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UPPER POROSITY WITH RESPECT TO MEASURES

Abstract

For subsets of a separable metric space X we introduce the notion of upper porosity with respect to a Borel regular probabilistic measure μ on X (called μ-upper porosity) that generalizes the concept of upper porosity of the measure μ. We explore several natural definitions and further provide a definition of even more general type of μ-upper porosity given by suitable porosity functions. As the main consequence of achieved results concerning general μ-upper porosities we get that every σ-μ-upper porous set can be decomposed to a σ-strongly upper porous set and a μ-null set.

1   Introduction.

Whereas the basic ideas concerning the notion of set porosity date back to around 1920 to works of A. Denjoy, the porosity of measures is a relatively new notion. The lower porosity of measures was introduced by J.-P. Eckmann, E. Järvenpää and M. Järvenpää in [1], the upper porosity of measures by M. E. Mera and M. Morán in [4].

Using the same idea as in [4], for suitable measures μ on a separable metric space X we define the notion of μ-upper porosity for subsets of X. In our concept, the upper porosity of the measure μ coincides with the μ-upper porosity of the space X. Further, we explore other natural definitions similar to μ-upper porosity and generalize this notion to μ-(g)-upper porosity where g is a suitable porosity function.

Using the ideas of [5], we prove that every σ-μ-(g)-upper porous set is a union of σ-(g)-upper porous and μ-null set. Then, using a result of [5] about approximation of σ-upper porous sets, we prove that a set is σ-μ-upper porous if and only if it is a union of a σ-strongly upper porous set and a μ-null set.

2   Basic Definitions.

Let X be a metric space. We will say that μ is a *Borel regular measure* on X if it is a complete measure for which all Borel sets are measurable and

$$\mu(M) = \sup\{\mu(F) : F \subset M, F \text{ is closed}\}$$

for every μ-measurable set M.

In the rest of this article, let (X, ϱ) be a fixed nonempty separable metric space and let μ be a Borel regular probabilistic measure on the space X.

We denote by $B(x, r)$ the open ball with center $x \in X$ and radius $r > 0$.

By μ^* we denote the outer measure corresponding to the measure μ. Thus $\mu^*(A) = \inf\{\mu(M) : A \subset M, M \ \mu\text{-measurable}\}$ for arbitrary $A \subset X$.

We denote by $\mathcal{G}$ the system of all real functions g with the following properties:

- $g(0) = 0$,

- g is increasing and continuous on $[0, h_1)$ for some $h_1 > 0$,

- there exist $A > 0$ and $h_2 > 0$ such that $g(x) \geq Ax$ for $x \in [0, h_2)$.

Especially,

$$g \in \mathcal{G} \Rightarrow \exists K > 0 \ \exists h > 0 \text{ such that } g^{-1}(x) \leq Kx \text{ for } x \in [0, h). \qquad (1)$$

The most important examples of such functions are $g(x) = x^\alpha$, $x \in [0, \infty)$, with $0 < \alpha \leq 1$.

In the following definitions, we remind classical notions of different types of upper porosities and σ-upper porosities of sets as well as introduce new notions of μ-upper porosities and σ-μ-upper porosities for subsets of X. Since they always follow the same definition scheme (porosity of the set at some point, porosity of the set and σ-porosity of the set), some of them are only briefly sketched.

Definition 2.1. (cf. [5, Definition 1.1]) Let $A \subset X$, $x \in X$ and $r > 0$. Let $\gamma(A, x, r)$ denote the supremum of numbers $s > 0$ for which there exists a point $z \in X$ such that $\varrho(x, z) + s \leq r$ and $B(z, s) \cap A = \emptyset$ (we put $\sup \emptyset = 0$).

(i) The set A is called *upper porous at x* if $\limsup\limits_{r \to 0_+} \frac{\gamma(A,x,r)}{r} > 0$.

(ii) The set A is called *upper porous* if it is upper porous at each point in A.

(iii) The set A is called *strongly upper porous* if $\limsup\limits_{r \to 0_+} \frac{\gamma(A,x,r)}{r} = \frac{1}{2}$ for every $x \in A$.

(iv) The set A is called *σ-upper porous* (resp. *σ-strongly upper porous*) if it is a countable union of upper porous (resp. strongly upper porous) sets.

Definition 2.2. (cf. [6, Definition 2.33]) For $g \in \mathcal{G}$, replacing $\frac{\gamma(A,x,r)}{r}$ by $\frac{g(\gamma(A,x,r))}{r}$ and upper porous by (g)-upper porous in Definition 2.1, we get notions of *(g)-upper porous set at x*, *(g)-upper porous set* and *σ-(g)-upper porous set*.

Remark 2.3. [7, Proposition 4.2] Let $0 < q < p < 1$, $M \subset X$ and denote $g_1(x) = x^q$, $g_2(x) = x^p$ for $x \in [0, \infty)$. Then M is σ-(g_1)-upper porous if and only if M is σ-(g_2)-upper porous.

