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FULL DIMENSIONAL SETS WITHOUT GIVEN PATTERNS

Abstract

We construct a d Hausdorff dimensional compact set in $\mathbb{R}^d$ that does not contain the vertices of any parallelogram. We also prove that for any given triangle (3 given points in the plane) there exists a compact set in $\mathbb{R}^2$ of Hausdorff dimension 2 that does not contain any similar copy of the triangle. On the other hand, we show that the set of the 3-point patterns of a 1-dimensional compact set of $\mathbb{R}$ is dense.

1 Introduction.

Assume that a compact set A is given in $\mathbb{R}^d$ and we would like to measure it from a geometrical point of view: considering the patterns (the similarity classes of all sets) that are contained by A.

Of course, the concepts of measure and dimension theory are also available. Are there connections between the measure and dimension theoretic size and the above-mentioned geometric size of the sets? A still open conjecture of Erdős [1] states that for any infinite set P there exists a set $A \subseteq \mathbb{R}$ of positive Lebesgue measure such that A does not contain any similar copy of P.

On the other hand, by a well known easy consequence of the Lebesgue Density Theorem, if a set is of positive Lebesgue measure in $\mathbb{R}^d$, then it contains some similar copy of every finite set. Does the conclusion also hold for sets of Hausdorff dimension d (from now on, these sets are said to be *full dimensional*)? We will prove that the answer is 'no.' First, we show that there exists a compact set of Hausdorff dimension d that does not contain the vertices of

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any parallelogram. Then we prove that for any given triangle, there exists a compact set of dimension 2 on the plane that does not contain the vertices of any triangle similar to the given one. These results are connected to and motivated by Keleti's theorems [5], [6], which refer to the real line.

However, I. Laba and M. Pramanik [7] showed that a full dimensional compact set $A \subseteq \mathbb{R}$ that satisfies certain conditions on the Fourier transform of a probabilistic measure supported by A must contain a nontrivial arithmetic progression of length 3.

Of course, a full dimensional compact set contains numerous patterns (since its cardinality is continuum). We will show that the set of the 3-point patterns of a full dimensional subset of $\mathbb{R}$ is dense in a very natural space of the 3-point patterns.

The whole area is somewhat connected to some very famous discrete problems and theorems. Denote by $r_k(n)$ the maximal number of elements that can be selected from the set $\{1, 2, \ldots, n\}$ without containing a nontrivial arithmetic progression of length k. There are many classical results on the magnitude of $r_k(n)$ (see [11], [12], [13]), but there are recent research as well (see [3], [4]).

First, we define what we mean by containing a pattern.

Definition 1.1. We say that $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ is a *similarity map*, if there exists some $c > 0$ such that for all $x, y \in \mathbb{R}^d$, $|\varphi(x) - \varphi(y)| = c|x-y|$). Let $A, P \subseteq \mathbb{R}^d$. We say that A *contains the pattern* P (or *contains* P *as a pattern*), if there exists a similarity map φ on $\mathbb{R}^d$ such that $\varphi(P) \subseteq A$.

2 Avoiding parallelograms and triangles.

Definition 2.1. We say that $[x_1, x_2, x_3, x_4]$ is a parallelogram, if there are at least 3 different points among $x_1, x_2, x_3, x_4 \in \mathbb{R}^d$ and $x_2 - x_1 = x_4 - x_3$.

Our main tool to guarantee the full Hausdorff dimension will be Lemma 2.2, which is the higher dimensional version of K. Falconer's lemma [2, Example 4.6]. First, we need a technical lemma.

