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MONOTONE NORMS ON $C(\Omega)$ AND MULTIPLICATIVE FACTORS

Abstract

Let $C(\Omega)$ be the algebra of continuous complex-valued functions on a topological space Ω and let ρ be a function norm on $C(\Omega)$. We give necessary and sufficient conditions on the set $A_\rho = \{f \in C(\Omega) : \rho(f) < \infty\}$ to be an algebra. Also, we prove that every complete function norm is quasi-submultiplicative provided A_ρ is an algebra and we give a characterization of the best multiplicative factor of ρ. Finally we characterize the infinity norm and we prove that every quasi-submultiplicative function norm on $C(\Omega)$ is equivalent to the infinity norm.

1 Introduction

Let Ω be a topological space and let $C(\Omega)$ be the algebra of continuous complex-valued functions. In a similar way as it was introduced in [3] we are going to consider a *function norm* ρ on $C(\Omega)$, i.e., a function $\rho : C(\Omega) \to [0, \infty]$ which satisfies the usual properties of a norm, including the monotonicity condition

$$f, g \in C(\Omega), |f| \leq |g| \Rightarrow \rho(f) \leq \rho(g).$$

It follows immediately from the definition that $\rho(|f|) = \rho(f)$ for all $f \in C(\Omega)$.

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Throughout this paper A_ρ will denote the following subspace of $C(\Omega)$,

$$A_\rho = \{f \in C(\Omega) : \rho(f) < \infty\},$$

and $\mathcal{B}(\Omega)$ will denote the space of bounded complex-valued functions on Ω. We are interested in giving necessary and sufficient conditions on $C(\Omega)$ for existence of a function norm such that A_ρ is an algebra. This problem was studied for other spaces in [1], and later in [2] and it is related to the existence of *submultiplicative norms*, see [4], [5] and [6].

A norm $\rho : C(\Omega) \to [0, \infty]$ will be called *σ-subadditive* if for all sequences of functions $f_n \in C(\Omega)$, $f_n \geq 0$ and $\sum_1^\infty f_n \in C(\Omega)$ it follows that

$$\rho\Big(\sum_1^\infty f_n\Big) \leq \sum_1^\infty \rho(f_n).$$

We will say that a norm ρ on $C(\Omega)$ is *complete* if (A_ρ, ρ) is complete. It is not difficult to see that every complete function norm on $C(\Omega)$ is σ-subadditive. See for example [3]. We will say that a norm ρ is *quasi-submultiplicative* if there exists a constant $K > 0$ such that

$$\rho(fg) \leq K\rho(f)\rho(g), \tag{1}$$

for all $f, g \in C(\Omega)$. In this case we will say that K is a *multiplicative factor* of ρ. The infimum of all multiplicative factors of ρ it is called the *best multiplicative factor of ρ*. Obviously given a quasi-submultiplicative function norm ρ, with $A_\rho \neq \{0\}$, its best multiplicative factor M is again a multiplicative factor of ρ, in particular $M > 0$.

One of our main results states that if ρ is a σ-subadditive function norm on $C(\Omega)$ and A_ρ is an algebra, A_ρ can not contains an unbounded function. The following example shows that there exist σ-subadditive function norms on $C(\Omega)$ where the subspace A_ρ admits unbounded functions. Let Ω be the interval $(0, 1)$ and

$$\rho(f) = \Big(\int_\Omega f^2 \, dx\Big)^{1/2} = \|f\|_2,$$

where dx stands for Lebesgue measure. Clearly, ρ is a σ-subadditive function norm on $C((0, 1))$. However there exist unbounded square integrable continuous functions on $(0, 1)$.

We will show that a function norm ρ is quasi-submultiplicative provided it is a complete norm function and A_ρ is an algebra. For the function norm ρ we are interested in obtaining an alternative, expression for the best multiplicative factor of ρ which is easier to handle. Along the way we will give a characterization of the best multiplicative factor analogous to the one in [2] in the case of function norms defined on measurable spaces. More explicitly, we prove for any non trivial ($A_\rho \neq \{0\}$) quasi-submultiplicative norm function ρ, that its best multiplicative factor is given by

$$M_\rho = \sup\{\|f\|_\infty : f \in C(\Omega), \rho(f) \leq 1\}, \tag{2}$$

where $\|.\|_\infty$ denote the infinity norm. Note that M_ρ is a well defined number in $[0, \infty]$ for any function norm ρ and because M_ρ is a finite number it will characterize quasi-submultiplicative norms.