Definition 2.4. Let $A \subset X$, $x \in X$, $r > 0$ and $\varepsilon > 0$. Let $\gamma(\mu, A, x, r, \varepsilon)$ denote the supremum of numbers $s > 0$ for which there exists a point $z \in X$ such that $\varrho(x,z) + s \leq r$ and $\mu^*(A \cap B(z,s)) \leq \varepsilon\mu(B(x,r))$ (we put $\sup \emptyset = 0$).

(i) The set A is called *μ-upper porous at x* if $\lim\limits_{\varepsilon \to 0_+} \limsup\limits_{r \to 0_+} \frac{\gamma(\mu,A,x,r,\varepsilon)}{r} > 0$.

(ii) The set A is called *μ-upper porous* if it is μ-upper porous at each point in A.

(iii) The set A is called *σ-μ-upper porous* if it is a countable union of μ-upper porous sets.

Definition 2.5. For $g \in \mathcal{G}$, replacing $\frac{\gamma(\mu,A,x,r,\varepsilon)}{r}$ by $\frac{g(\gamma(\mu,A,x,r,\varepsilon))}{r}$ and μ-upper porous by μ-(g)-upper porous in Definition 2.4, we get notions of *μ-(g)-upper porous set at x*, *μ-(g)-upper porous set* and *σ-μ-(g)-upper porous set*.

Remark 2.6. (i) The limits over $\varepsilon \to 0_+$ in Definitions 2.4 (resp. 2.5) exist since the functions

$$\varepsilon \mapsto \limsup\limits_{r \to 0_+} \frac{\gamma(\mu,A,x,r,\varepsilon)}{r} \quad \left(\text{resp. } \varepsilon \mapsto \limsup\limits_{r \to 0_+} \frac{g(\gamma(\mu,A,x,r,\varepsilon))}{r} \text{ for every } g \in \mathcal{G}\right)$$

are nonnegative and nondecreasing on $(0, \infty)$. Both these limits are non-negative, the first one of them is bounded from above by 1, the second one can take on the value $+\infty$ for some $g \in \mathcal{G}$.

(ii) Denote

$$\overline{p_\mu}(A, x) = \lim_{\varepsilon \to 0_+} \limsup_{r \to 0_+} \frac{\gamma(\mu, A, x, r, \varepsilon)}{r}$$

and

$$\overline{p_{\mu,(g)}}(A, x) = \lim_{\varepsilon \to 0_+} \limsup_{r \to 0_+} \frac{g(\gamma(\mu, A, x, r, \varepsilon))}{r}$$

for $A \subset X$, $x \in X$ and $g \in \mathcal{G}$. Then

$$\overline{p_\mu}(A, x) = \lim_{k \to \infty} \lim_{l \to \infty} \sup \left\{ \frac{\gamma(\mu, A, x, r, \varepsilon_k)}{r} : 0 < r < r_l \right\}$$

and

$$\overline{p_{\mu,(g)}}(A, x) = \lim_{k \to \infty} \lim_{l \to \infty} \sup \left\{ \frac{g(\gamma(\mu, A, x, r, \varepsilon_k))}{r} : 0 < r < r_l \right\}$$

for arbitrary decreasing sequences of positive numbers $\{\varepsilon_k\}_{k=1}^\infty$, $\{r_l\}_{l=1}^\infty$ such that $\lim_{k \to \infty} \varepsilon_k = \lim_{l \to \infty} r_l = 0$, since the functions

$$\varepsilon \mapsto \limsup_{r \to 0_+} \frac{\gamma(\mu, A, x, r, \varepsilon)}{r} \text{ and } \varepsilon \mapsto \limsup_{r \to 0_+} \frac{g(\gamma(\mu, A, x, r, \varepsilon))}{r} \text{ for every } g \in \mathcal{G}$$

are nondecreasing on $(0, \infty)$.

(iii) Every μ-null set is μ-upper porous (resp. μ-(g)-upper porous for every $g \in \mathcal{G}$).

(iv) Every upper porous (resp. (g)-upper porous) set is μ-upper porous (resp. μ-(g)-upper porous for every $g \in \mathcal{G}$). Hence every σ-upper porous (resp. σ-(g)-upper porous) set is σ-μ-upper porous (resp. σ-μ-(g)-upper porous for every $g \in \mathcal{G}$).

(v) The systems of all σ-μ-upper porous sets (resp. σ-μ-(g)-upper porous sets for every $g \in \mathcal{G}$) are σ-ideals of sets.

We will introduce two natural definitions closely related to μ-upper porosity. Both of them give stronger notions than μ-upper porosity (see Proposition 2.9). The first one is called (s)-μ-upper porosity (where '(s)' stands for 'strong'), the other one μ-approximate upper porosity.