Lemma 2.1. *Let $U \subseteq \mathbb{R}^d$ be bounded, $l > 0$ and let $B \subseteq U$ be a finite set. If $|B| > (2\mathrm{diam}(U)\sqrt{d}/l + 1)^d$, then there exist two points of B such that their distance is less than l (where $|B|$ denotes the cardinality of B).*

Proof. Let $l' < l$ such that $|B| > (2\mathrm{diam}(U)\sqrt{d}/l' + 1)^d$. We can cover U with $(2\mathrm{diam}(U)\sqrt{d}/l' + 1)^d$ cubes of sidelength $l'/(2\sqrt{d})$. There are two points of B that are in the same cube, their distance is at most $l' < l$. □

Lemma 2.2. *Let $F = \cap_{k=1}^{\infty} E_k \subseteq \mathbb{R}^d$, where every E_k is a compact set that consists of d dimensional cubes, E_0 is a single cube. Assume that the following holds for all $k \geq 1$: $E_k \subseteq E_{k-1}$ and each cube of E_{k-1} contains at least m_k^d cubes of E_k. Assume that for any two cubes of E_k, their distance is at least ε_k, where $0 < \varepsilon_k < \varepsilon_{k-1}$ and $\lim_{k\to\infty} \varepsilon_k = 0$. Assume that $m_k \varepsilon_k < 1$. Then*

$$\dim_{\mathrm{H}}(F) \geq \liminf_{k\to\infty} \frac{d \log(m_1 \cdot \ldots \cdot m_{k-1})}{-\log(m_k \varepsilon_k)}.$$

PROOF (CF. [2, EXAMPLE 4.6]). We can assume that each cube of E_{k-1} contains exactly m_k^d cubes of E_k. Let μ be the following probability measure (supported on F): for each cube C of E_k, let $\mu(C) = (m_1 \cdot \ldots \cdot m_k)^{-d}$. Let U be an arbitrary set of diameter less than ε_1. We estimate $\mu(U)$. Let k be such that $\varepsilon_k \leq \mathrm{diam}(U) < \varepsilon_{k-1}$.

Then U intersects at most one cube of E_{k-1}, therefore at most m_k^d cubes of E_k. By the previous lemma, it cannot intersect more than $(2\mathrm{diam}(U)\sqrt{d}/\varepsilon_k + 1)^d \leq (4\mathrm{diam}(U)\sqrt{d}/\varepsilon_k)^d$ cubes of E_k. Hence,

$$\begin{aligned}\mu(U) &\leq (m_1 \cdot \ldots \cdot m_k)^{-d} \min\{(4\mathrm{diam}(U)\sqrt{d}/\varepsilon_k)^d, m_k^d\} \\ &\leq (m_1 \cdot \ldots \cdot m_k)^{-d}((4\mathrm{diam}(U)\sqrt{d}/\varepsilon_k)^s m_k^{d-s})\end{aligned}$$

holds for all $0 \leq s \leq d$. Let $s < \liminf_{k\to\infty} \frac{d \log(m_1 \cdot \ldots \cdot m_{k-1})}{-\log(m_k \varepsilon_k)}$.

Then

$$\frac{\mu(U)}{(\mathrm{diam}(U))^s} \leq \frac{(4\sqrt{d})^s}{(m_1 \cdot \ldots \cdot m_{k-1})^d m_k^s \varepsilon_k^s},$$

which is bounded from above by some $K > 0$.

Therefore $(\mathrm{diam}(U))^s \geq \mu(U)/K$ for all U which is of diameter less than ε_1. Suppose that we cover F with a countable collection of sets $U_1, U_2, \ldots$, each U_n is of diameter less than ε_1. Then

$$\sum_{n=1}^{\infty} (\mathrm{diam}(U_n))^s \geq \sum_{n=1}^{\infty} \mu(U_n)/K \geq \mu(F)/K = 1/K,$$

which shows that $\dim_{\mathrm{H}}(F) \geq s$. □

In the following theorem, we generalize a construction of Keleti [5], who proved the theorem in $\mathbb{R}$. Then we discover that if $d = 2$, then our set has an other interesting property. This other property will be the starting point of some more observations.

