable. Let $\mathcal{C}$ denote the class of continuous functions. We denote by

$$C([a,b]) = \{f : [a,b] \to \mathbb{R} \,:\, f \in \mathcal{C}\}$$

and

$$C_{ap}([a,b]) = \{f : [a,b] \to \mathbb{R} \,:\, f \text{ is approximately continuous on } [a,b]\}\,.$$

Definition 1. ([1], p. 165). Let $F : [a,b] \to \mathbb{R}$, $x_o \in (a,b)$. If there is a measurable set $E \subset [a,b]$ such that

$$\liminf_{h\to 0+} \frac{m(E \cap (x_o, x_o + h))}{h} > \frac{1}{2} \text{ and } \liminf_{h\to 0+} \frac{m(E \cap (x_o - h, x_o))}{h} > \frac{1}{2}$$

and

$$\lim_{\substack{x\to x_o\\ x\in E}} F(x) = F(x_o)\,,$$

then F is said to be preponderantly continuous at x_o. The definition of the preponderant continuity of F at a and b is obvious. We denote by $C_{pr}([a,b]) = \{F : [a,b] \to \mathbb{R} \,:\, F \text{ is preponderantly continuous at each } x \in [a,b]\}\,.$

Let $(L_1([a,b]), \|\cdot\|_1)$ be the Banach space of all Lebesgue integrable functions on $[a,b]$. We denote by $\mathcal{B}_1$ the Baire one functions, and by $\mathcal{DB}_1$ the Darboux Baire one functions. A function $F : [a,b] \to \mathbb{R}$ is said to satisfy Lusin's condition (N), if $|F(Z)| = 0$ whenever $Z \subset [a,b]$ with $|Z| = 0$. For the definitions of VB and AC see [15].

Definition 2. ([12], p. 91). A function $f : [a,b] \to \mathbb{R}$ is said to be a stepfunction if we can subdivide $[a,b]$ by the points

$$c_0 = a < c_1 < c_2 < \cdots < c_n = b$$

into a finite number of subintervals, in the interior of which f is constant. Let

$$\mathcal{S}([a,b]) = \{f : [a,b] \to \mathbb{R} \,:\; f \text{ is a stepfunction}\}$$

Remark 1. By the proof of Theorem 6 of [12], p. 172 (see also the comments on pp. 199-200), it follows that $\mathcal{S}([a,b])$ and $C([a,b])$ are dense in the Banach space $(L_1([a,b]), \|\cdot\|_1)$.

Definition 3. Let $E \subseteq [a,b]$. A function $F : [a,b] \to \mathbb{R}$ is said to be ACG (respectively AC^*G, VBG, VB^*G) on E if there exists a sequence of sets $\{E_n\}$ with $E = \cup_n E_n$, such that F is AC (respectively AC^*, VB, VB^*) on each E_n. If in addition the sets E_n are supposed to be closed we obtain the classes $[ACG]$, $[AC^*G]$, $[VBG]$, $[VB^*G]$. Note that ACG and AC^*G used here differ from those of [15] (because in our definitions the continuity is not assumed).

Definition 4. Let P be a real set and $F_n : P \to \mathbb{R}$, $n = 1, 2, \ldots$.

- ([2], p. 38). The sequence $\{F_n\}_n$ is said to be UAC on P if it has the following property: for every $\epsilon > 0$ there is a $\delta_\epsilon > 0$ such that $\sum_{k=1}^m |F_n(\beta_k) - F_n(\alpha_k)| < \epsilon$ for all $n = 1, 2, \ldots,$ whenever $\{[\alpha_k, \beta_k]\}$, $k = 1, 2, \ldots, m$ is a finite set of nonoverlapping closed intervals with endpoints in P and $\sum_{k=1}^m (\beta_k - \alpha_k) < \delta_\epsilon$.

- The sequence $\{F_n\}_n$ is said to be $UACG$ on P, if $P = \cup P_k$ and $\{F_n\}_n$ is UAC on each P_k. If in addition each P_k is supposed to be closed then $\{F_n\}_n$ is said to be $[UACG]$ on P.

Remark 2. If P is a closed set then our condition "$[UACG]$ on P" is exactly the "$UACG$ on P" from [2], p. 38 (the assertion can be proved by using the technique of Theorem 9.1 of [15], p. 233). This was pointed out by Bullen in [2] (p. 308).

Also, the condition "$[UAC^*G]$ on P" in this paper is exactly the "UAC^*G on P" from [2], p. 38 (the assertion can be proved by using the technique of Theorem 9.1 of [15], p. 233).

Definition 5. Let $P \subset [a,b]$ and $F_n : [a,b] \to \mathbb{R}$, $n = 1, 2, \ldots$.

- ([2], p. 38). The sequence $\{F_n\}_n$ is said to be UAC^* on P if it has the following property: for every $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that

$$\sum_{k=1}^m \mathcal{O}(F_n; [\alpha_k, \beta_k]) < \epsilon\,, \quad n = 1, 2, \ldots\,,$$

  whenever $\{[\alpha_k, \beta_k]\}$, $k = 1, 2, \ldots, m$ is a set of nonoverlapping closed intervals with endpoints in P and $\sum_{k=1}^m (\beta_k - \alpha_k) < \delta_\epsilon$.

- The sequence $\{F_n\}_n$ is said to be UAC^*G on P, if $P = \cup P_k$ and $\{F_n\}_n$ is UAC^* on each P_k. If in addition each P_k is supposed to be closed then $\{F_n\}_n$ is said to be $[UAC^*G]$ on P.

