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# TWO-NORM CONVERGENCE IN THE $L_p$ SPACES

**Abstract**

In this paper we consider $L_p$, $1 \leq p \leq \infty$, as a two-norm space and prove a representation for two-norm continuous functionals defined on $L_p$, $1 \leq p \leq \infty$. Hence we have provided a unified approach for the scale of the $L_p$ space, including the case when $p = \infty$

## 1 Introduction.

The Banach dual of $C[0,1]$ is $BV[0,1]$, where $C[0,1]$ and $BV[0,1]$ denote the space of all continuous functions on $[0,1]$ and the space of all functions of bounded variation on $[0,1]$, respectively. However, the Banach dual of $BV[0,1]$ is not $C[0,1]$ if we endorse $BV[0,1]$ with its usual norm, namely, $|f(0)| + V(f;[0,1])$ where $V(f;[0,1])$ denotes the total variation of $f$ on $[0,1]$. Since $BV[0,1]$ is not separable, then the usual technique of proving such a representation theorem no longer applies. More precisely, the proof often contains the following two steps. First, we prove the representation for some elementary functions, for example, step functions. Second, we approximate a general function by a sequence of elementary functions. Thus the representation for general functions follows from a convergence theorem for the integral. If the space is non-separable, the second step does not work. Hildebrandt [1] and Khaing ([2], [3]) proved a representation theorem for $BV[0,1]$ by regarding $BV[0,1]$ as a two-norm space [7].

For $1 \leq p < \infty$, $L_p[0,1]$ is a space of all measurable functions $f$ such that $\int_0^1 |f(x)|^p \, dx < \infty$ and $L_\infty[0,1]$ is a space of all functions $f$ with $\operatorname{ess\,sup} |f| <$

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$\infty$, where $\operatorname{ess\,sup}|f| = \inf\{M : |f(x)| \leq M$ a.e. on $[0,1]\}$. As we know, the Riesz representation theorem is well-known. If $p$ and $q$ are two real numbers, $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, the Banach dual of $L_p[0,1]$ is $L_q[0,1]$. We also have that the Banach dual of $L_1[0,1]$ which is $L_\infty[0,1]$. However, the Banach dual of $L_\infty$ is not $L_1$ if we endorse $L_\infty[0,1]$ with the usual norm, $\operatorname{ess\,sup}|f|$.

In this paper, we consider $L_p$, $1 \leq p \leq \infty$, as a two-norm space and prove a representation theorem for two-norm continuous linear functionals on $L_p$, $1 \leq p \leq \infty$. Furthermore, we give a unified approach to the dual of $L_p$, $1 \leq p \leq \infty$.

## 2 Two-norm convergence in $L_\infty$.

Let $L_\infty$ denote the space of all essentially bounded functions on $[0,1]$. A function $f$ is **essentially bounded** if it is bounded almost everywhere. The two norms defined on $L_\infty$, as suggested by Orlicz [7], are the essential bound $\|f\|_\infty$ and $\int_0^1 |f(x)|dx$.

In what follows, when we say **absolutely integrable** we mean **Lebesgue integrable**. A sequence $\{f_n\}$ of functions is said to be **two-norm convergent** in $L_\infty$, if there is $M > 0$ such that $\|f_n\|_\infty \leq M$ for all $n$ and

$$\lim_{n\to\infty} \int_0^1 f_n(x) g(x)\ dx \text{ exists,}$$

for every absolutely integrable function $g$ on $[0,1]$.

We shall prove the completeness in Theorem 2. However, we need the big Sandwich Lemma and the concept of an absolutely continuous function. We state without proof the Big Sandwich Lemma [5].

**Lemma 1.** *If* $0 \leq a_n \leq b_{kn}$ *for all* $n, k$ *and*

$$\lim_{k\to\infty} \lim_{n\to\infty} b_{kn} = 0$$

*then* $\lim_{n\to\infty} a_n = 0$.