Theorem 2.3. *For any $d = 1, 2, \ldots$, there exists a full dimensional compact set $A \subseteq \mathbb{R}^d$ such that A does not contain the vertices of any parallelogram.*

Proof. Let $\delta_m = 1/(6^{m-1}m!)$. We define the compact sets $A_1, A_2, \dots$ by induction. The sets A_m will consist of pairwise disjoint, closed cubes:

$$A_m = \bigcup_{1 \le i_k \le k} \prod_{j=1}^{d} [n^{(j)}_{i_1,\dots,i_m}\delta_m, (n^{(j)}_{i_1,\dots,i_m}+1)\delta_m],$$

where $\prod$ denotes the Cartesian product and the integers $n^{(j)}_{i_1,\dots,i_m}$ are chosen later. Therefore, A_m is compact and it consists of $(m!)^d$ cubes. Denote the cubes of A_m by $I^m_1, \dots, I^m_{(m!)^d}$ (in an arbitrary order), and let the sequence $(J_1, J_2, \dots)$ be the sequence of all cubes that occur: $(J_1, J_2, \dots) = (I^1_1, \dots, I^{m-1}_{((m-1)!)^d}, I^m_1, \dots, I^m_{(m!)^d}, I^{m+1}_1, \dots)$.

Let $n^{(1)}_1 = \dots = n^{(d)}_1 = 0$. Then $A_1 = [0,1]^d$. Suppose that $A_1, \dots, A_m$ are already defined. We construct A_{m+1}.

If $(n^{(1)}_{i_1,\dots,i_m}\delta_m, \dots, n^{(d)}_{i_1,\dots,i_m}\delta_m) \notin J_m$, then for all $1 \le i \le m+1, 1 \le j \le d$, let

$$n^{(j)}_{i_1,\dots,i_m,i} = 6(m+1)n^{(j)}_{i_1,\dots,i_m} + 6i - 6.$$

If $(n^{(1)}_{i_1,\dots,i_m}\delta_m, \dots, n^{(d)}_{i_1,\dots,i_m}\delta_m) \in J_m$, then for all $1 \le i \le m+1, 1 \le j \le d$, let

$$n^{(j)}_{i_1,\dots,i_m,i} = 6(m+1)n^{(j)}_{i_1,\dots,i_m} + 6i - 3.$$

Let $A = \cap_{m=1}^{\infty} A_m$.

Claim 2.4. *A is compact and does not contain any parallelogram.*

Proof. The compactness is clear.

Suppose that there are three different elements among $x_1, x_2, x_3, x_4 \in A$. We need to show that $x_2 - x_1 \ne x_4 - x_3$. Assume that x_1 is different from each other, all the other cases are essentially the same. Choose m and j such that $x_1 \in I^m_j = J_M$, $x_2, x_3, x_4 \notin I^m_j = J_M$. By definition, the first coordinate of x_1 is $(6N_1 + 3)\delta_M + \varepsilon_1$, while the first coordinate of x_i $(i = 2, 3, 4)$ is $6N_i\delta_M + \varepsilon_i$, where N_1, N_2, N_3, N_4 are integers and $0 \le \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \le \delta_M$. Hence, $x_2 - x_1 \ne x_4 - x_3$. □

Claim 2.5. $\dim_{\mathrm{H}}(A) = d$.

Proof. Using the notations of Lemma 2.2, we have $E_{k-1} = A_k$, $m_k = k+1$. In the kth step we divide the cubes of A_k into smaller cubes and we choose some of them to give A_{k+1}. The minimal distance can be estimated from below by $\varepsilon_{k+1} = \delta_k/(\frac{5}{6}(k+1))$, because the sidelength of the cubes of A_k is δ_k, we divide the cubes to $(6(k+1))^d$ smaller cubes and then choose every

6th of them (in each coordinate), so we leave a space of length $\delta_k/(\frac{5}{6}(k+1))$. Lemma 2.2 gives

$$\dim_{\mathrm{H}}(A) \geq \liminf_{k\to\infty} d \cdot \frac{\log(k!)}{-\log\left(\frac{k}{\frac{5}{6}k}\cdot\frac{1}{6^{k-2}(k-1)!}\right)} = d,$$

while $\dim_{\mathrm{H}}(A) \leq d$ is clear.
This completes the proof of Theorem 2.3. □

The set constructed in Theorem 2.3 has an other interesting property, if $d = 2$.