2 Sequential definitions for $\mathcal{D}^*$–integral and $\mathcal{D}$–integral

Lemma 1. *Let $F : [a,b] \to \mathbb{R}$, P be a closed subset of $[a,b]$ and let $A \subset [a,b] \setminus P$ be a finite set. If $F \in AC$ on P then $F \in AC$ on $P \cup A$.*

Proof. For $\epsilon > 0$ let $\delta_\epsilon > 0$ be given by the fact that $F \in AC$ on P. Then $\eta_\epsilon := \inf\{\delta_\epsilon, d(P;A), |x-y| \,:\, x,y \in A,\ x \neq y\} > 0$ is a "δ" that satisfies the definition of F being AC on $P \cup A$. □

Lemma 2. *Let $F : [a,b] \to \mathbb{R}$ and let $\{P_i\}, i = 1,2,\ldots,n$ be a finite set of closed subset of $[a,b]$. Suppose that for each $i = 1,2,\ldots,n$ there exists a set H_i that is the union of a finite set of nonoverlapping closed intervals (some of them might be degenerate), such that $int(H_{i_1}) \cap int(H_{i_2}) \neq \emptyset$ for $i_1 \neq i_2$. If F is AC on each P_i then F is AC on Q, where $Q = \cup_{i=1}^n (P_i \cap H_i)$.*

Proof. Let $c = \inf(Q)$, $d = \sup(Q)$. By Lemma 1, we may suppose without loss of generality that each component of H_i contains at least two points of P_i and has the endpoints in P_i (if for example $[\alpha,\beta]$ is a component of H_i that contains at least two points of P_i then we replace $[\alpha,\beta]$ if necessary, with $[\alpha^{'},\beta^{'}]$, where $\alpha^{'} = \inf(P_i \cap [\alpha,\beta])$ and $\beta^{'} = \sup(P_i \cap [\alpha,\beta])$). Let $G := int([c,d] \setminus \cup_{i=1}^n H_i)$. Then G is an open set that contains only a finite number of components (i.e. maximal open intervals contained in G). Let η be the length of the shortest component (or components) of G. Let $\epsilon > 0$. For $\epsilon/2^i$, $i = 1,2,\ldots,n$, let $\delta_i > 0$ be given by the fact that F is AC on P_i. Let $\delta \in (0, \min_{i=1}^n\{\eta,\delta_i\})$ and let $\{[a_j,b_j]\}$, $j = 1,2,\ldots,m$ be a finite set of nonoverlapping closed intervals with endpoints in Q and $\sum_{j=1}^m (b_j - a_j) < \delta$.

We shall say that an interval $[a_j,b_j]$ is of the first kind if $(a_j,b_j) \subset int(\cup_{i=1}^n H_i)$, otherwise the interval will be of the second kind. Note that for any interval $[a_j,b_j]$ of the second kind, there exists a unique point $c_j \in (a_j,b_j) \cap Q$ such that

$$(a_j,c_j) \cup (c_j,b_j) \subset int(\cup_{i=1}^n H_i)\,. \tag{1}$$

Indeed, a_j belongs to some component $[\alpha^{'},\beta^{'}]$ of the figure $\cup_{i=1}^n H_i$, and b_j belongs to some other component $[\alpha^{''},\beta^{''}]$ of the same figure. Then $\beta^{'} = \alpha^{''} =: c_j$ (if not then $b_j - a_j > \eta$, a contradiction). Therefore we have (1). Since $|F(b_j) - F(a_j)| \le |F(c_j) - F(a_j)| + |F(b_j) - F(c_j)|$, we may suppose without loss of generality that each interval $[a_j,b_j]$ is of the first kind. Then

$$\sum_{j=1}^m |F(b_j) - F(a_j)| < \sum_{i=1}^n \sum_{\substack{j \\ (a_j,b_j) \subset int(H_i)}} |F(b_j) - F(a_j)| < \sum_{i=1}^n \frac{\epsilon}{2^i} < \epsilon\,.$$

Therefore $F \in AC$ on Q. □

Lemma 3 (Main lemma)**.** *Let $F : [a,b] \to \mathbb{R}$, $F \in [ACG]$ on $[a,b]$. Then there exists an increasing sequence of closed sets $\{Q_n\}$, $n = 1, 2, \ldots$ such that $[a,b] = \cup_{n=1}^{\infty} Q_n$ and F is AC on each Q_n. Moreover, if for each positive integer n we define $F_n : [a,b] \to \mathbb{R}$ such that $F_n(x) = F(x)$ on $Q_n \cup \{a,b\}$ and F_n is linear on the closure of each interval contiguous to $Q_n \cup \{a,b\}$, then each F_n is AC on $[a,b]$ and $\{F_n\}_n$ is $[UACG]$ on $[a,b]$.*

PROOF. Since $F \in [ACG]$ on $[a,b]$ it follows that there exists a sequence of closed sets $\{P_n\}$, $n = 1, 2, \ldots$ such that $[a,b] = \cup_{n=1}^{\infty} P_n$ and F is AC on each P_n. We may suppose without loss of generality that $a, b \in P_1$. For each positive integer i let $G^i := [a,b] \setminus (\cup_{k=1}^{i} P_k)$. We may suppose that each G^i has infinitely many components and is of the form $G^i = \cup_{j=1}^{\infty} (a_j^i, b_j^i)$. Let $G_m^i := \cup_{j=1}^{m} [a_j^i, b_j^i]$, $i = 1, 2, \ldots, m = 1, 2, \ldots$. Let $Q_1 := P_1$ and for each positive integer $n \geq 2$ let

$$Q_n := P_1 \cup (P_2 \cap G_n^1) \cup (P_3 \cap G_n^2) \cup \cdots \cup (P_n \cap G_n^{n-1})\,. \tag{2}$$