Recall that a point $x \in X$ is called a *dispersion point* of $D \subset X$ if

$$\lim_{r \to 0_+} \frac{\mu^*(D \cap B(x,r))}{\mu(B(x,r))} = 0.$$

Definition 2.7. Let $A \subset X$, $x \in X$, $r > 0$ and $\varepsilon > 0$. Let $\widehat{\gamma}(\mu, A, x, r, \varepsilon)$ denote the supremum of numbers $s > 0$ for which there exists a point $z \in X$ such that $\varrho(x, z) + s \leq r$ and $\mu^*(A \cap B(z, s)) \leq \varepsilon \mu(B(z, s))$ (we put $\sup \emptyset = 0$).

(i) The set A is called (s)-*μ-upper porous at* x if $\displaystyle\lim_{\varepsilon \to 0_+} \limsup_{r \to 0_+} \frac{\widehat{\gamma}(\mu, A, x, r, \varepsilon)}{r} > 0$.

(ii) The set A is called *μ-approximately upper porous at* x if there exists $D \subset X$ such that $A \setminus D$ is upper porous at x and x is a dispersion point of D.

(iii) The set A is called (s)-*μ-upper porous* (resp. *μ-approximately upper porous*) if it is (s)-μ-upper porous (resp. μ-approximately upper porous) at each point in A.

(iv) The set A is called σ-(s)-*μ-upper porous* (resp. σ-*μ-approximately upper porous*) if it is a countable union of (s)-μ-upper porous (resp. μ-approximately upper porous) sets.

Remark 2.8. Both (s)-μ-upper porous and μ-approximately upper porous sets have similar properties as μ-upper porous sets (cf. Remark 2.6). In particular, every μ-null or upper porous set is (s)-μ-upper porous and μ-approximately upper porous and the systems of σ-(s)-μ-upper porous and σ-μ-approximately upper porous sets are σ-ideals of sets.

Proposition 2.9. *Let $A \subset X$ and $x \in X$.*

(i) *If A is (s)-μ-upper porous at x, then A is μ-upper porous at x.*

(ii) *If A is μ-approximately upper porous at x, then A is μ-upper porous at x.*

Proof. (i) Obvious, since $\gamma(\mu, A, x, r, \varepsilon) \geq \widehat{\gamma}(\mu, A, x, r, \varepsilon)$ for every $r > 0$ and $\varepsilon > 0$.

(ii) Since A is μ-approximately upper porous at x, there exists $D \subset X$ such that $A \setminus D$ is upper porous at x and x is a dispersion point of D. There exist $C > 0$ and a sequence of positive numbers $\{r_n\}_{n=1}^{\infty}$ such that $\displaystyle\lim_{n \to \infty} r_n = 0$ and $\gamma(A \setminus D, x, r_n) > Cr_n$ for every $n \in \mathbb{N}$. Choose $\varepsilon > 0$. Since $A \cap D \subset D$ and x is a dispersion point of D, there exists $n_0 \in \mathbb{N}$ such that

$$\mu^*((A \cap D) \cap B(x, r_n)) \leq \varepsilon \mu(B(x, r_n))$$

for every $n \in \mathbb{N}$, $n \geq n_0$. It easily follows that $\gamma(\mu, A, x, r_n, \varepsilon) > Cr_n$ for every $n \in \mathbb{N}$, $n \geq n_0$. Since $\varepsilon > 0$ was arbitrary, A is μ-upper porous at x. $\quad\square$

The following example shows that a set that is μ-upper porous at some point must be neither (s)-μ-upper porous nor μ-approximately upper porous at this point.

Example 2.10. Put $X = A = \{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ and define a probabilistic measure μ on X by

$$\mu\left(\{0\}\right) = 0 \ \text{ and } \ \mu\left(\left\{\frac{1}{n}\right\}\right) = \frac{1}{2^n} \ \text{ for } \ n \in \mathbb{N}.$$

Then

(a) A is μ-upper porous at 0,

(b) A is not (s)-μ-upper porous at 0,

(c) A is not μ-approximately upper porous at 0.

PROOF. (a) Choose a decreasing sequence of positive numbers $\{\varepsilon_k\}_{k=1}^{\infty}$ such that $\lim\limits_{k\to\infty} \varepsilon_k = 0$. By Remark 2.6 (ii),

$$\overline{p_\mu}(A, 0) = \lim_{k\to\infty} \lim_{l\to\infty} \sup \left\{ \frac{\gamma(\mu, A, 0, r, \varepsilon_k)}{r} : 0 < r \leq \frac{1}{l} \right\}.$$

Fix $k \in \mathbb{N}$ and choose $l_k \in \mathbb{N}$ such that $\frac{1}{2^{l_k}} < \varepsilon_k$. Put

$$r_l = \frac{1}{l}, \ z_l = \frac{1}{4l} \ \text{ and } \ s_l = \frac{1}{4l}$$

for every $l \geq l_k$. Then $\varrho(0, z_l) + s_l < r_l$ and

$$\mu(A \cap B(z_l, s_l)) = \frac{1}{2^{2l}} < \varepsilon_k \frac{1}{2^l} = \varepsilon_k \mu(B(0, r_l)).$$

Thus $\gamma(\mu, A, 0, r_l, \varepsilon_k) \geq s_l$ and therefore

$$\lim_{l\to\infty} \sup \left\{ \frac{\gamma(\mu, A, 0, r, \varepsilon_k)}{r} : 0 < r \leq \frac{1}{l} \right\} \geq \frac{1}{4}.$$

Since $k \in \mathbb{N}$ was arbitrary, we get $\overline{p_\mu}(A, 0) \geq \frac{1}{4} > 0$.

