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SOME ESTIMATES OF COMMUTATORS

Abstract

By using the boundedness of the maximal and sharp operators on Morrey spaces, we have proved that the commutators $[M_p, b]$ and $[M^{\#}, b]$ are bounded on Morrey spaces $L^{q,\lambda}$ if and only if b is in BMO and the negative part of b is in L^∞.

1 Introduction

In [4], Coifman, Rochberg, and Weiss proved that a locally integrable function b in R^n is in BMO if and only if the commutator $[H, b]$,

$$[H, b](f) = H(bf) - bH(f)$$

is bounded in L^p, for some p (for all), $p \in (1, \infty)$.

Also in the L^p space setting, Bastero, Milman, and Ruiz in [2] showed that a locally integrable function b is in BMO and the negative part of b is in L^∞ if and only if the commutator $[M_p, b]$, defined by

$$[M_p, b] = M_p(bf) - bM_p(f)$$

is bounded in L^q for some q (for all), $q \in (p, \infty)$. In fact here, M_p can be replaced by a more generalized positive quasilinear operator.

The purpose of this work is devoted to the study of the relationship between the boundedness of $[M_p, b]([M^{\#}, b])$ on Morrey spaces and the function b. To the best of my knowledge, the setting of $L^{q,\lambda}$ for these problems is new, independent, and of particular interest.

Morrey space plays an important role in the study of regularity questions in PDE. And also in modern analysis, Morrey spaces can be a part of a family

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that includes L^p, BMO(the space of Bounded Mean Oscillations), and Hölder function spaces. Many people have studied this family of function spaces. For more details, we refer the interested readers to [3], [10], [11], [12], [14], [15] and references therein.

Let Q be a cube on R^n, $\lambda \in R$, and $f \in L^q_{loc}$. Then f is said to be in the function space $L^{q,\lambda}$ provided

$$\|f\|_{L^{q,\lambda}} = \sup_Q \left(\frac{1}{|Q|^\lambda} \int_Q |f(x)|^q dx \right)^{1/q} < \infty, \tag{1}$$

where the supremum is taken over all cubes on R^n and $|Q|$ is the volume of Q. It is well known that if $1 \le q < \infty$, then we have $L^{q,0} = L^q$ and $L^{q,1} = L^\infty$, if $\lambda < 0$, $L^{q,\lambda} = \{0\}$ and if $\lambda > 1$, $L^{q,\lambda}$ is the space of $\frac{(\lambda-1)}{q}-$Hölder continuous functions. The Morrey space is defined to be $L^{q,\lambda}$ when $0 < \lambda < 1$.

In the present work propositions 1-4 are basic facts about the Hardy-Littlewood maximal operators multiplied by a nonnegative function. From these propositions we know that the cancellation implied in the commutator $[b, M_p]$ is crucial. All these propositions and proofs are in section 2 along with some necessary background materials. Also main results of this note, Theorems 5-8, and their proofs are given in the section 3.

Throughout the whole paper all constants are denoted by C which may be different from each occurrence.

2 Preliminaries

For the sake of completeness, we recall the definitions and some properties we are going to use in our proofs. For a set $E \in R^n$ we denote the characteristic function of E by χ_E and $|E|$ the Lebesgue measure of E.

For a locally integrable function $f(t)$ and $1 \le p < \infty$, the Hardy-Littlewood maximal function is given by

$$M_p(f)(x) = \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q |f(t)|^p dt \right)^{1/p},$$

for all $x \in R^n$, where Q represents a cube with sides parallel to the coordinate axes. Note that for $p = 1$, $M_p = M$ is the classical Hardy-Littlewood maximal operator.

The Sharp operator is defined by

$$M^{\#}(f)(x) = f^{\#}(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(t) - f_Q| dt,$$

for all $x \in R^n$, where $f_Q = \frac{1}{|Q|}\int_Q f(t)dt$, the mean of f on Q.

If Q_0 is a fixed cube, then the Hardy-Littlewood maximal function relative to Q_0 is defined as

$$M_{p,Q_0}(f)(x) = \sup_{x\in Q\subset Q_0}\left(\frac{1}{|Q|}\int_Q |f(t)|^p dt\right)^{1/p},$$

for all $x \in Q_0$.

Remark As is well known, the sharp functions or operators were introduced by C. Fefferman and E.M. Stein.

