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REMARKS ON THE CONTINUITY OF FUNCTIONS OF TWO VARIABLES

Abstract

The continuity of $f(x, y)$ at (x_0, y_0) can be described by the behavior of f along a collection of paths toward (x_0, y_0) if the collection is rich enough. The collection of paths that are $\mathcal{C}^1$ and convex is rich enough but the collection of differentiable functions with bounded derivatives is not. The collection of $\mathcal{C}^n$ parameterized paths $(x(t), y(t))$ for any $n > 0$ is also rich enough to capture continuity.

1 Introduction

Although the definition of continuity for functions of two variables is straightforward, it is always tempting to describe continuity in terms of approaching the limiting point along paths and requiring the same limiting value along all paths. The collection of "all paths" may be a bit vague and it is certainly large. Any calculus book will have an example showing that approaching the limiting point along all lines through the point does not suffice to describe continuity, so the collection of lines is too small for this type of description of continuity. What is the smallest class of paths through the limiting point so that if a function is continuous on those paths, then the function is continuous?

Let $f(x, y)$ be a function of two variables defined on an open set D in $\mathbf{R}^2$ and $O \in D$. Rosenthal in [6] shows that the collection of twice differentiable curves is too small to describe continuity and that the collection of paths that are continuously differentiable and convex is large enough. He proves in his

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Theorem 1 a discontinuity result and a continuity result. He shows (1) the function f can be discontinuous at O even if f is continuous along every curve through O which is (at least) twice differentiable; and (2) if f is continuous along every continuously differentiable convex curve through O, then f is continuous at O (as a function of two variables).

In particular, this implies that f can be discontinuous at O even if f is continuous along every analytic curve through O; see [4, 5]. A similar remark is made by Vulfson [7]. Note that there still a gap between the necessary condition and sufficient condition for the continuity of f in terms of the continuity of f along curves through O. In this paper we will prove that a function f can be discontinuous at O even if f is continuous along every curve (convex or not) through O which is once differentiable with bounded derivative. This is almost equivalent to (2) above. In addition, we will show that with parametric curves the following necessary and sufficient condition holds: for any integer $n \geq 0$, f is continuous at $(0,0)$ if and only if $f(x(t), y(t))$ is continuous at $t = 0$ for all functions x, y of t with $(x(0), y(0)) = (0,0)$ that are n times differentiable at $t = 0$.

We do not know whether the preceding statement is still true if the class of n times differentiable functions is replaced by the class of infinitely differentiable functions or the class of analytic functions.

Investigations of alternative descriptions of continuity for functions of several variables occur many times in the literature. A few of the works that are related to the present paper include [1], [3], and [8]. In [1] Dzagnidze defines several versions of continuity. An example is strong continuity in the sense that, in each (unit) direction v, $\lim_{x\to 0} f(x) - f(\pi(x)) = 0$ where $\pi(x)$ is the project of x onto the line perpendicular to v, that is, $\pi(x) = x - (v \cdot x)v$. It is shown that f is continuous at O if and only if f is strongly continuous in each (unit) direction v. The authors in [3] show that a certain monotone function of two variables (in ordered topological spaces) is continuous if it is continuous in each of the variables. In [8] the author gives a motivating example to show that a function $f(x, y)$ defined on a bicylinder (two cylinders intersecting at a right angle), continuous in one variable and analytic in the other, need not be continuous.

2 Continuity of Multivariate Functions

We begin with some notation for classes of single-valued functions. For a non-negative integer n, let $\mathcal{C}^n$ be the set of all single-variable functions g whose derivatives up to order n are continuous on the interval $[-\delta, \delta]$ for some $\delta > 0$, which may depend on g. Let $\mathcal{C}^0 = \mathcal{C}$. For $\alpha \in (0, 1]$, denote by $\mathcal{C}^{n,\alpha}$ the set of all single-variable functions g whose derivatives up to order n are Hölder continuous on the interval $[-\delta, \delta]$ with exponent α; that is, there is a constant $B > 0$ such that for all $x, y \in [-\delta, \delta]$

$$|g^{(k)}(x) - g^{(k)}(y)| \leq B|x - y|^\alpha \text{ for } k = 0, 1, ..., n.$$

We want to consider a class $\mathcal{M}$ of functions motivated by the modulus of continuity. In [2], Guthrie demonstrates that the modulus of continuity (in a more general setting) can be chosen to be a continuous function of h and the point where continuity is discussed. For a function $z \in \mathcal{C}$, the modulus of continuity, $\omega(h) = \sup_{|x| \leq |h|} |z(x) - z(0)|$, has the following properties.