(b) Since $A = X$, every point in $X \setminus \{0\}$ has positive μ-measure and 0 is not an isolated point of X, it follows that $0 < \mu(B(z, s)) = \mu^*(A \cap B(z, s))$ for

every $z \in X$ and $s > 0$. Therefore $\mu^*(A \cap B(z,s)) \not\leq \varepsilon\mu(B(z,s))$ for arbitrary $0 < \varepsilon < 1$, $z \in X$ and $s > 0$. Hence $\widehat{\gamma}(\mu, A, 0, r, \varepsilon) = 0$ for arbitrary $r > 0$ and $0 < \varepsilon < 1$ and thus A is not (s)-μ-upper porous at 0.

(c) On the contrary, suppose that A is μ-approximately upper porous at 0. Then there exists $D \subset X$ such that $A \setminus D$ is upper porous at 0 and 0 is a dispersion point of D. Since $A = X$ and $A \setminus D$ is upper porous at 0, there exists an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $n_1 > 1$ and $\frac{1}{n_k} \in D$ for every $k \in \mathbb{N}$. Put $r_k = \frac{1}{n_k - 1}$ for every $k \in \mathbb{N}$. Then $\lim_{k \to \infty} r_k = 0$ and

$$\frac{\mu^*(D \cap B(0, r_k))}{\mu(B(0, r_k))} \geq \frac{\mu(\{\frac{1}{n_k}\})}{\mu(B(0, r_k))} = \frac{1}{4} > 0.$$

Hence 0 is not a dispersion point of D, which is a contradiction. $\qquad\square$

3 Decomposition Theorems.

Recall the notation

$$\overline{p_{\mu,(g)}}(A, x) = \lim_{\varepsilon \to 0_+} \limsup_{r \to 0_+} \frac{g(\gamma(\mu, A, x, r, \varepsilon))}{r} \qquad (2)$$

for $A \subset X$, $x \in X$ and $g \in \mathcal{G}$ introduced in Remark 2.6.

Lemma 3.1. *For every $A \subset X$ and $g \in \mathcal{G}$ the function $x \mapsto \overline{p_{\mu,(g)}}(A, x)$ is Borel measurable on X.*

PROOF. Let $A \subset X$ and $g \in \mathcal{G}$. By Remark 2.6 (ii),

$$\overline{p_{\mu,(g)}}(A, x) = \lim_{k \to \infty} \lim_{l \to \infty} \sup\left\{ \frac{g(\gamma(\mu, A, x, r, \frac{1}{k}))}{r} : 0 < r < \frac{1}{l} \right\}.$$

It suffices to prove that functions

$$u_{k,l} : x \mapsto \sup\left\{ \frac{g(\gamma(\mu, A, x, r, \frac{1}{k}))}{r} : 0 < r < \frac{1}{l} \right\}$$

are Borel measurable on X for arbitrary sufficiently big numbers $k, l \in \mathbb{N}$, since then the function $x \mapsto \overline{p_{\mu,(g)}}(A, x)$ is expressed as a countable limit of Borel measurable functions and therefore is Borel measurable.

Since $g \in \mathcal{G}$, there exists a number $h_1 > 0$ such that the function g is increasing and continuous on $[0, h_1)$. Choose arbitrary $k \in \mathbb{N}$ and $l \in \mathbb{N}$ so that $\frac{1}{l} < h_1$. We show that the function $u_{k,l}$ is lower semicontinuous

(and thus Borel measurable) on X. Consider an arbitrary $c \in \mathbb{R}$ and put $G_c = \{x \in X : u_{k,l}(x) > c\}$. We prove that G_c is open.