A function $G$ defined on $[0,1]$ is said to be **absolutely continuous** if for every $\epsilon > 0$ there is $\delta > 0$ such that

$$|(\mathcal{D}) \sum \{G(v) - G(u)\}| < \epsilon$$

whenever $(\mathcal{D}) \sum |v - u| < \delta$, where $\mathcal{D} = \{[u,v]\}$ denotes a partial division of $[0,1]$ in which $[u,v]$ stands for a typical interval in the partial division. We are using the notation of the Henstock integral [6, 4].

**Theorem 2.** *If $\{f_n\}$ is two-norm convergent in $L_\infty$, then there exists a function $f \in L_\infty$, such that*

$$\int_0^1 f_n g \to \int_0^1 fg, \text{ as } n \to \infty,$$

*for every absolutely integrable function $g$ on* $[0,1]$.

Proof. Let $x \in [0,1]$ and define

$$g(t) = \begin{cases} 1 & \text{for } 1 \le t < x \\ 0 & \text{otherwise.} \end{cases}$$

Let $F(x) = \lim_{n\to\infty} \int_0^x f_n$, for $x \in [0,1]$. Since,

$$|\int_u^v f_n| \le \|f_n\|_\infty |v-u| \le M(v-u).$$

By taking $n \to \infty$, we have

$$|F(v) - F(u)| \le M|v-u|.$$

Therefore $F$ is absolutely continuous. As a corollary, $F'$ exists almost everywhere. Put $f = F'$ almost everywhere. Moreover,

$$|\frac{F(v)-F(u)}{v-u}| \le M.$$

That means, $|f(x)| \le M$ almost everywhere in $[0,1]$.
For any step function $g$,

$$\lim_{n\to\infty} \int_0^1 f_n g = \lim_{n\to\infty} \int_0^1 fg.$$

For $g \in L_1$, there exists a sequence of step function $\{g_k\}$ such that $\int_0^1 |g_k - g| \to 0$, as $k \to \infty$. Applying the Big Sandwich Lemma, we have

$$\begin{aligned} |\int_0^1 f_n g - \int_0^1 fg| &\le |\int_0^1 f_n g - \int_0^1 f_n g_k| \\ &\quad + |\int_0^1 f_n g_k - \int_0^1 f g_k| + |\int_0^1 f g_k - \int_0^1 fg| \\ &\le 2M \int_0^1 |g_k - g| + |\int_0^1 f_n g_k - \int_0^1 f g_k| \to 0 \text{ as } n \to \infty. \end{aligned}$$

That means, there is a function $f \in L_\infty$ such that for every $g \in L_1$,

$$\lim_{n\to\infty} \int_0^1 f_n g = \int_0^1 fg. \qquad \square$$

It is clear from the definition, if $g$ is absolutely integrable on $[0,1]$ and $\{f_n\}$ is two-norm convergent to $f$ in $L_\infty$ then

$$\lim_{n\to\infty} \int_0^1 f_n(x)g(x)dx = \int_0^1 f(x)g(x)dx.$$

We shall define two-norm continuous functional and shall prove Theorem 3, and finally the representation theorem in Theorem 5.

A functional $F$ defined on $L_\infty$ is said to be **two-norm continuous** in $L_\infty$, if

$$F(f_n) \to F(f) \text{ as } n \to \infty$$

whenever $\{f_n\}$ is two-norm convergent to $f$ in $L_\infty$.

**Theorem 3.** *If $g$ is absolutely integrable on $[0,1]$ and*

$$F(f) = \int_0^1 f(x)g(x)dx \text{ for } f \in L_\infty,$$

*then $F$ defines a two-norm continuous linear functional on $L_\infty$.*

PROOF. The linearity of $F$ comes from the properties of the integral. The continuity of $F$ comes from the definition of two-norm convergence in $L_\infty$. $\square$

In the following lemma, we define

$$\gamma_t(x) = \begin{cases} 1 & \text{for } 0 \le x < t \\ 0 & \text{for } t \le x \le 1. \end{cases}$$

**Lemma 4.** *Let $F$ be a two-norm continuous linear functional on $L_\infty$. If $G(t) = F(\gamma_t)$ for $t \in [0,1]$ then $G$ is absolutely continuous.*