1. ω is an even function on $[-\delta, \delta]$ for some $\delta > 0$.

2. $\omega(0) = 0$, continuous at 0 and increasing $[0, \delta]$.

Let $\mathcal{M}$ be the set of all functions in $\mathcal{C}$ satisfying (1) and (2).

Example 2.1. *The set $\mathcal{M}$ contains the following functions:* $|h|^\alpha, |h|^\alpha |\ln(|h|)|^\beta$ *(for all numbers $\alpha > 0$ and β),* $\sin(|h|), 1 - \cos(h)$.

The functions in $\mathcal{M}$ form a natural set of comparison functions for the order of convergence. The class $\mathcal{O}(\omega)$ defined next is the set of all functions that are $O(\omega(h))$ as $h \to 0$ for some fixed $\omega \in \mathcal{M}$.

Definition 2.2. For $\omega \in \mathcal{M}$, let

1. $\mathcal{O}(\omega)$ be the set of all functions $g \in \mathcal{C}$ such that $|g(h) - g(0)| \leq B\omega(h)$ for some $B > 0$ and all h close to 0.

2. $\mathcal{O}^1(\omega)$ be the set of all functions g with $g, g' \in \mathcal{O}(\omega)$.

Observe that the class $\mathcal{O}^1(\omega)$ includes only those functions with continuous first derivatives in a neighborhood of 0.

Proposition 2.3. *Let g be a function defined near* 0 *and $\alpha \in (0, 1]$. Then the following hold.*

(1) If $g \in \mathcal{C}$, then $g \in \mathcal{O}(\omega)$ for some $\omega \in \mathcal{M}$.

(2) If $g \in \mathcal{C}^{0,\alpha}$, then $g \in \mathcal{O}(h^\alpha)$.

(3) If $g \in \mathcal{C}^{1,\alpha}$, then $g \in \mathcal{O}^1(h^\alpha)$. In particular, if g is twice differentiable at 0*, then $g \in \mathcal{O}^1(h)$.*

PROOF. (1) If g is any function in $\mathcal{C}$, then the modulus of continuity ω defined above shows that $g \in \mathcal{O}(\omega)$.

(2) If $g \in \mathcal{C}^{0,\alpha}$, then $|g(x) - g(y)| \leq B|x-y|^\alpha$ for $x, y \in (-\delta, \delta)$ and some $B, \delta > 0$. In particular, $|g(h) - g(0)| \leq B|h|^\alpha$ for $h \in (-\delta, \delta)$. So $g \in \mathcal{O}(h^\alpha)$.

(3) The hypotheses on g implies that $|g'(x) - g'(y)| \leq B\,|x-y|^\alpha$. Letting $x = h$ and $y = 0$ gives $|g'(h) - g'(0)| \leq B\,|h|^\alpha$ which implies $g \in \mathcal{O}^1(h^\alpha)$.

If g is twice differentiable at the origin, then $|g'(h) - g'(0)| \,/\, |h|$ is bounded near $h = 0$ so that $g \in \mathcal{O}^1(h)$. □

The next proposition gives some algebraic properties of $\mathcal{M}$.

Proposition 2.4. *If $\omega, \gamma \in \mathcal{M}$, $a, b \geq 0$, then*

1. $a\omega + b\gamma \in \mathcal{M}$.

2. $\omega \cdot \gamma \in \mathcal{M}$.

3. $\omega \circ \gamma \in \mathcal{M}$.