Choose an arbitrary point $x \in G_c$. By the definition of $u_{k,l}$ we find a number $r_x > 0$ such that $c < \frac{g(\gamma(\mu,A,x,r_x,\frac{1}{k}))}{r_x}$ and $0 < r_x < \frac{1}{l}$. By the definition of $\gamma(\mu, A, x, r_x, \frac{1}{k})$ a by continuity of g on $\left[0, \frac{1}{l}\right)$ we can find a number $s_x > 0$ and a point $z_x \in X$ such that the following three inequalities hold:

$$\frac{g(s_x)}{r_x} > c, \; \varrho(x, z_x) + s_x \leq r_x \text{ and } \mu^*(A \cap B(z_x, s_x)) \leq \frac{1}{k}\mu(B(x, r_x)).$$

Choose $\eta > 0$ such that $\eta + r_x < \frac{1}{l}$ and $\frac{g(s_x)}{\eta + r_x} > c$. We show that $y \in G_c$ for an arbitrary point $y \in B(x, \eta)$. Indeed, if we put $r_y = \eta + r_x$, $s_y = s_x$ and $z_y = z_x$, we get

$$0 < r_y < \frac{1}{l},$$

$$\varrho(y, z_y) + s_y \leq \eta + r_x = r_y$$

and

$$\mu^*(A \cap B(z_y, s_y)) \leq \frac{1}{k}\mu(B(x, r_x)) \leq \frac{1}{k}\mu(B(y, r_y))$$

where the last inequality follows by the monotonicity of the measure μ. Therefore $\gamma(\mu, A, y, r_y, \frac{1}{k}) \geq s_y = s_x$ and thus $\frac{g(\gamma(\mu,A,y,r_y,\frac{1}{k}))}{r_y} > c$ because g is increasing on $[0, \frac{1}{l})$. Hence $u_{k,l}(y) = \sup\left\{\frac{g(\gamma(\mu,A,y,r,\frac{1}{k}))}{r} : 0 < r < \frac{1}{l}\right\} > c$ and $y \in G_c$. Thus we have proven that G_c is open for arbitrary $c \in \mathbb{R}$ and so the function $u_{k,l}$ is lower semicontinuous on X. $\qquad\square$

Remark 3.2. By the previous lemma, the function $\varphi : x \mapsto \overline{p_{\mu,(g)}}(A, x)$ is Borel measurable on X for arbitrary $A \subset X$ and $g \in \mathcal{G}$. If $B \subset X$ is Borel measurable (resp. μ-measurable) and $\alpha \geq 0$ is arbitrary, then both sets

$$\{x \in B : \overline{p_{\mu,(g)}}(A, x) \leq \alpha\} \text{ and } \{x \in B : \overline{p_{\mu,(g)}}(A, x) > \alpha\}$$

are Borel measurable (resp. μ-measurable). In particular, for every Borel measurable (resp. μ-measurable) set $B \subset X$, the subset of B consisting of all points at which B is μ-(g)-upper porous is Borel measurable (resp. μ-measurable).

Remark 3.3. (see [3, Section 6.B]) For any set $A \subset X$ there exists a μ-measurable set $H(A) \subset X$ satisfying $A \subset H(A)$ and

$$\mu^*(A \cap M) = \mu(H(A) \cap M) \tag{3}$$

for any μ-measurable set $M \subset X$. The set $H(A)$ is called a *measurable cover* (or *measurable hull*) of A. This set is not uniquely determined. A μ-measurable set $Q \subset X$ is a measurable cover of A, if and only if $A \subset Q$ and $\mu_*(Q \setminus A) = 0$ (where μ_* denotes the inner measure corresponding to the measure μ).

Lemma 3.4. *For every $A \subset X$, $g \in \mathcal{G}$ and $\varepsilon > 0$ there exists a (g)-upper porous set $S \subset P = \{x \in A : \overline{p_{\mu,(g)}}(A, x) > 0\}$ such that $\mu^*(P \setminus S) < \varepsilon$.*

Proof. Fix $A \subset X$, $g \in \mathcal{G}$ and $\varepsilon > 0$ arbitrarily. Denote

$$\widehat{P} = \{x \in H(A) : \overline{p_{\mu,(g)}}(A, x) > 0\} \text{ and } \widehat{P}_q = \{x \in \widehat{P} : \overline{p_{\mu,(g)}}(A, x) \leq q\}$$

for $q > 0$. Since $H(A)$ is μ-measurable, by Remark 3.2, $\widehat{P}$ and $\widehat{P}_q$ (for every $q > 0$) are also μ-measurable.

Since $\bigcap_{q>0} \widehat{P}_q = \emptyset$ and μ is probabilistic (and thus finite), we can find $p > 0$ such that

$$\mu(\widehat{P}_{2p}) < \frac{\varepsilon}{2}.$$

Since $\widehat{P}_{2p} \subset H(A)$ and $\widehat{P}_{2p} \cap A = \{x \in P : \overline{p_{\mu,(g)}}(A, x) \leq 2p\}$, by (3) it follows that

$$\mu^*(\{x \in P : \overline{p_{\mu,(g)}}(A, x) \leq 2p\}) = \mu(\widehat{P}_{2p}) < \frac{\varepsilon}{2}.$$