Proposition 2.6. *If $d = 2$, then the above constructed A does not contain a rectangular isosceles triangle.*

Proof. We prove by contradiction. Suppose that $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A$ (throughout this proof, x_1, x_2, x_3 denote the first, y_1, y_2, y_3 denote the second coordinate of the vertices of the triangle) is a rectangular isosceles triangle, in which the right angle is at (x_2, y_2) and we get the point (x_1, y_1) by rotating (x_3, y_3) around (x_2, y_2) by angle $\frac{\pi}{2}$. Choose M such that $(x_1, y_1) \in I_j^m = J_M$, $(x_2, y_2), (x_3, y_3) \notin I_j^m = J_M$. Then $x_1 = (6N_1^x + 3)\delta_M + \varepsilon_1^x$,$y_1 = (6N_1^y + 3)\delta_M + \varepsilon_1^y$, while for $(j = 2, 3)$, $x_j = 6N_j^x\delta_M + \varepsilon_j^x$,$y_j = 6N_j^y\delta_M + \varepsilon_j^y$, where $0 \leq \varepsilon_1^x, \varepsilon_1^y \varepsilon_j^x, \varepsilon_j^y \leq \delta_M$ and $N_1^x, N_1^y, N_j^x, N_j^y$ are integers.

Then $(x_3, y_3) - (x_2, y_3) = (6(N_3^x - N_2^x)\delta_M + c_3^x, 6(N_3^y - N_2^y)\delta_M + c_3^y)$, where $-\delta_M \leq c_3^x, c_3^y \leq \delta_M$. Then, on the one hand, $(x_1, y_1) - (x_2, y_2)$ equals $(-6N_3^y - c_3^y, 6N_3^x + c_3^x)$ (since (x_1, y_1) is the rotated image of (x_3, y_3) around (x_2, y_2) by angle $\frac{\pi}{2}$). On the other hand, it is $((6(N_1^x - N_2^x)+3)\delta_M + c_1^x, (6(N_1^y - N_2^y) + 3)\delta_M + c_1^y)$, where $-\delta_M \leq c_1^x, c_1^y \leq \delta_M$, and this is a contradiction. □

Proposition 2.6 says that we can construct a compact set of dimension 2 on the plane that does not contain the rectangular isosceles triangle as a pattern. Can we avoid any other 3-point pattern on the plane? Keleti [6] gave affirmative answer on the real line. In the following, we prove that the same holds in $\mathbb{R}^2$, which is also considered as the complex plane $\mathbb{C}$ from now on.

Lemma 2.7. *Let $\alpha \neq 0$ complex, for which $|\alpha| < \frac{1}{12}$. Then there exists an axisparallel square containing at least $\frac{1}{18|\alpha|^2}$ Gaussian integer $j = j_1 + j_2 i \in \mathbb{Z} + i\mathbb{Z}$ such that $\alpha j \in [0,1] \times [0,1]$.*

Proof. If $\alpha > 0$ real, then take the axisparallel square Q of sidelength $\frac{1}{3}$ and centered at $(\frac{1}{2}, \frac{1}{2})$. This square contains at least $(\frac{1}{3\alpha} - 1)^2 > \frac{1}{9\alpha^2} - \frac{2}{3\alpha} > \frac{1}{18\alpha^2}$ complex numbers c such that $\frac{1}{\alpha}c$ is a Gaussian integer (and these Gaussian integers are in a square lattice). Now let $\alpha = |\alpha|e^{i\theta}$, where $0 \le \theta < 2\pi$. Rotate the above defined Q around $\left(\frac{1}{2}, \frac{1}{2}\right)$ by angle θ, denote it by Q^θ. In Q^θ, take the elements of the form $j\alpha = j|\alpha|e^{i\theta}$, where j is a Gaussian integer. As in the real case, there are at least $\frac{1}{18|\alpha|^2}$ of them in a square lattice. Since $Q^\theta \subset [0,1] \times [0,1]$, the claim follows. □

Theorem 2.8. *Let $P = (p_1, p_2, p_3) \subseteq \mathbb{R}^2$ triangle, that is, p_1, p_2, p_3 are distinct. Then there exists a compact $A \subseteq \mathbb{R}^2$ such that $\dim_{\mathrm{H}}(A) = 2$ and A does not contain a subset that is similar to P.*

Proof. Let p_1, p_2, p_3 be complex numbers as well.