PROOF. Parts 1 and 2 are straightforward. Part 3 would be simple calculus if differentiability of functions in $\mathcal{M}$ were assumed. That $\omega \circ \gamma$ is even and continuous at 0 is clear. For h and k positive and small enough, $\gamma(h+k) > \gamma(h)$, so since ω is increasing on some interval $[0, \delta]$, $\omega\circ\gamma(h+k) = \omega(\gamma(h+k)) > \omega(\gamma(h)) = \omega \circ \gamma(h)$ showing that $\omega \circ \gamma$ is in $\mathcal{M}$. □

The following theorem is similar in spirit to Rosenthal's Theorem 1 in [6]. Let D be an open set in $\mathbf{R^2}$ containing a point O, which is assumed to be the origin $O = (0, 0)$ for convenience. Let $f(x, y)$ be a function defined on D. The following theorem says that there are many functions f with the property that f is discontinuous at O even though f is continuous on every continuously differentiable curve in $\mathcal{O}^1(\omega)$ and through O. Thus the classes $\mathcal{O}^1(\omega)$ (for all $\omega \in \mathcal{M}$) are still too small to describe continuity of functions at O.

Theorem 2.5. *For every $\omega \in \mathcal{M}$, there is a function f defined on some open set D that contains O such that*

(1) f is discontinuous at O but continuous at every point in $D \setminus \{O\}$.

(2) for all $z \in \mathcal{O}^1(\omega)$ with $z(0) = 0$, $f(x, z(x))$ and $f(z(x), x)$ are continuous at $x = 0$.

PROOF. Let $g(x) = |x|\sqrt{\omega(x)}$, $x \in (-\delta, \delta)$. We define a function f in the square $D = (-\delta, \delta) \times (-\delta, \delta)$ in a way similar to those in [5] and [6].

$$f(x,y) = \begin{cases} 0, & \text{if } |y| \geq 3g(x) \text{ or } |y| \leq g(x); \\ 1 - |2-r|, & \text{if } |y| = rg(x), 1 < r < 3. \end{cases} \tag{1}$$

In particular, $f(0,0)=0$. Note that f is discontinuous at O since $f(x,2g(x)) \equiv 1$ as $x \to 0$ but is continuous at other points in D.

Now let $z \in \mathcal{O}^1(\omega)$. We show $f(x,z(x))$ is continuous at $x=0$. By the mean value theorem, for each x near 0, there exists a number r_x between 0 and x such that $z(x)=z'(r_x)x$. Consider the two cases separately: (1) $z'(0)\neq 0$ and (2) $z'(0)=0$.

In case (1), we may assume that (by reducing δ if necessary) that for all $x \in (-\delta,\delta)$:

$|z'(r_x)| \geq 0.6|z'(0)|$ and $\sqrt{\omega(x)} \leq 0.2|z'(0)|$.

So for $x \in (-\delta,\delta)$,

$$|z(x)| = |z'(r_x)x| \geq 0.6|z'(0)||x| \geq 3|x|\sqrt{\omega(x)} = 3g(x).$$

By the definition of f, $f(x,z(x))=0$. This implies that $f(x,z(x)) \to 0 = f(0,0)$ as $x \to 0$. So $f(x,z(x))$ is continuous at $x=0$.

In case (2), $|z'(r_x)| \leq B\omega(x)$ for some constant B and all x close 0. Since $\omega(0)=0$ and ω is continuous at 0, $B\sqrt{\omega(x)} \leq 1$ for x close to 0. So

$$|z(x)| \leq B|x|\omega(x) \leq |x|\sqrt{\omega(x)} \cdot B\sqrt{\omega(x)} \leq g(x).$$

By definition of f, $f(x,z(x))=0$ for x close to 0. This again implies that $f(x,z(x))$ is continuous at $x=0$.

Finally we show that $f(z(x),x)$ is also continuous at $x=0$. Because $z(0)=0$ and $z'(0)$ exists, there is a number B such that $|z(x)| \leq B|x|$ for x close to 0. It follows that

$$g(z(x)) = |z(x)|\sqrt{\omega(z(x))} \leq B|x|\sqrt{\omega(B|x|)} = |x| \cdot B\sqrt{\omega(B|x|)}.$$

So for x sufficiently close to 0, $B\sqrt{\omega(B|x|)} < 1/3$, which implies that $g(z(x)) \leq |x|/3$. By the definition of f, $f(z(x),x)=0$ for x sufficiently close to 0. Therefore, $f(z(x),x)$ is continuous at $x=0$. $\square$

Because the classes of $\mathcal{C}^{1,\alpha}$, $\mathcal{C}^2$, $\mathcal{C}^n$ $(n \geq 2)$, $\mathcal{C}^\infty$, and $\mathcal{C}^\omega$ (the set of analytic functions) are all subsets of $\mathcal{O}^1(h^\alpha)$ (see Proposition 2.4 above), the theorem implies the following result; part of it is proved in [4, 5, 6].