Clearly $\{Q_n\}$ is an increasing sequence of closed sets. We show that

$$[a,b] = \cup_{n=1}^{\infty} Q_n\,. \tag{3}$$

Let $x \in [a,b]$. If $x \in P_1$ then clearly $x \in \cup_{n=1}^{\infty} Q_n$. Suppose that $x \notin P_1$. Then there exists a positive integer $n \geq 2$ such that

$$x \in P_n \setminus (\cup_{i=1}^{n-1} P_i) = P_n \cap G^{n-1}\,.$$

Let m be such that $x \in (a_m^{n-1}, b_m^{n-1})$. Then $x \in P_n \cap G_m^{n-1}$. For $m \leq n$

$$G_m^{n-1} = \cup_{j=1}^{m} [a_j^{n-1}, b_j^{n-1}] \subseteq \cup_{j=1}^{n} [a_j^{n-1}, b_j^{n-1}] = G_n^{n-1}\,,$$

so $x \in P_n \cap G_n^{n-1} \subset Q_n$. If $m > n$ then $x \in P_n \cap G_m^{n-1} \subset Q_m$. Therefore $x \in Q_{max\{m,n\}} \subset \cup_{i=1}^{\infty} Q_i$, so we have (3).

We show that F is AC on each Q_n. Clearly F is AC on Q_1. If for each $i \in \{2, 3, \ldots, n\}$, the set $P_i \cap G_n^{i-1} = \emptyset$ then $Q_n = Q_1$, so F is AC on Q_n. Suppose that there exists $i \in \{2, 3, \ldots, n\}$ such that $P_i \cap G_n^{i-1} \neq \emptyset$ and denote by $\mathcal{A} = \{i \in \{2, 3, \ldots, n\} : P_i \cap G_n^{i-1} \neq \emptyset\} = \{i_1, i_2, \ldots, i_p\}$, with $i_1 < i_2 < \cdots < i_p$. Clearly $p \leq n-1$. For simplicity reasons, we shall suppose that $p = n-1$, hence $\mathcal{A} = \{2, 3, \ldots, n\}$. For each $i \in \mathcal{A}$ we denote by A_n^i the union of those components intervals of G_n^{i-1} whose interiors contain points of P_i. Clearly

$$A_n^i \cap P_i = P_i \cap G_n^{i-1}\,.$$

For simplicity reasons again we shall suppose that $A_n^i = G_n^{i-1}$. Let

$$H_n^n := G_n^{n-1}$$
$$H_n^{n-1} := G_n^{n-2} \setminus int(G_n^{n-1}) = G_n^{n-2} \setminus int(H_n^n)$$
$$H_n^{n-2} := G_n^{n-3} \setminus int(G_n^{n-2} \cup G_n^{n-1}) = G_n^{n-3} \setminus int(H_n^{n-1} \cup H_n^n)$$
$$\vdots$$
$$H_n^2 := G_n^1 - int(G_n^2 \cup G_n^3 \cup \cdots \cup G_n^{n-1}) = G_n^1 - int(H_n^3 \cup H_n^4 \cup \cdots \cup H_n^n)$$
$$H_n^1 := [a,b] \setminus int(G_n^1 \cup G_n^2 \cup \cdots \cup G_n^{n-1}) = [a,b] \setminus int(H_n^2 \cup H_n^3 \cup \cdots \cup H_n^n)$$

It follows that each H_n^i is the union of a a finite set of nonoverlapping closed intervals (some of them might be degenerate) and

$$int(H_n^i) \cap int(H_n^j) = \emptyset \quad \text{for } i \neq j\,.$$

We show that the set Q_n defined by (2) can be written as follows:

$$Q_n = (P_n \cap H_n^n) \cup (P_{n-1} \cap H_n^{n-1}) \cup \cdots \cup (P_2 \cap H_n^2) \cup (P_1 \cap H_n^1)\,. \tag{4}$$

We have

$$P_n \cap G_n^{n-1} = P_n \cap H_n^n\,;$$
$$P_{n-1} \cap G_n^{n-2} = P_{n-1} \cap \big(H_n^{n-1} \cup int(G_n^{n-1})\big) = P_{n-1} \cap H_n^{n-1}\,;$$
$$P_{n-2} \cap G_n^{n-3} = P_{n-2} \cap \big(H_n^{n-2} \cup int(G_n^{n-2} \cup G_n^{n-1})\big) = P_{n-2} \cap H_n^{n-2}\,;$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$P_2 \cap G_n^1 = P_2 \cap \big(H_n^2 \cup int(G_n^2 \cup G_n^3 \cup \cdots \cup G_n^{n-1})\big) = P_2 \cap H_n^2\,;$$
$$P_1 = P_1 \cap [a,b] = P_1 \cap \big(H_n^1 \cup int(G_n^1 \cup G_n^2 \cup \cdots \cup G_n^{n-1})\big) = P_1 \cap H_n^1\,;$$

therefore, from (2) we obtain (4). Now by Lemma 2, $F \in AC$ on Q_n.

We show the second part. Since $F \in AC$ on Q_n, each $F_n \in VB \cap \mathcal{C} \cap (N) = AC$ on $[a,b]$ (see the Banach-Zarecki Theorem). Let n be fixed. Since F_i is AC on $[a,b]$ for each i (particularly for $i \leq n-1$), $F_i = F$ on Q_n for each $i \geq n$ and $F \in AC$ on Q_n, it follows that $\{F_i\}_i$ is UAC on Q_n. Therefore $\{F_n\}_n$ is $[UACG]$ on $[a,b]$. □

Remark 3. If $F \in [ACG]$ on $[a,b]$ then there exists a sequence of closed sets $\{P_i\}_i$ such that $[a,b] = \cup_i P_i$ and $F \in AC$ on each P_i. In Lemma 3, it seemed natural to define the sets Q_n as follows: $Q_n = \cup_{i=1}^n P_n$. But in [3] (Remark 1, p. 756), the author gave a simple example of a continuous function on $[0,1]$, that is AC on two closed subsets E_1 and E_2 of $[0,1]$, and that **is not** AC on $E_1 \cup E_2$. Having this in mind, Lemma 3 is quite unexpected, and the technique used in the proof seems to be new.