Let M be a fixed even number. Let $\alpha = \frac{p_3 - p_1}{p_2 - p_1} \in \mathbb{C}$. It is clear that $\alpha \neq 0, 1$. Let $L > 0$ real and let $\delta_k = \frac{1}{L^k m_1 \cdot \ldots \cdot m_k}$. We will determine the numbers M, L, m_k later.

Our idea is the following. We start out from the unit square $I = [0,1] \times [0,1]$, our list in the beginning is (I, I, I). In the kth step, we have a list that consists of triples and we consider a certain triple of our list: (S_1, S_2, S_3), where S_1, S_2, S_3 are sets that consist of many squares. We take a *correction step*: we replace S_1, S_2, S_3 with S_1', S_2', S_3' with the following properties. 1) $S_i' \subseteq S_i$ for $i = 1, 2, 3$. 2) Each of S_1', S_2', S_3' consist of m_k^2 small, axisparallel squares. 3) The triple (S_1', S_2', S_3') is *correct*, that is, if $s_1 \in S_1', s_2 \in S_2', s_3 \in S_3'$, then (s_1, s_2, s_3) is not similar to P with the same orientation. 4) The sidelength of the small squares are δ_k. 5) The distance between two small squares is at least δ_k. Every other square X (other than S_1, S_2, S_3) is also replaced with X' that satisfies 1), 2), 4), 5). Then we write all triples that consist of the small squares to the end of our list, in an arbitrary order. Hence, we get a decreasing sequence of compact sets, let the intersection be A. If

$$\lim_{k \to \infty} \frac{2 \log(m_1 \cdot \ldots \cdot m_{k-1})}{-\log(m_k \delta_k)} = 2$$

holds for the sequence (m_k), then $\dim_{\mathrm{H}}(A) = 2$ by Lemma 2.2. The choice $m_k = \max(k, 3)$ is appropriate.

Let the squares X, Y, Z be given. In each of them, there are squares of sidelength δ_{k-1} and we want to take the correction step. We want to define X', Y', Z' such that if $x \in X', y \in Y', z \in Z'$, then $\frac{y-z}{x-z} \neq \alpha$.

Correction in the squares of Y: in every square of Y, take all the small squares of the form $\delta_k(M\alpha j_y + [0,1] \times [0,1])$, where j_y is a Gaussian integer. These small squares are pairwise disjoint and their distance is at least δ_k, if $M|\alpha| > 2\sqrt{2} + 1$, that is, $M > M_y$ for some M_y. The number of these values j_y is at least $1/18(M|\alpha|\frac{\delta_k}{\delta_{k-1}})^2 > 18m_k^2$, if $L > L_y$ (the conditions of Lemma

2.7 are also in condition $L > L_y$; this L_y can depend on M) and these points are in a square lattice. From these lattice points, we can choose those that are not on the perimeter and from the chosen lattice points, we can take the squares of sidelength $\delta_k([0,1] \times [0,1])$. Hence, we are able to choose m_k^2 small squares (the number of non-perimeter points is at least m_k^2, since $m_k \geq 3$).

Correction in the squares of X: in each square, take the following small squares: $\delta_k(Mj_x + [0,1] \times [0,1])$. If $M > M_x, L > L_x$, we can take this step as before.

Correction in the squares of Z: in each square, take the following small squares: $\delta_k(M\frac{\alpha}{\alpha-1}j_z + \frac{M}{2}\frac{\alpha}{\alpha-1} + [0,1] \times [0,1])$. If $M > M_x, L > L_x$, we can take this step as before.

In those squares that are not in X, Y or Z, take the small squares arbitrarily (taking care of the sidelength and distance δ_k).