Corollary 2.6. *Given any $\alpha \in (0,1]$, there is a function f defined on some open set D that contains O such that*

(1) f is discontinuous at O but continuous at every point in $D \setminus \{O\}$.

(2) for all $z \in \mathcal{C}^{1,\alpha}$ with $z(0)=0$, $f(x,z(x))$ and $f(z(x),x)$ are continuous at $x=0$. In particular, $f(x,z(x))$ and $f(z(x),x)$ are continuous at $x=0$ for all $z \in \mathcal{C}^n$ $(n \geq 2), \mathcal{C}^\infty, \mathcal{C}^\omega$ with $z(0)=0$.

So far we have proved the existence of a function that is discontinuous at O but which is continuous along certain types of curves. The conclusions of Theorem 2.5 and its corollary are "almost sufficient" in the following sense: given a function f that is discontinuous at O, there is a continuously differentiable convex curve through O so that f restricted to this curve has a discontinuity at O ([6]). Furthermore there are smooth parameterized functions through O so that f on those curves is discontinuous at O. This is the content of the next theorem.

Theorem 2.7. *Suppose a function $f(x, y)$ defined in D is discontinuous at $(0, 0)$. Then*

(1) there is a continuously differentiable convex function $z \in \mathcal{C}^1$ such that either $f(x, z(x))$ or $f(z(x), x)$ is discontinuous at $x = 0$.

(2) for any integer $n \geq 0$, there exist functions $x(t), y(t) \in \mathcal{C}^n$ such that $f(x(t), y(t))$ is discontinuous at $t = 0$.

PROOF. **Part (1)** is proved in [6] with a geometric construction. A more analytic proof is outlined as follows. (See Remark 2.8 for a much shorter construction in a special case.)

Step I. First note that the discontinuity of f at O implies that there is a sequence $(x_i, y_i) \to (0, 0)$ such that

$$\lim_{i\to\infty} f(x_i, y_i) \text{ exists (might be infinity) but} \neq f(0,0). \tag{2}$$

Step II. By passing to a subsequence and rotating/reflecting the plane about x-, y-axes or $y = \pm x$ if necessary, we can assume that the sequence (x_i, y_i) in (2) has the following properties.

(1) $x_i \geq y_i > 0$ and both x_i and y_i are decreasing to 0.

To see this, consider the regions in the plane divided by $x = 0, y = 0, y = \pm x$. One of them must contain infinitely many points in the sequence (x_i, y_i). By rotation and reflection, we may assume that it is the region defined by $x \geq y > 0$.

(2) $s_i = \frac{y_i}{x_i} \to 0$ and s_i is strictly decreasing.

To see this, by taking a subsequence we may assume that $\frac{y_i}{x_i} \to s \in [0, 1]$. By rotation/reflection and taking a subsequence, we may further assume that $s = 0$ and y_i/x_i is strictly decreasing.

(3) $m_i = \frac{y_i - y_{i+1}}{x_i - x_{i+1}} \to 0$ and m_i is decreasing.

To prove this, start with (x_1, y_1) and (x_2, y_2). Choose (x_i, y_i), $i > 2$, such that

$$\frac{y_2 - y_i}{x_2 - x_i} < \frac{y_1}{x_1} < \frac{y_1 - y_2}{x_1 - x_2}.$$

This is possible because $\frac{y_2}{x_2} < \frac{y_1}{x_1}$ and $x_i, y_i \to 0$ as $i \to \infty$. Rename (x_i, y_i) as (x_3, y_3). Repeating this process we get $m_{i+1} < \frac{y_i}{x_i} \leq m_i$ for all $i \geq 1$.