Lemma 4. *Let $F : [a,b] \to \mathbb{R}$ and let $\{P_n\}_n$ be an increasing sequence of closed sets such that $\cup_n P_n = [a,b]$. For each n, let $F_n : [a,b] \to \mathbb{R}$ such that $F_n(x) = F(x)$ if $x \in P_n$, and F_n is linear on the closure of each interval contiguous to $P_n \cup \{a,b\}$. Then $\{F_n\}_n$ converges pointwise to F on $[a,b]$. Moreover, if F is continuous on $[a,b]$ then $\{F_n\}_n$ converges uniformly to F on $[a,b]$.*

Proof. Let $x \in [a,b]$. Then there exists a positive integer n_x such that $x \in P_{n_x}$. Since $\{P_n\}_n$ is increasing,

$$F_n(x) = F(x), \quad \forall\, n \geq n_x\,.$$

It follows that $\{F_n(x)\}_n$ converges to $F(x)$.

Suppose that F is continuous on $[a,b]$. Let $\epsilon > 0$. Then there exists a $\delta_\epsilon > 0$ such that

$$\mathcal{O}(F;[\alpha,\beta]) < \epsilon\,, \quad \text{whenever} \;\; [\alpha,\beta] \subset [a,b] \;\; \text{and} \;\; \beta - \alpha < \delta_\epsilon\,.$$

Since $\{P_n\}_n$ is increasing and $\cup_n P_n = [a,b]$, it follows that there exists a positive integer n_ϵ such that

$$m([a,b] \setminus Q_n) < \delta_\epsilon\,, \quad (\forall)\, n \geq n_\epsilon\,.$$

Since $F_n - F = 0$ on Q_n and the length of each component interval of $[a,b] \setminus Q_n$ is less than δ_ϵ for $n \geq n_\epsilon$, it follows that

$$|F_n - F| < \epsilon \text{ on } [a,b]\,,\; (\forall)\, n \geq n_\epsilon\,,$$

hence $\{F_n\}_n$ converges uniformly to F on $[a,b]$. □

Definition 6. Let (X,τ_1) and (Y,τ_2) be topological spaces and let $A \subset X$. A function $f : A \to Y$ is said to be continuous at $a \in A$ if for each neighborhood W of $f(a)$ there is a neighborhood V of a such that $f(A \cap V) \subset W$. f is said to be continuous on a set $B \subset A$ if it is continuous at each point of B.

Lemma 5. *Let (X,τ_1) and (Y,τ_2) be two topological spaces, and let $\{X_i\}$, $i = 1,2,\ldots,n$ be a finite set of closed subsets of X. Let $f : \cup_{i=1}^n X_i \to Y$. If for each i, $f_{/X_i}$ is continuous on X_i then f is continuous on $\cup_{i=1}^n X_i$.*

Proof. Let $a \in \cup_{i=1}^n X_i$. Then $a \in X_i$ for some i. Let

$$I_a = \{i \in \{1,2,\ldots,n\} \,:\, a \in X_i\} \quad \text{and} \quad J_a = \{1,2,\ldots,n\} \setminus I_a$$

Let W be a neighborhood of $f(a)$. For $i \in I_a$, since $f_{/X_i}$ is continuous at a, there exists a neighborhood V_i such that $f(X_i \cap V_i) \subset W$. Since $\cup_{i \in J_a} X_i$ is a closed set that does not contain a, it follows that

$$V := \left(\cap_{i \in I_a} V_i\right) \setminus \left(\cup_{i \in J_a} X_i\right)$$

is a neighborhood of a. But

$$V \cap \left(\cup_{i=1}^{n} X_i\right) = \cup_{i \in I_a} (V \cap X_i)\,,$$

hence

$$f\big(V \cap \left(\cup_{i=1}^{n} X_i\right)\big) = \cup_{i \in I_a} f(V \cap X_i) \subset W\,.$$

It follows that f is continuous at a. $\square$

Lemma 6. *Let $F : [a,b] \to \mathbb{R}$ and $\{P_i\}_i$, $i = 1, 2, \dots, n$ be closed subsets of $[a,b]$. If $F \in AC^*$ on each P_i then $F \in AC$ on $Q := \cup_{i=1}^{n} P_i \cup \{a,b\}$.*

Proof. Let $G : [a,b] \to \mathbb{R}$ such that $G(x) = F(x)$ for $x \in Q$ and G is linear on the closure of each interval contiguous to Q. Let $\alpha, \beta \in Q$, $\alpha < \beta$. Then

$$\mathcal{O}(G; [\alpha,\beta]) = \mathcal{O}(F; Q \cap [\alpha,\beta]) \le \mathcal{O}(F; [\alpha,\beta])\,.$$

It follows that $G \in AC^*$ on each P_i. Clearly $F_{/P_i}$ is continuous on P_i, so by Lemma 5, $F_{/Q}$ is continuous on Q. It follows that G is continuous on $[a,b]$ (see for example Lemma 2 of [12], p. 101). By Lemma 3 of [3], $G \in AC^*$ on Q. Therefore $F \in AC$ on Q. $\square$