Finally, we will give a simple characterization of the infinity norm and we will prove that for every finite complete function norm ρ on $C(\Omega)$ the multiples $\lambda\rho$ are submultiplicative norms for $\lambda \geq M_\rho$. In general it is easier to decide that a norm is monotone and complete. Then the previous result gives us a method to obtain submultiplicative norms.

2 Results and Proofs

Theorem 1. *Let ρ be a function norm on $C(\Omega)$.*

(a) If ρ is σ-subadditive, then A_ρ is an algebra if and only if $A_\rho \subset \mathcal{B}(\Omega)$,

(b) If $A_\rho = C(\Omega)$, then $C(\Omega) \subset \mathcal{B}(\Omega)$.

(c) If ρ is quasi-submultiplicative, then $A_\rho \subset \mathcal{B}(\Omega)$.

Proof. (a) We assume that A_ρ is an algebra. Suppose that there exists an unbounded function $f \in A_\rho$. Since $\rho(|f|) = \rho(f)$ we assume that $f \geq 0$. The next argument is similar to the one used in Theorem 1 of [4]. Thus we get a sequence of elements $t_n \in \Omega$ such that $f(t_{n+1}) > f(t_n) + 3$ for each $n \in \mathbb{N}$, $f(t_1) > 2$ and $\frac{f(t_n)}{n^2}$ tends to infinity. Let (I_n) be the sequence of pairwise disjoint closed intervals $I_n = [f(t_n)-1, f(t_n)+1]$ and for each $n \in \mathbb{N}$ we choose a continuous function $g_n : \mathbb{R} \to \mathbb{R}$ with $\text{supp}(g_n) \subset I_n g_n \geq 0$ and $\|g_n\|_\infty = 1$. Let $h_n(t) = g_n(f(t))$. Then $h_n \in C(\Omega)$, $0 \leq h_n(t) \leq f(t)$ for all $t \in \Omega$, so $h_n \in A_\rho$ and $(fh_n)(t) \geq (f(t_n) - 1)h_n(t)$ for all $t \in \Omega$. Thus $0 < \rho(h_n) < \infty$. Let $g = \sum_1^\infty \frac{h_n}{n^2\rho(h_n)}$. Clearly $g \in C(\Omega)$. Since ρ is σ-subadditive, we have that $g \in A_\rho$. Then $fg \in A_\rho$. On the other hand, by the monotonicity we get

$$\rho(fg) \geq \frac{\rho(fh_n)}{n^2\rho(h_n)} \geq \frac{f(t_n) - 1}{n^2},$$

which is a contradiction.

Assume now $A_\rho \subset \mathcal{B}(\Omega)$, f and g belong to A_ρ. Thus we have $\rho(fg) \le \|f\|_\infty \rho(g) < \infty$. Therefore $fg \in A_\rho$.

(b) Suppose that there exists an unbounded function $f \in C(\Omega)$. By defining a function g as in (a), we obtain as before a contradiction.

(c) Suppose that there exists an unbounded function $f \in A_\rho$. We choose g_n and h_n as in (a). Then for some $K > 0$ we get $f(t_n - 1)\rho(h_n) \le \rho(fh_n) \le K\rho(f)\rho(h_n)$ and we obtain that $f(t_n - 1) \le K\rho(f)$, which is a contradiction. □

Remark. We note that part (c) of Theorem 1, is not a consequence of [4] or [5], because they used that $A_\rho = C(\Omega)$. Also, the example given in the introduction allows us to observe that monotonicity does not implies quasi-submultiplicative. In fact, ρ is not quasi-submultiplicative, otherwise by (c) of Theorem 1, the set A_ρ should not admit an unbounded function, which is false. On the other hand there exist non-monotone quasi-submultiplicative norms. In order to see this, let ρ be the Minkowski's functional associated to a bounded balanced convex absorbing set P in $\mathbb{R}^2$. If in addition P is a closed set in $\mathbb{R}^2$, it is not difficult to see that ρ is monotone if and only if P is symmetric, i.e., if $(x_1, x_2) \in P$, then $(\epsilon_1 x_1, \epsilon_2 x_2) \in P$ where $\epsilon_i = \pm 1, i = 1, 2$. Then if we consider a set P nonsymmetric, the norm ρ is not monotone. However it is quasi-submultiplicative, since ρ is equivalent to the submultiplicative norm $\|\cdot\|_\infty$.