Step III. Let $n_1 = m_1$ and $n_i = \frac{m_i + m_{i-1}}{2}$ for $i \geq 2$. Let $y = q_i(x)$ be the equation of the line through (x_i, y_i) with slope n_i, $i \geq 1$. Then $q(x) = \sup\{q_i(x), i \geq 1\}$ for $x \in [-1, x_1]$ is a piecewise linear convex function $q(x_i) = y_i$ for all $i = 1, 2, 3, \ldots$. We can smooth the corners of q by using arcs of circles tangent to the graph of q to get a function p that is convex, $\mathcal{C}^1$ and $p(x_i) = y_i$ for all $i = 1, 2, 3, \ldots$.

Here are the details of the smoothing. In brief, we show how to insert an arc of a circle that joins smoothly two adjacent lines. For $j = i, i+1$ let l_j denote the line $y_j = q_j(x)$. Let $(\hat{x}_i, \hat{y}_i)$ be the intersection of the lines l_i and l_{i+1}. Consider the two line segments, one from (x_i, y_i) to $(\hat{x}_i, \hat{y}_i)$ and the other from (x_{i+1}, y_{i+1}) to $(\hat{x}_i, \hat{y}_i)$. Construct the perpendicular bisector to the shorter segment. Reflect this bisector across the angle bisector between the lines l_i and l_{i+1}. The result is two lines perpendicular to the lines l_i and l_{i+1}, respectively. These lines meet along the angle bisector. Use this intersection point as the center of a circle tangent to both the lines l_i and l_{i+1}. Then the piecewise defined function "segment of line l_i followed by a portion of a circle followed by a segment of line l_{i+1}" is (a part of) the desired $\mathcal{C}^1$ convex curve $y = p(x)$.

Part (2). Let n be the fixed integer in the statement of the theorem. First note that by passing to a subsequence we may assume that

$$x_i, y_i \leq 2^{-ni}$$

for $i = 1, 2, 3, \ldots$. Now let $\phi : [0,1] \to R$ be a $\mathcal{C}^{n+1}$ function such that $\phi(0) = 0, \phi(1) = 1$ and $\phi^{(k)}(0) = \phi^{(k)}(1) = 0$ for $k = 1, ..., n+1$. There are many such functions such as polynomials but the specific form of ϕ is not important here. Denote

$$B_k = \max_{0 \leq t \leq 1} |\phi^{(k)}(t)|.$$

We next define a function that pieces together compressed versions of ϕ on a collection of adjacent intervals of decreasing length. Define $x(t)$ as follows.

$$x(t) = \begin{cases} 0 & t \in [-1, 0] \\ x_i \phi(2^{i+1}(t - 2^{-i-1})) & \\ \quad + x_{i+1}\phi(2^{i+1}(2^{-i} - t)) & t \in [2^{-i-1}, 2^{-i}], i = 0, 1, 2, \ldots \end{cases}$$

Then for all i's,

$$\begin{aligned} x(2^{-i-1}) &= x_i \phi(0) + x_{i+1}\phi(1) = x_{i+1} \\ x(2^{-i}) &= x_i \phi(1) + x_{i+1}\phi(0) = x_i \end{aligned}$$

so that x interpolates the x_i's at 2^{-i}. Note that

$$x^{(k)}(t) = 2^{(i+1)k} x_i \phi^{(k)}(2^{i+1}(t - 2^{-i-1})) + 2^{(i+1)k} x_{i+1} \phi^{(k)}(2^{i+1}(2^{-i} - t)).$$

So $x^{(k)}(2^{-i-1}) = x^{(k)}(2^{-i}) = 0$ for all positive integers $k \le n+1$ because $\phi^{(k)}(0) = \phi^{(k)}(1) = 0$. In addition,

$$|x^{(k)}(t)| \le B_k 2^{(i+1)k}[x_i + x_{i+1}] \le B_k 2^{(i+1)k} 2^{-ni+1} \le 2^{k+1} B_k$$

for $t \in [-1, 1]$. So $x(t)$ is $\mathcal{C}^n$ on $[-1, 1]$ and $x(2^{-i}) = x_i$. Similarly we can define a $\mathcal{C}^n$ function $y(t)$ with $y(2^{-i}) = y_i$. Because of (2), the function $f(x(t), y(t))$ must be discontinuous at $t = 0$. $\square$