Lemma 7. *Let $F : [a,b] \to \mathbb{R}$, $F \in [AC^*G]$ on $[a,b]$. Suppose that $\{P_i\}_i$ is a sequence of closed sets such that $\cup_{i=1}^{\infty} P_i = [a,b]$ and $F \in AC^*$ on each P_i. For every positive integer n let $Q_n := \cup_{i=1}^{n} P_i \cup \{a,b\}$ and let $F_n : [a,b] \to \mathbb{R}$ such that $F_n(x) = F(x)$ for $x \in Q_n$ and F_n is linear on the closure of each interval contiguous to Q_n. Then each F_n is AC on $[a,b]$ and the sequence $\{F_n\}_n$ is $[UAC^*G]$ on $[a,b]$.*

Proof. By Lemma 6, $F \in AC$ on Q_n, therefore $F_n \in AC$ on $[a,b]$ (see the Banach-Zarecki Theorem). Fix some P_i. We show that $\{F_n\}_n \in UAC^*$ on P_i. For $\epsilon > 0$ let $\delta_j > 0$, $j = 1, 2, \dots, i-1$ be given by the fact that $F_j \in AC = AC^*$ on $[a,b]$. For $j \ge i$ we have that $F_j = F$ on P_i. Let $\alpha, \beta \in P_i$, $\alpha < \beta$. Then

$$\mathcal{O}(F_j; [\alpha,\beta]) = \mathcal{O}(F; [\alpha,\beta] \cap P_i) < \mathcal{O}(F; [\alpha,\beta])\,. \tag{5}$$

For ϵ, let $\delta_i > 0$ be given by the fact that $F \in AC^*$ on P_i. Let

$$\delta := \inf\{\delta_j \,:\, j = 1, 2, \dots, i\}\,.$$

Let $\{[a_k, b_k]\}_k$, $k = 1, 2, \ldots, m$ be a finite set of nonoverlapping closed intervals with endpoints in P_i, such that $\sum_{k=1}^{m}(b_k - a_k) < \delta$. By (5), it follows that for each n we have $\sum_{k=1}^{m} \mathcal{O}(F_n; [a_k, b_k]) < \epsilon$, hence $\{F_n\}_n$ is UAC^* on P_i. □

Lemma 8. *Let $\mathcal{A}([a,b])$ be a dense subset of $(L_1[a,b], \|\cdot\|_1)$. Let $f : [a,b] \to \overline{\mathbb{R}}$ be such that there exists $F : [a,b] \to \mathbb{R}$ such that F is $[ACG]$ (respectively $[AC^*G]$) on $[a,b]$ and $F'_{ap} = f$ (respectively $F' = f$) a.e. on $[a,b]$. Then there exists a sequence $\{g_n\} \in \mathcal{A}([a,b])$ such that if*

$$G_n(x) := (\mathcal{L}) \int_a^x g_n(t)\, dt, \quad (\forall)\, x \in [a,b]$$

then we have:

(i) g_n converges pointwise to f a.e on $[a,b]$;

(ii) G_n converges pointwise to F on $[a,b]$;

(iii) G_n is $[UACG]$ (respectively $[UAC^*G]$) on $[a,b]$.

Moreover if F is continuous on $[a,b]$ (therefore f is $\mathcal{D}$– integrable (respectively $\mathcal{D}^*$–integrable) on $[a,b]$) then (ii) can be replaced by

(ii$'$) G_n converges uniformly to F on $[a,b]$.

PROOF. By Lemma 3 (respectively Lemma 7) there is an increasing sequence $\{Q_n\}_n$ of closed sets such that $[a,b] = \cup_{n=1}^{\infty} Q_n$. Also, for each n, the function F_n is AC on $[a,b]$ and $\{F_n\}_n$ is $[UACG]$ (respectively $[UAC^*G]$ on $[a,b]$, where $F_n = F$ on Q_n and F_n is linear on the closure of each interval contiguous to $Q_n \cup \{a,b\}$. Let $f_n : [a,b] \to \mathbb{R}$, $f_n(x) := F'_n(x)$ whenever F_n is derivable at x and $f_n(x) := 0$ elsewhere. Clearly $\{f_n\}$ is a sequence of Lebesgue integrable functions on $[a,b]$. Since

$$f(x) = F'_{ap}(x) = F'_n(x) = f_n(x) \;\; a.e. \;\text{ on }\; Q_n$$

(respectively

$$f(x) = F'(x) = F'_n(x) = f_n(x) \;\; a.e. \;\text{ on }\; Q_n\,)$$

and $\{Q_n\}$ is increasing with $[a,b] = \cup_{n=1}^{\infty} Q_n$, it follows that

$$f_n \;\text{ converges pointwise to } f \text{ } a.e. \text{ on } [a,b] \tag{6}$$

Since $\mathcal{A}([a,b])$ is dense in $(L_1[a,b], \|\cdot\|_1)$ it follows that for each positive integer n there exists a function $g_n \in \mathcal{A}([a,b])$ such that

$$(\mathcal{L}) \int_a^b |g_n(t) - f_n(t)|\, dt < \frac{1}{2^n} \tag{7}$$

By (7), using a consequence of Beppo Levi's Theorem (see for example the corollary at page 142 of [12]), we obtain that

$$\lim_{n\to\infty} |g_n(t) - f_n(t)| = 0 \; a.e. \text{ on } [a,b] .$$

By (6), $\{g_n\}$ converges pointwise to f *a.e.* on $[a,b]$, so we obtain (i).

By Lemma 4, $\{F_n\}_n$ converges pointwise to F on $[a,b]$. Moreover, if F is continuous then $\{F_n\}_n$ converges uniformly to F on $[a,b]$. By (7) we have:

$$|G_n(x) - F(x)| \le |G_n(x) - F_n(x)| + |F_n(x) - F(x)| <$$

$$< (\mathcal{L}) \int_a^b |g_n(t) - f_n(t)| \, dt + |F_n(x) - F(x)| < \frac{1}{2^n} + |F_n(x) - F(x)| .$$

Hence, we obtain (ii) and $(ii^{'})$.