In the remainder of this section we study existence and characterization of multiplicative factors.

Theorem 2. *Let ρ be a function norm on $C(\Omega)$.If any of the two conditions holds*

(a) ρ is a complete norm and A_ρ is a subalgebra of $C(\Omega)$, or

(b) $A_\rho = C(\Omega)$ where Ω is a T_1-space with a dense set of isolated points without accumulation points.

Then there exists a constant K such that $\|f\|_\infty \le K\rho(f)$, for all $f \in C(\Omega)$.

Proof. Suppose that (a) holds and the theorem is false. Then there is a non-negative function sequence f_n, with $\|f_n\|_\infty = a_n, \rho(f_n) = 1$ and $\sum_1^\infty \frac{1}{(a_n)^{\frac{1}{2}}} < \infty$. We can assume without lost of generality that $a_1 \ge 4$ and $a_{n+1} > a_n$ for all $n \in \mathbb{N}$. Let t_n a sequence in Ω be such that $f_n(t_n) > \frac{a_n}{2}$, and set $J_n = [f_n(t_n) - 1, f_n(t_n) + 1]$. For each $n \in \mathbb{N}$, let g_n be a non negative function in $C(\mathbb{R})$ with $\text{supp}(g_n) \subset J_n$, $g_n(f_n(t_n)) = \frac{a_n}{4} = \|g_n\|_\infty$. We define a function

$h_n \in C(\Omega)$ by $h_n(t) = g_n(f_n(t))$. Given $t \in \Omega$, if $f_n(t) \in J_n$, then

$$f_n(t) \geq f_n(t_n) - 1 > \frac{a_n}{2} - 1 \geq \frac{a_n}{4} = \|g_n\|_\infty \geq g_n(f_n(t)) = h_n(t),$$

while $f_n(t) \notin J_n$, implies that $h_n(t) = 0$. Therefore $h_n \leq f_n$ and hence $\rho(h_n) \leq \rho(f_n) = 1$.

As $\sum_{n=1}^{\infty} \frac{\rho(h_n)}{\sqrt{a_n}} \leq \sum_1^{\infty} \frac{1}{/\sqrt{a_n}} < \infty$, the function $s_k := \sum_{n=1}^{k} \frac{h_n}{\sqrt{a_n}}$ belongs to A_ρ and (s_k) is a Cauchy sequence. Since A_ρ is a complete space, there exists $s \in A_\rho$ such that $\rho(s_k - s) \to 0$. Moreover, as in part (b) of the proof of theorem 4.8 in [3], we have $s \geq s_k$, for every k. Then

$$\|s\|_\infty \geq \|s_k\|_\infty \geq \frac{\|h_k\|_\infty}{\sqrt{a_k}} \geq \frac{\sqrt{a_k}}{4} \to \infty.$$

Therefore s is not bounded, contrary to (a) of Theorem 1.

Now we assume (b) and suppose the theorem is false. Then there exists a sequence of functions $f_n \in A_\rho$, $f_n \geq 0$ such that $\rho(f_n) = 1$ and $\|f_n\|_\infty \to \infty$ for $n \to \infty$. Thus we can get a sequence of isolated points t_n such that $f_n(t_n) \to \infty$ and t_n has no accumulation points. If δ_n is the characteristic function of the unitary set $\{t_n\}$, then the function δ_n is continuous. We let $h = \sum_1^\infty f_n(t_n)\delta_n$. Since the set $\{t_n : n \in \mathbb{N}\}$ has no accumulation points, it follows that $h \in C(\Omega)$. On the other hand, $\|h\|_\infty \geq f_n(t_n)$ for all $n \in \mathbb{N}$. Therefore $h \notin B(\Omega)$, contrary to part (b) of Theorem 1. □

In particular, the hypothesis (b) of Theorem 2 holds on Ω when $\Omega = \mathbb{Z}$, the set of integers with the discrete topology.

Corollary 3. *Let ρ be a function norm on $C(\Omega)$.*

(a) If ρ is complete, then A_ρ is a subalgebra of $C(\Omega)$ if and only if ρ is quasi-submultiplicative.