Remark 2.8. In the proof of Part (1), if we are interested in merely a continuously differentiable function (not necessarily convex), then the easiest definition of p might be

$$p(x) = \begin{cases} 0 & x \in [-1, 0] \\ y_{i+1} + \frac{m_i}{h_i}(x - x_{i+1})^2 & \\ \quad - \frac{2m_i}{h_i^2}(x - x_{i+1})^2(x - x_i) & x \in [x_{i+1}, x_i], i = 1, 2, \ldots \end{cases}$$

where $h_i = x_i - x_{i+1}$. Note that p on $[x_{i+1}, x_i]$ is just the cubic polynomial satisfying

$$p(x_{i+1}) = y_{i+1}, p(x_i) = y_i, p'(x_{i+1}) = p'(x_i) = 0.$$

It is easy to see that p is continuously differentiable on $[-1, 1]$.

The following corollary to Theorems 2.5 and 2.7 gives some necessary and sufficient conditions for the continuity of a two-variable function. The corollary is essentially a statement of the contrapositive of these theorems thus turning a "negative" result on the existence of a discontinuous function into a "positive" statement about continuous functions. The new aspect of the corollary is the observation that while the collection of graphs of $\mathcal{C}^1$ functions, either $(x, z(x))$ or $(z(x), x)$ is sufficient to describe continuity, the collection of parameterized curves $(x(t), y(t))$ for functions x and y in $\mathcal{C}^n$, for any positive integer n, also suffices to describe continuity of $f(x, y)$.

Corollary 2.9. *The function $f(x, y)$ is continuous at $(0, 0)$ if and only if one of the following holds.*

(1) for every function $z \in \mathcal{C}^1$ with $z(0) = 0$, $f(x, z(x))$ and $f(z(x), x)$ is continuous at $x = 0$.

(2) for every convex function $z \in \mathcal{C}^1$ with $z(0) = 0$, $f(x, z(x))$ and $f(z(x), x)$ is continuous $x = 0$.

(3) for some natural number n and all functions $x(t), y(t) \in \mathcal{C}^n$ with $(x(0), y(0)) = (0, 0)$, $f(x(t), y(t))$ is continuous at $t = 0$.

Remark 2.10. Note that (1) and (2) are false if $\mathcal{C}^1$ is replaced by $\mathcal{C}^2$ or $\mathcal{O}^1(\omega)$ for any $\omega \in \mathcal{M}$ as shown by the construction in Theorem 2.5.

3 Concluding Remarks

Rosenthal [6] includes several applications of his main theorem. These applications pertain to various properties that a function can have at or near a point p_0. Such properties include existence of a tangent plane, existence of a limit, boundedness, maxima, and others. The applications have the form: a function has property P in case it has property P along every $\mathcal{C}^1$ convex curve (or every parametric $\mathcal{C}^n$ curve for any $n \geq 2$) through p_0 ; the function does not necessarily have property P if it has property P along every $\mathcal{C}^2$ curve through p_0. Theorem 2.7 can be used to show that the set of curves $\mathcal{C}^2$ can be replaced with the larger set $\mathcal{C}^{1,\alpha}$ and this set of curves is still not large enough to ensure property P for an arbitrary function. The examples needed are essentially the same as in [6].

Finally we note that the results in section 2 can be extended to functions of more than two variables. For example, the following can be proved.

Theorem 3.1. *$f(x, y, z)$ is continuous at $(0, 0, 0)$ if and only if one of the following holds.*

(1) for all functions $u, v \in \mathcal{C}^1$ with $u(0) = v(0) = 0$, $f(x, u(x), v(x))$ and $f(u(x), x, v(x))$, $f(u(x), v(x), x)$ are continuous $x = 0$.

(2) for some natural number n and all functions $x(t), y(t), z(t) \in \mathcal{C}^n$ with $x(0) = y(0) = z(0) = 0$, $f(x(t), y(t), z(t))$ is continuous at $t = 0$.

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