Proposition 1. *Let* $b > 0$, $\phi \geq 0$, *and* $1 < p < q < \infty$. *Then*

$$\phi M_{p'}(b) \in L^\infty \Longrightarrow \phi M(bf) \in L^q, \forall f \in L^q$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Consider

$$\begin{aligned}
|\phi M(bf)| &= \phi \sup_{x\in Q}\frac{1}{|Q|}\int_Q b(t)|f(t)|dt \\
&\leq \phi\left(\sup_{x\in Q}\frac{1}{|Q|}\int_Q |b(t)|^{p'}dt\right)^{1/p'}\left(\sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(t)|^p dt\right)^{1/p} \\
&= \phi M_{p'}(b)(x) M_p(f)(x).
\end{aligned}$$

With $M_p(f) = (M(|f|^p)^{1/p}$, the desired result follows from the L^p boundedness of M in [5]. □

We can extend this result in $L^{p,\lambda}$ to have the following proposition.

Proposition 2. *Let* $b > 0$, $\phi \geq 0$, *and* $1 < p < q < \infty$. *Then*

$$\phi M_{p'}(b) \in L^\infty \Longrightarrow \phi M(bf) \in L^{q,\lambda}, \forall f \in L^{q,\lambda}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. By the proof of Proposition 1 and [3], we conclude the proof. □

Conversely we are not able to get $\phi M_{p'}(b) \in L^\infty$ but instead we get $\phi b \in L^\infty$ which is a little bit weaker than $\phi M_{p'}(b) \in L^\infty$ since $\phi b \leq \phi M_{p'}(b)$, a.e.

Proposition 3. *Let* $b > 0$, $\phi \geq 0$, *and* $1 < p < \infty$. *Then*

$$\phi M(bf) \in L^p, \forall f \in L^p \Longrightarrow \phi b \in L^\infty.$$

PROOF. For any cube Q_0, let $f = \chi_{Q_0}$. Then

$$\|f\|_p = |Q_0|^{1/p}.$$

And by the assumption

$$\|\phi M(b\chi_{Q_0}\|_p \leq C|Q_0|^{1/p},$$

that is,

$$\int \phi(x)^p (M(b\chi_{Q_0})(x))^p dx \leq C|Q_0|$$
$$\frac{1}{|Q_0|}\int_{Q_0} \phi(x)^p (M(b\chi_{Q_0})(x))^p dx \leq C$$
$$\frac{1}{|Q_0|}\int_{Q_0} \phi^p (M_{1,Q_0}(b)(x))^p dx \leq C$$
$$\frac{1}{|Q_0|}\int_{Q_0} (\phi b)^p dx \leq C.$$

Since Q_0 is arbitrary, Lebesgue's differentiation theorem implies

$$(\phi b)^p \in L^\infty \quad \text{i.e.} \quad \phi b \in L^\infty.$$

Here we have used the facts $M(b\chi_{Q_0}) = M_{1,Q_0}(b)$ and $|b| \leq M_{1,Q_0}(b), \forall x \in Q_0$. □

In the same manner, this result can be extended to $L^{p,\lambda}$.

Proposition 4. *Let $b > 0$, $\phi \geq 0$, and $1 < p < \infty$. Then*

$$\phi M(bf) \in L^{p,\lambda}, \forall f \in L^{p,\lambda} \Longrightarrow \phi b \in L^\infty.$$

PROOF. For any cube Q_0, choose $f = \chi_{Q_0}$, then $\|\chi_{Q_0}\|_{L^{p,\lambda}} = |Q_0|^{(1-\lambda)/p}$ Since $\phi M(bf) \in L^{p,\lambda}$, we have

$$\frac{1}{|Q_0|^\lambda}\int_{Q_0} (\phi M_{Q_0}(b))^p \, dx \leq C|Q_0|^{1-\lambda}$$

$$\frac{1}{|Q_0|}\int_{Q_0} (\phi b)^p \, dx \leq C.$$

Similar to the proof of proposition 3, $\phi b \in L^\infty$ follows. □

Observing all these, we have the following theorems.

3 Theorems and the proofs

In this section, we will give main theorems and their proofs.