PROOF. Suppose $G$ is not absolutely continuous on $[0,1]$. Then there is $\epsilon > 0$ such that for every $\delta$ there exists a partial division $\mathcal{D} = \{[u,v]\}$ satisfying

$$(\mathcal{D})\sum |v-u| < \delta \text{ and } |(\mathcal{D})\sum\{G(v)-G(u)\}| \ge \epsilon.$$

For each $n$, take $\delta = \frac{1}{n}$ and $\mathcal{D} = \mathcal{D}_n$. For every $x \in [0,1]$, put

$$f_n(x) = (\mathcal{D}_n)\sum |\gamma_v - \gamma_u|.$$

Then $\|f_n\|_\infty \leq 1$ for all $n$ and for every integrable function $g$ on $[0,1]$

$$\int_0^1 |f_n(x)g(x)|dx = (\mathcal{D}_n)\sum M|v-u| \downarrow 0 \text{ as } n \to \infty,$$

where $M$ is the essentially bound of $g$ on $[0,1]$. That is, $\{f_n\}$ is two-norm convergent to zero function in $L_\infty$. Yet we have

$$F(f_n) = |(\mathcal{D}_n)\sum\{G(v) - G(u)\}| \geq \epsilon \text{ for all } n.$$

It contradicts the fact that $F$ is two-norm continuous. Hence $G$ is absolutely continuous on $[0,1]$. □

**Theorem 5.** *If $F$ is a two-norm continuous linear functional on $L_\infty$ then there is an absolutely integrable function $g$ such that*

$$F(f) = \int_0^1 f(x)g(x)dx \quad \text{for} \quad f \in L_\infty.$$

Proof. In view of Lemma 4 and using the notation introduced there, we obtain

$$F(\gamma_t) = G(t) = \int_0^t g = \int_0^1 \gamma_t g,$$

where $g = G'$ almost everywhere on $[0,1]$. Since $F$ is linear,

$$F(f) = \int_0^1 fg$$

for any step function $f$. Take $f \in L_\infty$, there is a sequence $\{f_n\}$ of step functions two-norm convergent to $f \in L_\infty$. Hence the general case of the theorem follows from the definition of two-norm continuity of $F$ and the Dominated Convergence Theorem. □

We remark that the representation theorem (Theorem 5) remains valid if the two-norm convergence in $L_\infty$, as defined above, is replaced by boundedness in $\|f\|_\infty$ and convergence in $\int_0^1 |f|$ as given by Orlicz [7]. The proof follows the same argument as above.

## 3 A unified approach for $L_p$, $1 \leq p \leq \infty$.

For $1 \leq p < \infty$, $L_p$ denotes the space of all measurable functions such that $\int_0^1 |f|^p < \infty$ and

$$\|f\|_p = [\int_0^1 |f|^p]^{\frac{1}{p}}.$$

We restate the norm convergence in $L_p$ and the norm continuous functional on $L_p$, $1 \leq p \leq \infty$. A sequence $\{f_n\}$ of functions in $L_p$, $1 \leq p \leq \infty$, is said to be **norm convergent** to $f \in L_p$ if $\|f_n - f\|_p \to 0$ as $n \to \infty$.

In this section we regard $L_p$, $1 \leq p \leq \infty$, as a two-norm space based on the result of Section 2.

A sequence $\{f_n\}$ of functions is said to be **two-norm convergent** to $f$ in $L_p$, $1 \leq p \leq \infty$ if there is $M > 0$ such that $\|f_n\|_p \leq M$ for all $n$ and

$$\lim_{n\to\infty} \int_0^1 f_n(x)g(x)\ dx \text{ exists,}$$

for every $g \in L_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

The completeness will be proved in Theorem 7 using Lemma 6.