(iii) Since $\{F_i\}_i$ is $[UACG]$ (respectively $[UAC^*G]$) on $[a,b]$, there exists a sequence $\{P_n\}_n$ of closed sets, such that $\{F_i\}_i$ is UAC (respectively UAC^*) on each P_n. Let n be fixed and $\epsilon > 0$. For $\epsilon/2$, let $\eta_\epsilon > 0$ be given by the fact that $\{F_i\}_i$ is UAC (respectively UAC^*) on P_n. Let m_ϵ be a positive integer such that $1/2^{m_\epsilon} < \epsilon/2$. For each $i = 1,2,\ldots,m_\epsilon - 1$ let $\delta_{i,\epsilon}$ be given for ϵ by the fact that G_i is AC on $[a,b]$. Let

$$\delta_\epsilon = \inf_{i=1}^{m_\epsilon - 1} \{\eta_\epsilon, \delta_{i,\epsilon}\} .$$

Let $\{[a_j, b_j]\}_j$, $j = 1,2,\ldots,p$ be a finite set of nonoverlapping closed intervals with endpoints in P_n, such that

$$\sum_{j=1}^{p} (b_j - a_j) < \delta_\epsilon .$$

For each $j = 1,2,\ldots,p$ let $[\alpha_j, \beta_j] \subseteq [a_j, b_j]$. For $i \ge m_\epsilon$ it follows that

$$\sum_{j=1}^{p} |G_i(\beta_j) - G_i(\alpha_j)| <$$

$$< \sum_{j=1}^{p} |G_i(\beta_j) - F_i(\beta_j) + F_i(\beta_j) - F_i(\alpha_j) + F_i(\alpha_j) - G_i(\alpha_j)| <$$

$$< \sum_{j=1}^{p} |F_i(\beta_j) - F_i(\alpha_j)| + \sum_{j=1}^{p} \Big| (\mathcal{L}) \int_{\alpha_j}^{\beta_j} (g_i(t) - f_i(t)) \, dt \Big| <$$

$$< \sum_{j=1}^{p} |F_i(\beta_j) - F_i(\alpha_j)| + \frac{1}{2^i} \le \sum_{j=1}^{p} |F_i(\beta_j) - F_i(\alpha_j)| + \frac{\epsilon}{2}$$

(see (7) and the fact that $1/2^i < 1/2^{m_\epsilon} < \epsilon/2$). We obtain that

$$\sum_{j=1}^{p} |G(b_j) - G(a_j)| \le \sum_{j=1}^{p} |F(b_j) - F(a_j)| + \frac{\epsilon}{2}$$

and

$$\sum_{j=1}^{p} \mathcal{O}(G; [a_j, b_j]) \le \sum_{j=1}^{p} \mathcal{O}(F; [a_j, b_j]) + \frac{\epsilon}{2}\,.$$

Therefore $\{G_i\}_i$ is UAC (respectively UAC^*) on Q_n. It follows that $\{G_i\}_i$ is $[UACG]$ (respectively $[UAC^*G]$) on $[a, b]$. □

Definition 7. Let $f_n : [a, b] \to \overline{\mathbb{R}}$, $n = 1, 2, \ldots$ be $\mathcal{D}^*$–integrable (respectively $\mathcal{D}$–integrable) functions, and let

$$F_n(x) := (\mathcal{D}^*) \int_a^x f_n(t)\, dt, \quad x \in [a, b]$$

(respectively

$$F_n(x) := (\mathcal{D}) \int_a^x f_n(t)\, dt, \quad x \in [a, b] \Big).$$

Let $f : [a, b] \to \overline{\mathbb{R}}$. The sequence $\{f_n\}_n$ is said to be $(\mathcal{D}^*)$-controlled convergent (respectively $(\mathcal{D})$-controlled convergent) to f on $[a, b]$ if the following conditions are satisfied:

1) $\lim_{n\to\infty} f_n = f$ *a.e.* on $[a, b]$;

2) $\{F_n\}_n$ is uniformly convergent on $[a, b]$;

3) $\{F_n\}_n$ is $[UAC^*G]$ (respectively $[UACG]$) on $[a, b]$.

Remark 4. The $\mathcal{D}^*$ version of Definition 7 is in fact Definition 7.4 of [9], p. 39. The $\mathcal{D}$ version is extracted from the hypotheses of Theorem 4.7., a) of [2], p. 40 (see also Bullen's comments on p. 308 of [2]).

Theorem 1 (Džvaršeišvili)**.** (Theorem 47, p. 40 of [2]).
Let $f_n : [a, b] \to \overline{\mathbb{R}}$, $n = 1, 2, \ldots$ be $(\mathcal{D}^*)$-integrable (respectively $(\mathcal{D})$-integrable functions). If $\{f_n\}_n$ is $(\mathcal{D}^*)$-controlled convergent (respectively $(\mathcal{D})$-controlled

convergent) to a function $f : [a, b] \to \overline{\mathbb{R}}$ then f is $(\mathcal{D}^*)$-integrable (respectively $(\mathcal{D})$-integrable) on $[a, b]$ and

$$\lim_{n\to\infty} (\mathcal{D}^*) \int_a^b f_n(t)\, dt = (\mathcal{D}^*) \int_a^b f(t)\, dt$$

(respectively

$$\lim_{n\to\infty} (\mathcal{D}) \int_a^b f_n(t)\, dt = (\mathcal{D}) \int_a^b f(t)\, dt\Big).$$

Remark 5. For the $\mathcal{D}^*$ version of Theorem 1 see also Theorem 7.4 of [9], p. 39.