Let $M > M_x, M_y, M_z$, $L > L_x, L_y, L_z$. Furthermore, let $M|\alpha|/2 > 4|\alpha|+4$, it can happen that this condition enlarges L again.

Take the correction step for each k. We claim that the intersection does not contain P as a pattern (with the same orientation). We prove by contradiction. Suppose that for some $x, y, z \in A$, $\frac{y-z}{x-z} = \alpha$. Choose k such that x, y, z are in distinct squares of the inductive definition of sidelength δ_k. Let these squares be X, Y, Z. What happens when we correct (X, Y, Z)? For some $0 \leq \varepsilon_x^1, \varepsilon_x^2, \varepsilon_y^1, \varepsilon_y^2, \varepsilon_z^1, \varepsilon_z^2 \leq 1$:

$$M\alpha j_y + (\varepsilon_y^1, \varepsilon_y^2) = \alpha(Mj_x + (\varepsilon_x^1, \varepsilon_x^2)) - (\alpha - 1)\left(M\frac{\alpha}{\alpha-1}\left(j_z + \frac{1}{2}\right) + (\varepsilon_z^1, \varepsilon_z^2)\right),$$

hence,

$$M\alpha(j_y - j_x + j_z) + \frac{M\alpha}{2} = \alpha(\varepsilon_x^1, \varepsilon_x^2) - (\alpha - 1)(\varepsilon_z^1, \varepsilon_z^2) - (\varepsilon_y^1, \varepsilon_y^2).$$

The absolute value of the left-hand side is at least $M|\alpha|/2$, the absolute value of the right-hand side is at most $4|\alpha| + 4$, which is a contradiction.

In each step, after correcting (X, Y, Z) with respect to α, correct it with respect to $\overline{\alpha}$. Therefore, the constructed set A does not contain any subset similar to P, either with the same orientation, or with the other. □

3 Avoiding "too many" patterns.

In fact, using the method seen in the previous section, a full dimensional compact set can avoid countably many patterns. In this section, we show that the patterns contained in a full dimensional set are dense in a sense.

Definition 3.1. Let $A \subseteq \mathbb{R}$ (or $\mathbb{R}^2 = \mathbb{C}$) compact. Let

$$\mathcal{T}(A) = \left\{ \frac{z-x}{y-x} : x, y, z \in A, x \neq y \right\}.$$

Notation. Let $0 < a, b < 1$ real numbers. Then let

$$h(a,b) = s, \text{ if } a^s + b^s = 1.$$

It can be easily seen that h is well-defined and positive, since $a^t + b^t$ is a continuous and strictly decreasing function of t and $a^0 + b^0 = 2$, $\lim_{t\to\infty} a^t + b^t = 0$.

Theorem 3.1. *Let* $0 < a < b < 1$, $A \subseteq \mathbb{R}$ *compact such that* $\mathcal{T}(A) \cap (a,b) = \emptyset$. *Then*

$$\dim_{\mathrm{H}}(A) \leq h(a, 1-b) < 1.$$

Corollary 3.2. *If* $A \subseteq \mathbb{R}$ *compact and* $\dim_{\mathrm{H}}(A) = 1$, *then* $\mathcal{T}(A)$ *is dense in* $\mathbb{R}$.

PROOF OF THEOREM 3.1. It is clear that $h(a, 1-b) < 1$.

We can assume that $\min(A) = 0$, $\max(A) = 1$. Let $s = h(a, 1-b)$, $\delta > 0$ be given. We will give the closed intervals $I_1, \dots, I_m$ such that their union covers A, the length of each interval is at most δ and $\sum_{i=1}^m \lambda(I_i)^s \leq 1$ (where λ denotes the Lebesgue measure and the length of the interval). On level 0, take the interval $[0,1]$. On level 1, take the covering $A \subseteq [0,a] \cup [b,1]$. On level 2, construct the following covering: let $a' = \max(A \cap [0,a]) \leq a$ and take $A \cap [0,a'] \subseteq [0, aa'] \cup [(1-b)a']$, then cover $A \cap [(1-b), 1]$ the same way. The length of the covering intervals are at most $a^2, a(1-b), (1-b)a, (1-b)^2$.