Theorem 5. *Let b be a nonnegative BMO function. Then $[M_p, b]$ is bounded on $L^{q,\lambda}$, $1 \le p < q < \infty$.*

Proof. From [2] we know that under such assumption, $[M_p, b]$ is bounded on L^q, that is,

$$|[M_p, b](f)\|_{L^q} \le C\|f\|_{L^q}.$$

Let Q be a fixed cube and $g \in L^{q,\lambda}$. Then function $f = g\chi_Q/|Q|^{\lambda/q} \in L^q$, and

$$\|f\|_{L^q} = \left(\frac{1}{|Q|^\lambda}\int_Q |g(t)|^q dt\right)^{1/q}.$$

Also

$$\|[M_p, b](f)\|_{L^q} \ge \left(\int_Q |M_p(bf)(t) - b(t)M_p(f)(t)|^q dt\right)^{1/q}.$$

For $t \in Q$, we have

$$M_p(bf)(t) = \frac{1}{|Q|^{\lambda/q}} M_p(bg)(t)$$

and

$$M_p(f)(t) = \frac{1}{|Q|^{\lambda/q}} M_p(g)(t).$$

So

$$\|[M_p, b](f)\|_{L^q} \ge \left(\frac{1}{|Q|^\lambda}\int_Q |M_p(bg)(t) - b(t)M_p(g)(t)|^q dt\right)^{1/q}.$$

Therefore

$$\left(\frac{1}{|Q|^\lambda}\int_Q |M_p(bg)(t) - b(t)M_p(g)(t)|^q dt\right)^{1/q} \le C\left(\frac{1}{|Q|^\lambda}\int_Q |g(t)|^q dt\right)^{1/q}.$$

Hence

$$\|[M_p, b](g)\|_{L^{q,\lambda}} \le C\|g\|_{L^{q,\lambda}}.$$

□

Theorem 6. *Let b be a nonnegative BMO function. Then $[M^\#, b]$ is bounded on $L^{q,\lambda}$, $1 \le p < q < \infty$.*

PROOF. Let's consider

$$
\begin{aligned}
|[M^{\#}, b](f)(x)| &= |M^{\#}(bf)(x) - b(x)M^{\#}(f)(x)| \\
&= \left|\sup_{x\in Q} \frac{1}{|Q|}\int_Q |b(t)f(t) - (bf)_Q| dt - b(x) \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(t) - f_Q| dt\right| \\
&\leq \sup_{x\in Q} \frac{1}{|Q|}\int_Q |b(t)f(t) - b(x)f(t) - (bf)_Q + b(x)f_Q|\, dt \\
&\leq \sup_{x\in Q} \frac{1}{|Q|}\int_Q |b(t)f(t) - b(x)f(t)| dt + \sup_{x\in Q} \frac{1}{|Q|}\int_Q |b(x)f_Q - (bf)_Q| dt \\
&\leq 2\sup_{x\in Q} \frac{1}{|Q|}\int_Q |b(t) - b(x)||f(t)| dt \\
&\leq 2\sup_{x\in Q} \left(\frac{1}{|Q|}\int_Q |b(t) - b(x)|^{r'} dt\right)^{1/r'} \left(\frac{1}{|Q|}\int_Q |f(t)|^r dt\right)^{1/r} \\
&\leq 2\|b\|_{BMO} M_r(f)(x),
\end{aligned}
$$

where $p < r < q$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Therefore

$$\|[M^{\#}, b](f)\|_{L^{q,\lambda}} \leq C\|b\|_{BMO}\|M_r(f)\|_{L^{q,\lambda}} \leq C\|b\|_{BMO}\|f\|_{L^{q,\lambda}}.$$

Note that M_r is bounded on $L^{q,\lambda}$. □

Theorem 7. *Let b be a real valued, locally integrable function in R^n. Then the following are equivalent:*

(a) The commutator $[M_p, b]$ is bounded in $L^{q,\lambda}$, for some $1 \leq p < q < \infty$,

(b) $b \in BMO$ and $b^- \in L^\infty$,

(c) For some $q \in (1, \infty)$, we have

$$\sup_Q \frac{1}{|Q|}\int_Q |b(t) - M_{b,Q}(b)(t)|^q dt < \infty.$$

PROOF. We'll prove this theorem by the cycle (b) $\Longrightarrow$ (a) $\Longrightarrow$ (c) $\Longrightarrow$ (b).

For this inclusion (b) $\Rightarrow$ (a),we know that if b is in BMO, then $|b|$ is also in BMO. Considering the fact

$$|[M_p, b](f) - [M_p, |b|](f)| \leq 2b^- M_p(f)$$

and Theorem 5, we obtain (a).