Theorem 2. *Let $\mathcal{A}([a, b])$ be a dense linear subspace of $(L_1[a, b], \|\cdot\|_1)$. The following conditions are equivalent*

1) $f : [a, b] \to \overline{\mathbb{R}}$ is $\mathcal{D}^*$–integrable (respectively $\mathcal{D}$–integrable) on $[a, b]$;

2) There exists a sequence $\{f_n\}_n \subset \mathcal{A}([a, b])$ such that $\{f_n\}_n$ is $\mathcal{D}^*$–controlled convergent (respectively $\mathcal{D}$–controlled convergent) to f on $[a, b]$;

and we have

$$(\mathcal{D}^*) \int_a^b f(t)\, dt = \lim_{n\to\infty} (\mathcal{L}) \int_a^b f_n(t)\, dt \tag{8}$$

(respectively

$$(\mathcal{D}) \int_a^b f(t)\, dt = \lim_{n\to\infty} (\mathcal{L}) \int_a^b f_n(t)\, dt\Big). \tag{9}$$

Proof. 1) $\Rightarrow$ 2) Let $f : [a, b] \to \overline{\mathbb{R}}$ be $\mathcal{D}^*$–integrable (respectively $\mathcal{D}$–integrable) on $[a, b]$. Then there exists a function $F : [a, b] \to \mathbb{R}$, $F(a) = 0$ such that $F \in [AC^*G] \cap \mathcal{C}$ (respectively $F \in [ACG] \cap \mathcal{C}$) on $[a, b]$ and $F' = f$ (respectively $F'_{ap} = f$) *a.e.* on $[a, b]$. By Lemma 8 and Definition 7, there exists a sequence $\{f_n\}_n \subset \mathcal{A}([a, b])$ that is $\mathcal{D}^*$–controlled convergent (respectively $\mathcal{D}$–controlled convergent) to f on $[a, b]$. Also

$$(\mathcal{D}^*) \int_a^b f(t)\, dt = F(b) = \lim_{n\to\infty} (\mathcal{L}) \int_a^b f_n(t)\, dt$$

(respectively

$$(\mathcal{D}) \int_a^b f(t)\, dt = F(b) = \lim_{n\to\infty} (\mathcal{L}) \int_a^b f_n(t)\, dt\Big).$$

Therefore we have obtained (8) (respectively (9)).

2) $\Rightarrow$ 1) and relation (8) (respectively (9)) follow by Theorem 1. $\square$

Remark 6. The $\mathcal{D}^*$ version in Theorem 2, for $\mathcal{A}([a,b]) = \mathcal{S}([a,b])$, is in fact The Equivalence Theorem of [10], p. 224 (see also [9], pp. 58–59). The $\mathcal{D}$ version of Theorem 2 seems to be new.

3 Sequential definitions for some general descriptive type integrals

Definition 8. Let $\mathcal{A}^*([a,b])$ (respectively $\mathcal{B}^*([a,b])$) be a class of functions (not necessarily a linear space) having the following properties:

1) $C([a,b]) \subsetneq \mathcal{A}^*([a,b])$ (respectively $C([a,b]) \subsetneq \mathcal{B}^*([a,b])$);

2) If $F_1, F_2 \in [AC^*G] \cap \mathcal{A}^*([a,b])$ (respectively $F_1, F_2 \in [ACG] \cap \mathcal{B}^*([a,b])$) and $(F_1 - F_2)^{'} = 0$ (respectively $(F_1 - F_2)^{'}_{ap} = 0$) *a.e.* on $[a,b]$ then $F_1 - F_2$ is a constant on $[a,b]$.

A function $f : [a,b] \to \overline{\mathbb{R}}$ is said to be $[\mathcal{A}^*\mathcal{D}^*]$–integrable (respectively $[\mathcal{B}^*\mathcal{D}]$–integrable) on $[a,b]$ if

- there exists $F : [a,b] \to \mathbb{R}$ such that $F \in [AC^*G] \cap \mathcal{A}^*([a,b])$ (respectively $F \in [ACG] \cap \mathcal{B}^*([a,b])$)on $[a,b]$ and
- $F^{'} = f$ (respectively $F^{'}_{ap} = f$) *a.e.* on $[a,b]$.

Then F is called the indefinite $[\mathcal{A}^*\mathcal{D}^*]$–integral (respectively $[\mathcal{B}^*\mathcal{D}]$–integral) of f, and its increment $F(b) - F(a)$ is called the definite $[\mathcal{A}^*\mathcal{D}^*]$–integral (respectively $[\mathcal{B}^*\mathcal{D}]$–integral) of f on $[a,b]$, denoted by

$$[\mathcal{A}^*\mathcal{D}^*] \int_a^b f(t)\,dt \quad \text{(respectively} \quad [\mathcal{B}^*\mathcal{D}] \int_a^b f(t)\,dt \Big).$$

Remark 7.

(i) Condition 2) in Definition 8 assures us that the $[\mathcal{A}^*\mathcal{D}^*]$–integral (respectively $[\mathcal{B}^*\mathcal{D}]$–integral) is well defined.

(ii) If $\mathcal{A}^*([a,b]) = \mathcal{B}^*([a,b] = C_{ap}([a,b])$ then $[\mathcal{A}^*\mathcal{D}^*]$ is in fact Ridder's α–integral, and $[\mathcal{B}^*\mathcal{D}]$ is Ridder's β–integral (that is also called the Kubota AD–integral [6], [13]).