Continue this method. Suppose that S is a covering interval of a certain level. Let $m = \min(A \cap S)$, $M = \max(A \cap S)$. Take the interval $[m, M]$, throw out the open interval $(a(M-m)+m, (1-b)(M-m)+m)$, and cover $A \cap S$ with the remaining two intervals.

Choose a level k such that $a^k, (1-b)^k \leq \delta$. On this level, the length of each interval (used in the covering) is at most δ and the sum of the sth power of the length of the intervals is at most

$$\sum_{l=0}^{k} \binom{k}{l} (a^l (1-b)^{k-l})^s = (a^s + (1-b)^s)^k = 1,$$

which completes the proof. □

Our next aim is to prove a weak converse.

Theorem 3.3. *Let $0 < a < b < 1$. Then there exists a compact $A \subseteq \mathbb{R}$ such that $\mathcal{T}(A) \cap (a,b) = \emptyset$ and*

$$\dim_{\mathrm{H}}(A) = h\left(\frac{ab}{1-a+ab}, 1 - \frac{b}{1-a+ab}\right).$$

Proof. Let $a' = \frac{ab}{1-a+ab}$, $b' = \frac{b}{1-a+ab}$. Take the self-similar set defined by the similarity maps $f_1(x) = a'x$, $f_2(x) = (1-b')x + (1-b')$. Since $a' < b'$ holds, $f_1(A)$ and $f_2(A)$ are disjoint, hence we can apply the well-known theorem on the dimension of self-similar sets, we obtain $\dim_{\mathrm{H}}(A) = h(a', 1-b')$.

The self-similar set A can be constructed as a limit of a decreasing sequence of sets: we start out from $[0,1]$ and in each step, we throw out from each interval $[t, t+t_1]$ a smaller open interval $(t + t_1 a', t + t_1 b')$.

It is easy to calculate that if I_1, I_2 are the two remaining parts of I, then for all $x \in I_1$, $z \in I_2$, $y \in I_1 \cup I_2$, $x < y < z$: $\frac{y-x}{z-x} \notin (a,b)$. □

Corollary 3.4. *If $s < \frac{\log 2}{\log 3}$, then there exists a compact $A \subseteq \mathbb{R}$, for which $\dim_{\mathrm{H}}(A) \geq s$ and $\mathcal{T}(A)$ is not dense in $\mathbb{R}$.*

Proof. For each $a < \frac{1}{2}$, $b = 1 - a$, take the compact set A given by the previous theorem, for which $\mathcal{T}(A) \cap (a,b) = \emptyset$. It is easy to calculate that $\dim_{\mathrm{H}}(A)$ tends to $\frac{\log 2}{\log 3}$ as a tends to $\frac{1}{2}$. □

Problem 1. *What can we say about the sets of dimension at least $\frac{\log 2}{\log 3}$?*

How can we estimate the dimension of $\mathcal{T}(A)$ from above? Using classical results about the dimension of product sets (see [8, Theorem 8.10]), the following statements can be easily shown. In the statements, $\dim_{\mathrm{P}}$ denotes the packing dimension.

Proposition 3.5. *Let $A \subseteq \mathbb{R}$ compact. Then $\dim_{\mathrm{H}}(\mathcal{T}(A)) \leq \dim_{\mathrm{H}}(A) + 2\dim_{\mathrm{P}}(A)$.*

Corollary 3.6. *Let $A \subseteq \mathbb{R}$ compact. If $\dim_{\mathrm{H}}(A) + 2\dim_{\mathrm{P}}(A) < 1$, then $\mathcal{T}(A) \neq \mathbb{R}$.*

Next, we examine $\mathcal{T}(A)$ in the complex case. The following is an immediate consequence of [8, Theorem 10.11] (proved in [9]).