(b) If Ω is a T_1 space with a dense set of isolated points without accumulation points, such that $A_\rho = C(\Omega)$, then ρ is quasi-submultiplicative.

PROOF. We only prove (a) since (b) follows by analogous arguments. Suppose that A_ρ is a subalgebra of $C(\Omega)$. Let $f, g \in A_\rho$. By Theorem 2 there exists a constant M such that $\|h\|_\infty \leq M\rho(h)$ for all $h \in C(\Omega)$. It follows that $\rho(fg) \leq \|f\|_\infty \rho(g) \leq M\rho(f)\rho(g)$, i.e. ρ is quasi-submultiplicative. The remaining of the statement is obvious. □

The condition that ρ be complete cannot be substituted by the weaker condition of σ-subadditive, though $A_\rho = C(\Omega)$, as the next example shows.

Example. Let $\Omega = [0,1]$ and define a function norm ρ on $C(\Omega)$ by $\rho(f) = \int_\Omega |f| d\mu = \|f\|_1$. Clearly, $A_\rho = C(\Omega)$ and ρ is σ-subadditive. We can construct a sequence $(f_n) \in C(\Omega)$ such that $\frac{\|f_n\|_2}{\|f_n\|_1}$ is arbitrarily large. Thus there is no constant K such that $\rho(f^2) \leq K(\rho(f))^2$ for all $f \in C(\Omega)$. Consequently ρ is not quasi-submultiplicative.

Next we will give a characterization of the best multiplicative factor. If $f \in C(\Omega)$and K is a nonnegative real number, we consider the following subset of Ω, $A(f,K) = \{t \in \Omega : f(t) > K\rho(f)\}$.

Lemma 4. *Let ρ be a quasi-submultiplicative function norm on $C(\Omega)$ and let $f \in C(\Omega)$. If K is a multiplicative factor of ρ and $A(f,K)$ is nonempty, then there exists a function $b \in A_\rho, b \neq 0$ such that $\rho(bf) = K\rho(b)\rho(f)$.*

PROOF. We may assume without lost of generality that $f \geq 0$. Since $A(f,K) \neq \emptyset$, the function $f \in A_\rho$ is by Theorem 1 a bounded function. Now set

$$r = \inf\{f(t) : t \in A(f,K)\}.$$

We have two cases, $r > K\rho(f)$ or $r = K\rho(f)$.
In the first case, we take a nonnegative function $g \in C(\mathbb{R})$ such that $g(x) = 0$ for $x \leq K\rho(f)$ and $\|g\|_\infty = g(r) = r$. Now, we define $b(t) = g(f(t))$ for $t \in \Omega$. Clearly $b \in C(\Omega)$ and $b \neq 0$. If $t \in A(f,K)$, we have $K\rho(f) \leq (f)(t)$, otherwise we have $b(t) = 0$. Thus $(bf)(t) \geq K\rho(f)b(t)$ for all $t \in \Omega$.
In the second case there is $t_0 \in \Omega$ such that $K\rho(f) < f(t_0) \leq \|f\|_\infty$. Here we choose a nonnegative function $g \in C(\mathbb{R})$ with $\text{supp}(g) \subset [r, \|f\|_\infty]$, $g(f(t_0)) \neq 0$ and $\|g\|_\infty = r$. We define a function b in $C(\Omega)$ by $b(t) = g(f(t))$. Then if $t \in A(f,K)$, we have $K\rho(f) < f(t) \leq \|f\|_\infty$, while for $t \notin A(f,K)$ we have $b(t) = 0$. Again we get $K\rho(f)b(t) \leq (bf)(t)$ for all $t \in \Omega$.

