Real Analysis Exchange
Vol. 24(1), 1998/99, pp. 423–426

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NORM CONVERGENCE AND UNIFORM INTEGRABILITY FOR THE HENSTOCK-KURZWEIL INTEGRAL

Abstract

We show that a uniformly integrable, pointwise convergent sequence of Henstock-Kurzweil integrable functions converges in the Alexiewicz norm. In particular, this implies that a sequence satisfying the conditions in the Dominated Convergence Theorem is norm convergent.

The Dominated Convergence Theorem (DCT) for the Lebesgue (or McShane) integral immediately implies that the dominated sequence is convergent with respect to the L^1-norm ([Sw1] 3.2.16). Since the DCT for the Henstock-Kurzweil (gauge) integral allows for conditionally convergent integrals ([M] p. 89, 100), it is not the case that the DCT for the Henstock-Kurzweil integral immediately implies the convergence of the dominated sequence with respect to the Alexiewicz norm on the space of Henstock-Kurzweil integrable functions. However, in this note we show that the Uniform Henstock Lemma recently established by Lee, Chew and Lee ([LCL] Lemma 3) can be employed to establish the norm convergence of the dominated sequence in the DCT for the Henstock-Kurzweil integral. Indeed, we use the Uniform Henstock Lemma to show that a uniformly integrable, pointwise convergent sequence is convergent with respect to the Alexiewicz norm. The analogous result was established for the vector-valued McShane integral in [Sw2]; however, the techniques employed there are not applicable to the Henstock-Kurzweil integral.

Throughout this note we will employ the notation and definitions for the Henstock-Kurzweil integral given in [LPY]. Let $I = [a, b]$ be an interval in $\mathbb{R}$ and let $\mathcal{HK}(I)$ be the space of all functions which are Henstock-Kurzweil integrable over I. If $f \in \mathcal{HK}(I)$, the Alexiewicz norm of f is defined by $\|f\| = \sup\left\{\left|\int_a^x f\right| : a \le x \le b\right\}$ ([A], [LPY] 11.1). In contrast to the L^1-norm on the space of Lebesgue integrable functions, the space $\mathcal{HK}(I)$ is not complete with respect to the Alexiewicz norm ([LPY] 11.1).

Key Words: Henstock-Kurzweil integral, uniform integrability, Alexiewicz norm
Mathematical Reviews subject classification: 26A39
Received by the editors August 11, 1997

A sequence $\{f_k\}$ in $\mathcal{HK}(I)$ is said to be uniformly (Henstock-Kurzweil) integrable over I if for every $\varepsilon > 0$ there exists a gauge $\delta : I \to (0, \infty)$ such that

$$\left| \int_I f_k - \sum_{i=1}^{m} f_k(t_i) |I_i| \right| < \varepsilon$$

for every k whenever $\mathcal{D} = \{(I_i, t_i) : 1 \le i \le m\}$ is a δ-fine tagged partition of I and $|J|$ denotes the length of an interval J. It is known that if $\{f_k\} \subset \mathcal{HK}(I)$ is uniformly integrable over I and $\{f_k\}$ converges pointwise to the function $f : I \to \mathbb{R}$, then f is integrable over I and $\lim \int_I f_k = \int_I f$ ([G1] Theorem 2, [G2] 13.16). We use the Uniform Henstock Lemma to show that this result can be improved to show that the sequence $\{f_k\}$ converges to f in the Alexiewicz norm.

For the convenience of the reader we give a statement of the Uniform Henstock Lemma established in [LCL], Lemma 3.

Lemma 1. *Let $f \in \mathcal{HK}(I)$ and $\varepsilon > 0$. If δ is a gauge on I such that*

$$\left| \int_I f - \sum_{k=1}^{m} f(t_k) |I_k| \right| < \varepsilon$$

for every δ-fine tagged partition $\mathcal{D} = \{(I_i, t_i) : 1 \le i \le m\}$, then

$$\left| \sum_{i=1}^{m} \left\{ f(t_i) |I_i \cap J| - \int_{I_i \cap J} f \right\} \right| \le 3\varepsilon$$

and

$$\sum_{i=1}^{m} \left| f(t_i) |I_i \cap J| - \int_{I_i \cap J} f \right| \le 6\varepsilon$$

for every subinterval J of I and every δ-fine partial tagged partition $\{(I_i, t_i) : 1 \le i \le m\}$ of I.