To prove inclusion (a) $\Rightarrow$ (c), for a fixed cube Q, let $f(t) = \chi_Q(t) \in L^{q,\lambda}$, then $\|f\|_{L^{q,\lambda}} = |Q|^{\frac{1-\lambda}{q}}$. By (a) we have

$$\|[M_p, b](f)\|_{L^{q,\lambda}} \le C\|f\|_{L^{q,\lambda}} = C|Q|^{\frac{1-\lambda}{q}}.$$

Also since on Q, $M_p(b\chi_Q) = M_{p,Q}(b)$ and $M_p(\chi_Q) = \chi_Q$, we have

$$\begin{aligned}\|[M_p, b](f)\|_{L^{q,\lambda}} &\ge \left(\frac{1}{|Q|^\lambda}\int_Q |M_p(b\chi_Q)(t) - b(t)M_p(\chi_Q)(t)|^q dt\right)^{1/q} \\ &\ge \left(\frac{1}{|Q|^\lambda}\int_Q |M_{p,Q}(b)(t) - b(t)|^q dt\right)^{1/q}.\end{aligned}$$

Therefore

$$\left(\frac{1}{|Q|^\lambda}\int_Q |M_{p,Q}(b)(t) - b(t)|^q dt\right)^{1/q} \le C|Q|^{\frac{1-\lambda}{q}},$$

that is,

$$\frac{1}{|Q|}\int_Q |b(t) - M_{p,Q}(b)(t)|^q dt \le C.$$

Hence

$$\sup_Q \frac{1}{|Q|}\int_Q |M_{p,Q}(b)(t) - b(t)|^q dt < \infty.$$

As to (c) $\Rightarrow$ (b), the proof here is due to [2]. Let Q be a fixed cube, by (c) and Hölder inequality we have

$$\frac{1}{|Q|}\int_Q |b(t) - M_{p,Q}(b)(t)| dt \le \left(\frac{1}{|Q|}\int_Q |b(t) - M_{p,Q}(b)(t)|^q dt\right)^{1/q} \le C.$$

Let $E = \{x \in Q; b(x) \le b_Q\}$ and $F = \{x \in Q; b(x) > b_Q\}$, then the equality

$$\int_E |b(t) - b_Q| dt = \int_F |b(t) - b_Q| dt$$

is true. Since for $x \in E$ we have $b(x) \le b_Q \le M_{p,Q}(b)(x)$ and also

$$\begin{aligned}\frac{1}{|Q|}\int_Q |b(t) - b_Q| dt = \frac{2}{|Q|}\int_E |b(t) - b_Q| dt &\le \frac{2}{|Q|}\int_E |b(t) - M_{p,Q}(b)(t)| dt \\ &\le \frac{2}{|Q|}\int_Q |b(t) - M_{p,Q}(b)(t)| dt \le C,\end{aligned}$$

we know that $b \in BMO$.

To show that $b^- \in L^\infty$, note that $M_{p,Q}(b) \geq |b|$ in Q and therefore we have, in Q,

$$0 \leq b^- = |b| - b^+ \leq M_{p,Q}b - b^+ + b^- = M_{p,Q}(b) - b.$$

Combining this and (c) we see that for any cube Q,

$$(b^-)_Q \leq C.$$

Thus the boundedness of b^- follows from Lebesgue's differentiation theorem. $\square$

Theorem 8. *Let b be a real valued, locally integrable function in R^n. Then the following are equivalent:*

(a) The commutator $[M^\#, b]$ is bounded in $L^{q,\lambda}$, for some $1 \leq q < \infty$,

(b) $b \in BMO$ and $b^- \in L^\infty$,

(c) For some $q \in [1, \infty)$, we have

$$\sup_Q \frac{1}{|Q|} \int_Q |b(t) - 2(b\chi_Q)^\#(t)|^q dt < \infty.$$

PROOF. We'll show this theorem by the following order. (b) $\Longrightarrow$ (a) $\Longrightarrow$ (c) $\Longrightarrow$ (b).

To show (b) $\Rightarrow$ (a), note the facts that

$$|[M^\#, b](f) - [M^\#, |b|](f)| \leq 2M^\#(b^- f) + 2b^- M^\#(f),$$

and if $b \in BMO$, then $|b| \in BMO$, and $M^\#(f) \leq 2M(f)$ for all locally integrable functions f, (a) follows immediately.