Put

$$Q = \{x \in P : \overline{p_{\mu,(g)}}(A, x) > 2p\}.$$

Then

$$\mu^*(P \setminus Q) < \frac{\varepsilon}{2}.$$

Take $x \in Q$ and a sequence of positive numbers $\{\eta_j\}_{j=1}^{\infty}$ such that

$$\sum_{j=1}^{\infty} \eta_j < \frac{\varepsilon}{2}.$$

Since $\overline{p_{\mu,(g)}}(A, x) > 2p$, for arbitrary $\eta > 0$ and $\delta > 0$ there exists $0 < r < \frac{\delta}{2}$ such that

$$\varrho(x, z) + g^{-1}(2pr) \leq r$$

and

$$\mu^*(A \cap B(z, g^{-1}(2pr))) \leq \eta\mu(B(x, r))$$

for some point $z \in X$. It follows by the triangle inequality for the metric ϱ that $B(x, r) \subset B(z, 2r)$ and thus $x \in B(z, 2r)$. By the monotonicity of measure μ we get $\mu(B(x, r)) \leq \mu(B(z, 2r))$. To simplify the notation we put $s = 2r$.

It follows from the previous estimates that the set Q is covered by a system of nonempty open balls

$$\{B(z,s) : s < \delta,\ \mu^*(A \cap B(z, g^{-1}(ps))) \leq \eta \mu(B(z,s))\}.$$

For $j \in \mathbb{N}$ we gradually choose $\eta = \eta_j$ and $\delta = \frac{1}{j}$ and using the covering theorem ([2, Theorem 1.2]) we get (for every $j \in \mathbb{N}$) a disjoint system of nonempty open balls $\{B(z_i, s_i) : i \in I_j\}$ such that

$$Q \subset \bigcup_{i \in I_j} B(z_i, 5s_i),$$

$$s_i < \frac{1}{j}$$

and

$$\mu^*(A \cap B(z_i, g^{-1}(ps_i))) \leq \eta_j \mu(B(z_i, s_i))$$

for every $i \in I_j$.

Fix $j \in \mathbb{N}$ and put

$$G_j = \bigcup_{i \in I_j} (A \cap B(z_i, g^{-1}(ps_i))).$$

Since $\{B(z_i, s_i) : i \in I_j\}$ is a disjoint system of nonempty open balls and the space X is separable, the set of indices I_j is countable. By the σ-subadditivity of the outer measure μ^*, disjointness of the system $\{B(z_i, s_i) : i \in I_j\}$ and the fact that μ is probabilistic, we directly estimate

$$\mu^*(G_j) \leq \sum_{i \in I_j} \mu^*(A \cap B(z_i, g^{-1}(ps_i))) \leq \eta_j \sum_{i \in I_j} \mu(B(z_i, s_i)) \leq \eta_j.$$

If we put $G = \bigcup_{j=1}^{\infty} G_j$, we get

$$\mu^*(G) \leq \sum_{j=1}^{\infty} \mu^*(G_j) \leq \sum_{j=1}^{\infty} \eta_j < \frac{\varepsilon}{2}.$$

We show that $Q \setminus G$ is (g)-upper porous. Clearly,

$$
\begin{aligned}
Q \setminus G \;\; &= \;\; Q \setminus \bigcup_{j=1}^{\infty} G_j \\[2mm]
&= \;\; \bigcap_{j=1}^{\infty} (Q \setminus G_j) \\[2mm]
&= \;\; \bigcap_{j=1}^{\infty} \Big(Q \setminus \bigcup_{i \in I_j} (A \cap B(z_i, g^{-1}(ps_i))) \Big) \\[2mm]
&= \;\; \bigcap_{j=1}^{\infty} \bigcap_{i \in I_j} (Q \setminus (A \cap B(z_i, g^{-1}(ps_i)))) \\[2mm]
&= \;\; \bigcap_{j=1}^{\infty} \bigcap_{i \in I_j} (Q \setminus B(z_i, g^{-1}(ps_i)))
\end{aligned}
$$

where the last equality holds since $Q \subset A$.

Fix $x \in Q \setminus G$ and $j \in \mathbb{N}$. Since Q is covered by the system of open balls $\{B(z_i, 5s_i) : i \in I_j\}$, there exists an index $i_j \in I_j$ such that $x \in B(z_{i_j}, 5s_{i_j})$. The open ball $B(z_{i_j}, g^{-1}(ps_{i_j}))$ is disjoint with the set $Q \setminus G$, thus we get

$$
\gamma(Q \setminus G, x, 5s_{i_j} + g^{-1}(ps_{i_j})) \geq g^{-1}(ps_{i_j})
$$

where $0 < s_{i_j} < \frac{1}{j}$. Repeating this procedure for every $j \in \mathbb{N}$ we get

$$
\begin{aligned}
\limsup_{r \to 0_+} \frac{g(\gamma(Q \setminus G, x, r))}{r}
\;\; &\geq \;\; \limsup_{j \to \infty} \frac{g(\gamma(Q \setminus G, x, 5s_{i_j} + g^{-1}(ps_{i_j})))}{5s_{i_j} + g^{-1}(ps_{i_j})} \\[2mm]
&\geq \;\; \lim_{j \to \infty} \frac{ps_{i_j}}{5s_{i_j} + g^{-1}(ps_{i_j})} \\[2mm]
&= \;\; p \lim_{j \to \infty} \frac{1}{5 + p \dfrac{g^{-1}(ps_{i_j})}{ps_{i_j}}} \\[2mm]
&\geq \;\; \frac{p}{5 + pK} > 0,
\end{aligned}
$$

because $\lim\limits_{j \to \infty} s_{i_j} = 0$ and $g \in \mathcal{G}$ (by (1) there exist $K > 0$ and $h > 0$ such that $g^{-1}(x) \leq Kx$ for every $x \in [0, h)$).