Thus, in both cases we obtain a function $b \in A_\rho, b \neq 0$ such that $(bf)(t) \geq K\rho(f)b(t)$ for all $t \in \Omega$. Finally, since ρ is monotone and K is a multiplicative factor we obtain $K\rho(f)\rho(b) \leq \rho(bf) \leq K\rho(f)\rho(b)$, and this concludes the proof. □

Theorem 5. *Let ρ a quasi-submultiplicative function norm on $C(\Omega)$ with $A_\rho \neq \{0\}$. Then the best multiplicative factor is given by (2).*

PROOF. Since $A_\rho \neq \{0\}$, it is easy to see that there exists a best multiplicative factor and it is given by $M = \sup\{\rho(fg) : \rho(f) \leq 1, \rho(g) \leq 1\}$ and $M > 0$. Now $\rho(fg) \leq \|f\|_\infty \rho(g)$ for all $f, g \in A_\rho$ which implies that $M \leq M_\rho$. We are going to show that $M \geq M_\rho$. Let $f \neq 0$ be a function in A_ρ and $\epsilon > 0$. Then the set $A(f, M+\epsilon)$ is empty. In fact if this where not so, by Lemma 4 there exists $b \in A_\rho, b \neq 0$ such that $\rho(bf) = (M+\epsilon)\rho(b)\rho(f)$, which is contradiction.

Hence, we must have $A(f, M+\epsilon) = \emptyset$. Therefore $\|f\|_\infty \leq (M+\epsilon)\rho(f)$. Thus $M_\rho \leq M + \epsilon$, for every $\epsilon > 0$. □

Corollary 6. *Let ρ be a function norm on $C(\Omega)$ with $A_\rho \neq \{0\}$ and which satisfies the conditions (a) or (b) of Theorem 2. Then ρ is quasi-submultiplicative and the best multiplicative factor is given by (2).*

PROOF. It follows immediately from Corollary 3 and Theorem 5. □

Corollary 7. *Let ρ be a norm on $C(\Omega)$ and $A_\rho \neq \{0\}$. Then ρ is quasi-submultiplicative (submultiplicative) if and only if $M_\rho < \infty$, ($M_\rho \leq 1$). Moreover if $M_\rho < \infty$ and $\lambda > 0$, the function norm $\lambda\rho$ is submultiplicative if and only if $\lambda \geq M_\rho$.*

PROOF. If $M_\rho < \infty$, ($M_\rho \leq 1$) the monotonicity of ρ implies that ρ is quasi-submultiplicative (submultiplicative). The converse statement follows by Theorem 5. Observe that $\lambda > 0$ and $A_\rho \neq \{0\}$. Then $M_{\lambda\rho} = \frac{1}{\lambda} M_\rho$. Thus the proof is completed. □

Theorem 8. *(a) Let ρ be a function norm on $C(\Omega)$. If $1 \in A_\rho$ and M_ρ satisfies $\rho(1)M_\rho = 1$, then $\|f\|_\infty = M_\rho\rho(f)$, for all $f \in C(\Omega)$.*

(b) The infinity norm is the unique submultiplicative function norm on $C(\Omega)$ such that $\rho(1) = 1$.

(c) Every quasi-submultiplicative function norm ρ on $C(\Omega)$ such that $1 \in A_\rho$, is equivalent to infinity norm.

PROOF. Since $\|f\|_\infty \leq M_\rho\rho(f) \leq M_\rho\rho(1)\|f\|_\infty$, we have (a).
Now, by Theorem 5 the best multiplicative factor for a quasi-submultiplicative function norm ρ is given by M_ρ. As ρ is submultiplicative $M_\rho \leq 1$. On the other hand, since $\rho(1) \leq M_\rho(\rho(1))^2$ and $\rho(1) = 1$, we have $M_\rho \geq 1$. Thus, (b) follows from (a). Finally (c) is a direct consequence of Theorem 5 and of the monotonicity of ρ. □

References

[1] R. Arens, M. Goldberg and W. A. Luxemburg, *multiplicative factors for function norms*, J. Math. Anal. Appl., **177**, (1993), 368–385.

[2] H. H. Cuenya, *multiplicative factors for function seminorms*, J. Math. Anal. Appl., **195** (1995), 440–448.

[3] W. A. J. Luxemburg, A. C. Zaanen, *Notes on Banach functions spaces, I*, Indag.Math., **25** (1963), 135–147.

[4] M. J. Meyer, *Some algebras without submultiplicative norms or positive functionals*, Studia Math., **116(3)** (1995), 299–302.

[5] A. R. Pruss, *A remark on non existence of an algebra norm for the algebra of continuous functions on a topological space admitting an unbounded continuous function*, Studia Math., **116 (3)** (1995), 295–297.

[6] B. Yood, *On the non existence of norms for some algebras of functions*, Studia Math., **111(1)** (1994) 97–101.