The lemma is stated differently in [LCL], but the proof given establishes Lemma 1 which is a more convenient form when dealing with uniformly integrable sequences.

Theorem 1. *Let $\{f_k\} \subset \mathcal{HK}(I)$ be uniformly integrable and suppose that $\{f_k\}$ converges pointwise to the function $f : I \to \mathbb{R}$. Then f is integrable and $\|f_k - f\| \to 0$.*

PROOF. By the remarks above f is integrable so we may assume that $f = 0$. Let $\varepsilon > 0$. Let δ be a gauge on I such that

$$\left| \int_I f_k - \sum_{i=1}^{m} f_k(t_i) |I_i| \right| < \varepsilon$$

for every k whenever $\{(I_i, t_i) : 1 \leq i \leq m\} = \mathcal{D}$ is a δ-fine tagged partition of I. Fix such a δ-fine tagged partition of I. Choose n such that $k \geq n$ implies $|f_k(t_i)| < \varepsilon/m\,|I_i|$ for $1 \leq i \leq m$. Suppose that J is an arbitrary subinterval of I and $k \geq n$. From Lemma 1, we have

$$\begin{aligned} \left| \int_J f_k \right| &\leq \left| \sum_{i=1}^{m} \left\{ \int_{I_i \cap J} f_k - f_k(t_i) |I_i \cap J| \right\} \right| + \sum_{i=1}^{m} |f_k(t_i)| \, |I_i \cap J| \\ &\leq 3\varepsilon + \varepsilon = 4\varepsilon. \end{aligned}$$

Hence, if $k \geq n$, then $\|f_k\| \leq 4\varepsilon$.

The proof of the DCT for the Henstock-Kurzweil integral given by McLeod yields the following version of the DCT ([M] p. 89, 100).

Theorem 2. *Let $\{f_k\} \subset \mathcal{HK}(I)$, $g \in \mathcal{HK}(I)$ and suppose $\{f_k\}$ converges pointwise to f on I. If $|f_k - f_j| \leq g$ on I for every k, j, then f is integrable and $\{f_k\}$ is uniformly integrable.*

It follows from Theorem 2 that the sequence $\{f_k\}$ converges to f in the Alexiewicz norm establishing the analogue of the conclusion in the DCT for the Lebesgue integral and the L^1-norm. [It should be noted that the domination hypothesis in Theorem 3 allows the functions in the sequence $\{f_k\}$ to be conditionally integrable ([M] p. 89) in contrast with the usual domination hypothesis found in the DCT for the Lebesgue integral ([Sw] 3.2.16).]

The Uniform Henstock Lemma is established for intervals in $\mathbb{R}^n$ in [LCL], and the proof of Theorem 2 is also valid for $\mathbb{R}^n$ with only the usual complications of notation in $\mathbb{R}^n$.

Also, it is easy to extend the definition of the Henstock-Kurzweil integral to functions with values in a Banach space. Henstock's Lemma is still valid in this setting and the first inequality in Lemma 1 can also be obtained from the proof of the Uniform Henstock Lemma in [LCL] (however, see [C] for the second inequality). The proof of Theorem 2 then carries forward to this setting.

In conclusion we also note another application of the Uniform Henstock Lemma. Namely, the proof of Theorem 2 shows that the step functions are dense in $\mathcal{HK}(I)$ with respect to the Alexiewicz norm [a step function is a linear combination of characteristic functions of intervals].

Theorem 3. *Given $f \in \mathcal{HK}(I)$ and $\varepsilon > 0$, there is a step function g such that $\|f - g\| \leq \varepsilon$.*

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