**Lemma 6.** *A function $f \in L_p$, $1 < p < \infty$, if and only if if*

$$\sup_{\mathcal{D}}\ (\mathcal{D}) \sum \frac{|F(v) - F(u)|^p}{|v-u|^{p-1}} < \infty,$$

*where the supremum is taken over all of divisions $\mathcal{D} = \{[u,v]\}$ of $[0,1]$, in which $[u,v]$ stands for a typical interval in the division.*

For a proof, see Riesz [8]. $\square$

**Theorem 7.** *If $\{f_n\}$ is two-norm convergent in $L_p$, $1 \leq p \leq \infty$, then there exists a function $f \in L_p$, such that*

$$\int_0^1 f_n g \to \int_0^1 fg, \text{ as } n \to \infty,$$

*for every $g \in L_q$, $\frac{1}{p} + \frac{1}{q} = 1$.*

PROOF. The case when $p = \infty$ follows from Theorem 2.
For $1 \leq p < \infty$. Take $F(x) = \lim_{n\to\infty} \int_0^x f_n$ for $x \in [0,1]$. Since

$$|\int_u^v f_n| \leq (\int_u^v |f_n|^p)^{\frac{1}{p}} (v-u)^{\frac{1}{q}},$$

then

$$\frac{|F_n(v) - F_n(u)|^p}{|v-u|^{p-1}} \leq \int_u^v |f_n|^p.$$

Therefore, for every partition $\mathcal{D}$ of $[0,1]$, we have

$$(\mathcal{D})\sum \frac{|F_n(v)-F_n(u)|^p}{|v-u|^{p-1}} \le (\mathcal{D})\sum \int_u^v |f_n|^p = \int_0^1 |f_n|^p \le M^p.$$

By taking $n \to \infty$,

$$(\mathcal{D})\sum \frac{|F(v)-F(u)|^p}{|v-u|^{p-1}} \le M^p.$$

Hence $f \in L_p$ by Lemma 6.

$$\begin{aligned} |\int_0^1 f_n g - \int_0^1 fg| &\le |\int_0^1 f_n g - \int_0^1 f_n g_k| \\ &\quad + |\int_0^1 f_n g_k - \int_0^1 f g_k| + |\int_0^1 f g_k - \int_0^1 fg| \\ &\le 2M\|g_k - g\|_q + |\int_0^1 f_n g_k - \int_0^1 f g_k| \to 0 \text{ as } n,k \to \infty. \end{aligned}$$

That means, there is a function $f \in L_p$, $1 \le p < \infty$, such that

$$\lim_{n\to\infty} \int_0^1 f_n g = \int_0^1 fg.$$

As a result, there is a function $f \in L_p$, $1 \le p \le \infty$, such that

$$\lim_{n\to\infty} \int_0^1 f_n g = \int_0^1 fg. \qquad \square$$

From the definition of two-norm convergence in $L_p$, if $g \in L_q$, $1 \le q \le \infty$ and $\{f_n\}$ is two-norm convergent to $f$ in $L_p$ then

$$\lim_{n\to\infty} \int_0^1 f_n(x)g(x)dx = \int_0^1 f(x)g(x)dx.$$

The connection between the two-norm convergence and norm convergence is given in Theorem 8 below.

**Theorem 8.** *Let $1 \le p \le \infty$. If $\{f_n\}$ is norm convergent to $f$ in $L_p$, then $\{f_n\}$ is two-norm convergent to $f$ in $L_p$.*

Proof. For every $g \in L_q$, we have $f_n g, fg \in L_1$. There exists a positive integer $n_o$ such that for every positive integer $n \ge n_o$, $\|f_n - f\|_p < 1$. Therefore, for $n \ge n_o$,

$$\|f_n\|_p \le \|f_n - f\|_p + \|f\|_p < 1 + \|f\|_p.$$

Take $M = \sup\{\|f_1\|_p, \|f_2\|_p, \ldots, \|f_{n_o-1}\|_p, 1 + \|f\|_p\}$, then

$$\sup \|f_n\|_p \leq M, \text{ for every } n.$$

Moreover,

$$|\int_0^1 f_n g - \int_0^1 fg| \leq \|f_n - f\|_p \|g\|_q \to 0, \text{ as } n \to \infty. \qquad \square$$

Due to the two-norm convergence we define above, we need to define the concept of two-norm continuity.