Lemma 3.7. *If $A \subseteq \mathbb{R}^n$ compact, then for μ^s-almost every $x \in A$, $\gamma_{n,n-m}$-almost every $W \in G(n, n-m)$:*

$$\dim_{\mathrm{H}}(A \cap (W + x)) \geq s - m.$$

(Here, $G(n,n-m)$ denotes the Grassmann manifold consisting of the $(n-m)$-dimensional subspaces of the linear space $\mathbb{R}^n$, while $\gamma_{n,n-m}$ is the natural measure on this manifold, which is preserved under the actions of the orthogonal group.)

Theorem 3.8. *Let $0 < a < b < 1$ and let $A \subseteq \mathbb{C}$ compact such that $\mathcal{T}(A) \cap (a,b) = \emptyset$. Then*

$$\dim_{\mathrm{H}}(A) \leq 1 + h(a, 1-b) < 2.$$

PROOF. It is clear that $1 + h(a, 1-b) < 2$.

Assume that $\dim_{\mathrm{H}}(A) > 1 + h(a, 1-b)$. Choose s such that $\dim_{\mathrm{H}}(A) > s > 1 + h(a, 1-b)$. Thus $\mu^s(A) > 0$. By Lemma 3.7, for some $x \in A$ and L line that passes through the origin, $\dim_{\mathrm{H}}(A \cap (L+x)) = s - 1 > h(a, 1-b)$. Then by Theorem 3.1, for some $x, y, z \in L \cap A$, $\frac{z-x}{y-x} \in (a,b)$. □

Corollary 3.9. *If $A \subseteq \mathbb{C}$ compact and $\dim_{\mathrm{H}}(A) = 2$, then $\mathcal{T}(A) \cap \mathbb{R}$ is dense in $\mathbb{R}$.*

Problem 2. *Is it true that if $A \subseteq \mathbb{C}$ compact and $\dim_{\mathrm{H}}(A) = 2$, then $\mathcal{T}(A)$ is dense in $\mathbb{C}$? Is it true that if $A \subseteq \mathbb{C}$ compact and $\dim_{\mathrm{H}}(A) > 1$, then $\mathcal{T}(A)$ is dense in $\mathbb{C}$?*

The condition $\dim_{\mathrm{H}}(A) > 1$ is obviously necessary: if A is a real set of dimension 1, then $\mathcal{T}(A)$ is real as well, therefore nowhere dense in $\mathbb{C}$.

Proposition 3.5 and Corollary 3.6 can be easily modified:

Proposition 3.10. *Let $A \subseteq \mathbb{C}$ compact. Then $\dim_{\mathrm{H}}(\mathcal{T}(A)) \leq \dim_{\mathrm{H}}(A) + 2\dim_{\mathrm{P}}(A)$.*

Corollary 3.11. *Let $A \subseteq \mathbb{C}$ compact. If $\dim_{\mathrm{H}}(A) + 2\dim_{\mathrm{P}}(A) < 2$, then $\mathcal{T}(A) \neq \mathbb{C}$.*

Earlier we proved that even in a full dimensional compact set on the plane we cannot guarantee any single triangle as a pattern. Then we saw that we cannot avoid "too many" patterns. One can ask if there are geometrically defined sets of patterns that we cannot avoid simultaneously.

Proposition 3.12 (Mattila [10])**.** *Let $A \subseteq \mathbb{C}$ compact. If $\mu^s(A) > 0$ and $s > 1$, then A contains the vertices of a rectangular triangle.*

PROOF. Apply Lemma 3.7. We have that for μ^s-almost every $x \in A$ and for almost every $L \in G(2,1)$, $\dim_{\mathrm{H}}(A \cap (L+x)) \geq s-1$. Choose an $x \in A$ with the property that for almost every $L \in G(2,1)$, $A \cap (L+x)$ contains points other than x. Then there are two lines $L_1, L_2 \in G(2,1)$ such that they are perpendicular and $A \cap (L_1 + x), A \cap (L_2 + x)$ contain points other than x. □

There are still several open problems. One more example:

Problem 3. *Is it true that if* $A \subseteq \mathbb{C}$ *compact and* $\dim_{\mathrm{H}}(A) = 2$*, then* A *contains the vertices of an isosceles triangle?*

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