Finally, put $S = Q \setminus G$. Then S is (g)-upper porous and

$$
\mu^*(P \setminus S) \leq \mu^*(P \setminus Q) + \mu^*(G) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
$$

$\square$

Theorem 3.5. *Let $A \subset X$ and $g \in \mathcal{G}$. The set A is σ-μ-(g)-upper porous if and only if $A = S \cup N$ where S is σ-(g)-upper porous and N is μ-null.*

PROOF. First suppose that A can be written as a union of a σ-(g)-upper porous set S and a μ-null set N. By Remark 2.6, S is σ-μ-(g)-upper porous, N is μ-(g)-upper porous and σ-μ-(g)-upper porous sets form a σ-ideal, thus $A = S \cup N$ is also σ-μ-(g)-upper porous.

Conversely, suppose that A is σ-μ-(g)-upper porous. Then there exist μ-(g)-upper porous sets A_n, $n \in \mathbb{N}$, such that $A = \bigcup\limits_{n=1}^{\infty} A_n$. Fix $n \in \mathbb{N}$. By Lemma 3.4 there exist (g)-upper porous sets S_n^m such that

$$S_n^m \subset A_n = \{x \in A_n : \overline{p_{\mu,(g)}}(A_n, x) > 0\}$$

and

$$\mu^*(A_n \setminus S_n^m) < \frac{1}{m}$$

for every $m \in \mathbb{N}$. Put $S_n = \bigcup\limits_{m=1}^{\infty} S_n^m$. The set S_n is a σ-(g)-upper porous subset of A_n and $\mu^*(A_n \setminus S_n) = 0$. Define $S = \bigcup\limits_{n=1}^{\infty} S_n$. Then the set S is clearly a σ-(g)-upper porous subset of A and $\mu^*(A \setminus S) = 0$. Since $A = S \cup (A \setminus S)$, we have found a decomposition of A to the σ-(g)-upper porous set S and the μ-null set $A \setminus S$. $\qquad\square$

Corollary 3.6. *A set $A \subset X$ is σ-μ-upper porous if and only if it can be expressed as $A = S \cup N$ where S is σ-upper porous and N is μ-null.*

PROOF. This is a direct consequence of the previous theorem with $g(x) = x$ for $x \in [0, \infty)$. $\qquad\square$

In the following example, we show that a μ-upper porous set needs not be a union of an upper porous and a μ-null set.

Example 3.7. Put $X = [0, 1]$, $A = \{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ and define a probabilistic measure $\widetilde{\mu}$ on X by $\widetilde{\mu} = \frac{\mu + \delta_0}{2}$ where μ is a probabilistic measure from Example 2.10 and δ_0 denotes the Dirac measure at 0. Then

(a) A is $\widetilde{\mu}$-upper porous,

(b) $A \neq P \cup N$ whenever P is upper porous and N $\widetilde{\mu}$-null.

Proof. (a) Since every $z \in A$, $z \neq 0$, is isolated in A and not isolated in X, A is $\widetilde{\mu}$-upper porous at z and thus it suffices to show that A is $\widetilde{\mu}$-upper porous at 0. In Example 2.10, we proved that A is μ-upper porous at 0. Using exactly the same sequences $\{\varepsilon_k\}_{k=1}^{\infty}$, $\{l_k\}_{k=1}^{\infty}$ and $\{r_l\}_{l=l_k}^{\infty}$, $\{z_l\}_{l=l_k}^{\infty}$, $\{s_l\}_{l=l_k}^{\infty}$ for every $k \in \mathbb{N}$ as in the mentioned Example 2.10 and considering balls $\{B(z_l, s_l)\}_{l=l_k}^{\infty}$ for every $k \in \mathbb{N}$, a direct computation yields the estimate $\overline{p_{\widetilde{\mu}}}(A, 0) \geq \frac{1}{4} > 0$.

(b) Suppose that $A = P \cup N$ where P is upper porous and N $\widetilde{\mu}$-null. Since every point in A has positive $\widetilde{\mu}$-measure, necessarily $N = \emptyset$ and $P = A$. Since $\{\frac{1}{n}\}_{n=1}^{\infty} \subset P$ is not upper porous at 0 and $0 \in P$, P is not upper porous which is a contradiction. □

In the case of σ-μ-upper porosity, we can further strengthen the result in Theorem 3.5 using a theorem about approximation of σ-upper porous sets.