A functional $F$ defined on $L_p$ is said to be **two-norm continuous** in $L_p$, $1 \leq p < \infty$, if

$$F(f_n) \to F(f) \text{ as } n \to \infty$$

whenever $\{f_n\}$ is two-norm convergent to $f$ in $L_p$, $1 \leq p < \infty$.

**Theorem 9.** *If $g \in L_q$, $1 \leq q \leq \infty$, and*

$$F(f) = \int_0^1 f(x)g(x)dx \text{ for } f \in L_p,$$

*then $F$ defines a two-norm continuous linear functional on $L_p$.*

The proof is similar to that of Theorem 3.

As a result of Theorem 8, we can derive a connection between norm continuous and two norm continuous a functional on $L_p$, $1 \leq p \leq \infty$. We restate the definition of norm continuous functionals as follows.

A functional $F$ defined on $L_p$, $1 \leq p \leq \infty$, is said to be a **norm continuous functional** on $L_p$, if for every sequence $\{f_n\}$ that is norm convergent to $f$ in $L_p$ then $\{F(f_n)\}$ converges to $F(f)$.

**Theorem 10.** *Let $1 \leq p \leq \infty$. If $F$ is two-norm continuous functional on $L_p$, then $F$ is norm continuous functional on $L_p$.*

PROOF. Let $\{f_n\}$ be norm convergent to $f$ in $L_p$. By Theorem 8, $\{f_n\}$ is two-norm convergent to $f$ in $L_p$. Since $F$ is two-norm continuous on $L_p$, we have

$$F(f) = \lim_{n\to\infty} F(f_n).$$

That is, $F$ is norm continuous on $L_p$. $\square$

In $L_p$ space, $1 \leq p < \infty$, the Riesz Representation Theorem says that every norm continuous linear functional on $L_p$ determines a function $g \in L_q$, $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$F(f) = \int_0^1 fg, \quad \text{for every } f \in L_p.$$

Finally, we derive the representation theorem of two-norm functional on $L_p$, $1 \leq p \leq \infty$ in Theorem 11.

**Theorem 11.** *Let $1 \leq p \leq \infty$. If $F$ is two-norm continuous linear functional on $L_p$, then there exists a function $g \in L_q$, $\frac{1}{p} + \frac{1}{q} = 1$, such that*

$$F(f) = \int_0^1 fg, \quad \text{for every } f \in L_p.$$

PROOF. The proof follows from Theorem 5, Theorem 10, and the Riesz Representation Theorem. □

**Corollary 12.** *Let $1 \leq p < \infty$. A linear functional on $L_p$ is two-norm continuous if and only if it is norm continuous.*

The corollary does not hold for $p = \infty$. Indeed, the two-norm convergence in $L_\infty$ does not imply the norm convergence as shown in the following example.

**Example 13.** *Let $\{I_k\}$ be a collection of pairwise disjoint open intervals in $[0, 1]$ with the union is not of measure 1, in other words, the set $X = [0, 1] \setminus \cup_{n=1}^{\infty} I_n$ is not measure zero. Furthermore, $|I_{n+1} \cup I_{n+2} \cup I_{n+3} \cup \dots| \to 0$ as $n \to \infty$. Let*

$$f_n(x) = \begin{cases} 0, & x \in \cup_{k=1}^{n} I_k \\ 1, & \textit{otherwise,} \end{cases}$$

*and*

$$f(x) = \begin{cases} 1, & x \in X \\ 0, & \textit{otherwise.} \end{cases}$$

*Since $\|f_n - f\|_\infty = 1$ and will not tend to $0$, then $f_n$ is not norm-convergent. On the other hand, $\|f_n\|_\infty \leq 1$ for all $n$, and*

$$\lim_{n\to\infty} \int_0^1 f_n g = \int_0^1 fg,$$

*for every $g \in L_1$. Thus, $\{f_n\}$ is two-norm convergent to $f$.*

In conclusion, we have proved completeness theorem and the representation theorem for two-norm continuous linear functionals in $L_p$, $1 \leq p \leq \infty$.

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