(iii) If $\mathcal{A}^*([a,b]) = \mathcal{B}^*([a,b]) = C_{pr}([a,b])$ then $[\mathcal{A}^*\mathcal{D}^*]$ and $[\mathcal{B}^*\mathcal{D}]$ seem to be new integrals. PROOF. Clearly $C_{pr}([a,b])$ satisfies the conditions 1) of Definition 8). We show 2) of the same definition. If $F \in C_{pr}([a,b])$ then $F \in \mathcal{DB}_1$ on $[a,b]$ (see for example [1], p. 166). Let $F_1, F_2 \in C_{pr}([a,b])$.

Then $F_1 + F_2 \in \mathcal{B}_1$ on $[a,b]$ and $F_1 + F_2$ satisfies the Young condition i.e., for each x there exist two sequences $x_n \nearrow x$, $y_n \searrow x$, such that

$$(F_1 + F_2)(x) = \lim_{n\to\infty}(F_1 + F_2)(x_n) = \lim_{n\to\infty}(F_1 + F_2)(y_n)\,.$$

By Theorem 1.1 (i), (ii) of [1], pp. 8–9, it follows that $F_1 + F_2 \in \mathcal{DB}_1$ on $[a,b]$. Suppose that $F_1, F_2 \in [ACG] \cap C_{pr}([a,b])$ and $(F_1 - F_2)'_{ap} = 0$ *a.e.* on $[a,b]$. Then $F_1 - F_2 \in \mathcal{DB}_1 \cap (N)$ on $[a,b]$. By Theorem 1 of [8], p. 61 (this theorem states the following: *a function $F \in \mathcal{DB}_1 \cap (N)$ on $[a,b]$, with $F'(x) \geq 0$ a.e. where F is derivable, is increasing and AC on* $[a,b]$), we obtain that $F_1 - F_2$ is a constant. □

Definition 9. Let $f_n : [a,b] \to \overline{\mathbb{R}}$, $n = 1,2,\dots$ be $[\mathcal{A}^*\mathcal{D}^*]$-integrable (respectively $[\mathcal{B}^*\mathcal{D}]$-integrable) functions, and let $F_n : [a,b] \to \mathbb{R}$ be the $[\mathcal{A}^*\mathcal{D}^*]$ (respectively $[\mathcal{B}^*\mathcal{D}]$) indefinite integral of f_n. Let $f : [a,b] \to \overline{\mathbb{R}}$. The sequence $\{f_n\}_n$ is said to be $[\mathcal{A}^*\mathcal{D}^*]$–controlled convergent (respectively $[\mathcal{B}^*\mathcal{D}]$–controlled convergent) to f on $[a,b]$ if the following conditions are satisfied:

1) $\lim_{n\to\infty} f_n = f$ *a.e.* on $[a,b]$;

2) There exists $F : [a,b] \to \mathbb{R}$, $F \in \mathcal{A}^*([a,b])$ (respectively $F \in \mathcal{B}^*([a,b])$) such that $\{F_n\}_n$ converges pointwise to F on $[a,b]$;

3) $\{F_n\}_n$ is $[UAC^*G]$ (respectively $[UACG]$) on $[a,b]$.

Theorem 3 (A Džvaršeišvili type theorem)**.** *Let $f_n : [a,b] \to \overline{\mathbb{R}}$, $n = 1,2,\dots$ be $[\mathcal{A}^*\mathcal{D}^*]$ (respectively $[\mathcal{B}^*\mathcal{D}]$) integrable functions, such that $\{f_n\}_n$ is $[\mathcal{A}^*\mathcal{D}^*]$–controlled convergent (respectively $[\mathcal{B}^*\mathcal{D}]$–controlled convergent) to a function $f : [a,b] \to \overline{\mathbb{R}}$ on $[a,b]$. Then f is an $[\mathcal{A}^*\mathcal{D}^*]$ (respectively $[\mathcal{B}^*\mathcal{D}]$) integrable function on $[a,b]$ and*

$$\lim_{n\to\infty}[\mathcal{A}^*\mathcal{D}^*]\int_a^b f_n(t)\,dt = [\mathcal{A}^*\mathcal{D}^*]\int_a^b f(t)\,dt$$

(respectively

$$\lim_{n\to\infty}[\mathcal{B}^*\mathcal{D}]\int_a^b f_n(t)\,dt = [\mathcal{B}^*\mathcal{D}]\int_a^b f(t)\,dt\Big).$$

Proof. The proof of the first part is similar to that of Theorem 3 of [4], and the proof of the second part is similar to that of Theorem 2 of [4]. □

Theorem 4. *Let $\mathcal{A}([a,b])$ be a dense linear subspace of $(L_1[a,b], \|\cdot\|_1)$. The following conditions are equivalent*

1) $f : [a, b] \to \overline{\mathbb{R}}$ *is* $[\mathcal{A}^*\mathcal{D}^*]$*–integrable (respectively* $[\mathcal{B}^*\mathcal{D}]$*–integrable) on* $[a, b]$*;*

2) There exists a sequence $\{f_n\}_n \subset \mathcal{A}([a, b])$ *such that* $\{f_n\}_n$ *is* $[\mathcal{A}^*\mathcal{D}^*]$*–controlled convergent (respectively* $[\mathcal{B}^*\mathcal{D}]$*–controlled convergent) to* f *on* $[a, b]$*;*

and we have

$$[\mathcal{A}^*\mathcal{D}^*]\int_a^b f(t)\,dt = \lim_{n\to\infty}(\mathcal{L})\int_a^b f_n(t)\,dt \tag{10}$$

(respectively

$$[\mathcal{B}^*\mathcal{D}]\int_a^b f(t)\,dt = \lim_{n\to\infty}(\mathcal{L})\int_a^b f_n(t)\,dt\Big). \tag{11}$$

Proof. 1) ⇒ 2) and (10) (respectively (11)) follow by Lemma 8 and Definition 9.

2) ⇒ 1) and (10) (respectively (11)) follow by Theorem 3. □

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