Theorem 3.8. *Let $A \subset X$. The set A is σ-μ-upper porous if and only if $A = S \cup N$ where S is σ-strongly upper porous and N is μ-null.*

Proof. First suppose that A can be written as a union of a σ-strongly upper porous set S and a μ-null set N. Since every σ-strongly upper porous set is also σ-upper porous, using Remark 2.6 we easily get that $A = S \cup N$ is σ-μ-upper porous.

Conversely, suppose that A is σ-μ-upper porous. By Corollary 3.6 there exist a σ-upper porous set S_1 and a μ-null set N_1 so that $A = S_1 \cup N_1$. By [5, Theorem 2.2 (iii)] there exists a σ-strongly upper porous set $S \subset S_1$ such that $\mu(S_1 \setminus S) = 0$. Put $N = N_1 \cup (S_1 \setminus S)$. Then $A = S \cup N$ and this decomposition has all desired properties. □

Remark 3.9. Let $n \in \mathbb{N}$, $n \geq 1$, $X = [0,1]^n \subset \mathbb{R}^n$ and denote by λ_n the normalized n-dimensional Lebesgue measure on X. Since, in this particular setting, every σ-upper porous set is λ_n-null, results presented in Corollary 3.6 and Theorem 3.8 can be shortened to the following proposition:

An arbitrary set $A \subset X$ is σ-λ_n-upper porous if and only if it is λ_n-null.

This statement is almost trivial and can be proved directly. Indeed, suppose that A is λ_n-upper porous and choose $x \in A$ arbitrarily. Then there exists $0 < c < 1$ such that for every $\varepsilon > 0$ there exists a sequence of positive numbers $\{r_k\}_{k=1}^{\infty}$ such that $\lim_{k \to \infty} r_k = 0$ and $\gamma(\lambda_n, A, x, r_k, \varepsilon) > cr_k$ for every $k \in \mathbb{N}$. Fix $0 < \varepsilon < c^n$ and consider the corresponding sequence $\{r_k\}_{k=1}^{\infty}$. Put $\delta = \varepsilon + 1 - c^n$. Then $0 < \delta < 1$. Fix $k \in \mathbb{N}$. We can find $s_k > cr_k$ and

$z_k \in X$ such that $\varrho(x, z_k) + s_k \leq r_k$ and $\lambda_n^*(A \cap B(z_k, s_k)) \leq \varepsilon \lambda_n(B(x, r_k))$. We estimate:

$$
\begin{aligned}
\lambda_n^*(A \cap B(x, r_k)) &\leq \lambda_n^*(A \cap B(z_k, s_k)) + \lambda_n(B(x, r_k) \setminus B(z_k, s_k)) \\
&\leq \varepsilon \lambda_n(B(x, r_k)) + (1 - c^n) \lambda_n(B(x, r_k)) \\
&= \delta \lambda_n(B(x, r_k)).
\end{aligned}
$$

Since previous estimates can be done for every $k \in \mathbb{N}$ and $\lim_{k \to \infty} r_k = 0$, x is not the point of outer density for A. Since $x \in A$ was arbitrary and almost every point of any subset of $\mathbb{R}^n$ is the point of outer density for this set, we conclude that $\lambda_n(A) = 0$.

However, results presented in Corollary 3.6 and Theorem 3.8 are interesting for other Borel regular measures and the general result of Theorem 3.5 is much deeper even in the case of normalized Lebesgue measure on the unit cube in $\mathbb{R}^n$, since for some $g \in \mathcal{G}$ there exist σ-(g)-upper porous sets with positive λ_n-measure (e.g., if $0 < q < 1$ and $g(x) = x^q$ for $x \in [0, \infty)$ then there exists a perfect (g)-upper porous set $E \subset \mathbb{R}$ of positive Lebesgue measure ([6, Proposition 2.41])).

The following proposition, which states that σ-ideals of σ-μ-upper porous, σ-(s)-μ-upper porous and σ-μ-approximately upper porous sets coincide, is a simple consequence of the decomposition theorem.

Proposition 3.10. *Let $A \subset X$. The following statements are equivalent:*

(i) A is σ-μ-upper porous.

(ii) A is σ-(s)-μ-upper porous.

(iii) A is σ-μ-approximately upper porous.

PROOF. Clearly, *(ii)*⇒*(i)* and *(iii)*⇒*(i)* by Proposition 2.9.

Let $A \subset X$ be σ-μ-upper porous. By Corollary 3.6, A can be expressed in the form $A = S \cup N$ where S is σ-upper porous and N is μ-null. By Remark 2.8, S and N are both σ-(s)-μ-upper porous and σ-μ-approximately upper porous and systems of all such sets form σ-ideals. Hence A is both σ-(s)-μ-upper porous and σ-μ-approximately upper porous. Thus *(i)*⇒*(ii)* and *(i)*⇒*(iii)*.   □

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