In order to prove (a) $\Rightarrow$ (c), let Q be a fixed cube. We know that $(\chi_Q)^\# = \frac{1}{2}$, for all $x \in Q$. Assume that $f = \chi_Q \in L^{q,\lambda}$. Then by (a), we have

$$\|[M^\#, b](\chi_Q)\|_{L^{q,\lambda}} \leq C\|\chi_Q\|_{L^{q,\lambda}},$$

namely,

$$\left(\frac{1}{|Q|^\lambda} \int_Q |(b\chi_Q)^\#(t) - b(t)(\chi_Q)^\#(t)|^q dt\right)^{1/q} \leq C|Q|^{\frac{1-\lambda}{q}}$$

$$\frac{1}{|Q|} \int_Q |b(t) - 2(b\chi_Q)^\#(t)|^q dt \leq C.$$

So

$$\sup_Q \frac{1}{|Q|}\int_Q |b(t)-2(b\chi_Q)^{\#}(t)|^q dt < \infty.$$

Finally (c) $\Rightarrow$ (b), the proof here is like the proof of Theorem 7 and also due to [2].

For a cube Q, let $x \in Q$ and choose a cube Q_1 containing Q and with volume $|Q_1| = 2|Q|$. Then

$$\begin{aligned}\frac{1}{2|Q|}\int_Q \left|b(t)-\frac{1}{2}b_Q\right| dt + \frac{1}{4}|b_Q| &= \frac{1}{2|Q|}\left(\int_Q \left|b(t)-\frac{1}{2}b_Q\right| dt + \frac{1}{2}|Q_1\backslash Q||b_Q|\right)\\ &= \frac{1}{|Q_1|}\int_{Q_1} |b\chi_Q(t)-(b\chi_Q)_{Q_1}|dt\\ &\le (b\chi_Q)^{\#}(x).\end{aligned}$$

On the other hand

$$\begin{aligned}|b_Q| &\le \frac{1}{|Q|}\int_Q \left|b(t)-\frac{1}{2}b_Q\right| dt + \frac{1}{|Q|}\int_Q |\frac{1}{2}b_Q|dt\\ &= \frac{1}{|Q|}\int_Q \left|b(t)-\frac{1}{2}b_Q\right| dt + \frac{1}{2}|b_Q|,\end{aligned}$$

and so

$$\frac{1}{2}|b_Q| \le \frac{1}{|Q|}\int_Q \left|b(t)-\frac{1}{2}b_Q\right| dt.$$

Therefore, we get

$$|b_Q| \le 2(b\chi_Q)^{\#}(x), \forall x \in Q.$$

Now we are able to show that b is in BMO. Let $E = \{x \in Q; b(x) \le b_Q\}$, then

$$\begin{aligned}\frac{1}{|Q|}\int_Q |b(x)-b_Q|dx &= \frac{2}{|Q|}\int_E (b_Q - b(x))dx\\ &\le \frac{2}{|Q|}\int_E (2(b\chi_Q)^{\#}(x)-b(x))dx\\ &\le \frac{2}{|Q|}\int_E |2(b\chi_Q)^{\#}(x)-b(x)|dx\\ &\le \frac{2}{|Q|}\int_Q |2(b\chi_Q)^{\#}(x)-b(x)|dx \le C.\end{aligned}$$

To prove b is bounded, We start with the inequality

$$\forall x \in Q, \quad |b_Q| - b^+(x) + b^-(x) \le 2(b\chi_Q)^\#(x) - b(x).$$

Averaging on Q, we have

$$\begin{aligned}
|b_Q| - \frac{1}{|Q|}\int_Q b^+(x)dx + \frac{1}{|Q|}\int_Q b^-(x)dx &= \frac{1}{|Q|}\int_Q \left(|b_Q| - b^+(x) + b^-(x)\right) dx \\
&\le \frac{1}{|Q|}\int_Q \left(2(b\chi_Q)^\#(x) - b(x)\right) dx \\
&\le \frac{1}{|Q|}\int_Q \left|2(b\chi_Q)^\#(x) - b(x)\right| dx \\
&\le C.
\end{aligned}$$

Letting $|Q| \to 0$ with $x \in Q$, Lebesgue differentiation theorem assures that

$$2b^- = |b(x)| - b^+(x) + b^-(x) \le C,$$

and we are done. □

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