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CONVEXITY IN REAL ANALYSIS

Abstract

We treat the classical notion of convexity in the context of hard real analysis. Definitions of the concept are given in terms of defining functions and quadratic forms, and characterizations are provided of different concrete notions of convexity. This analytic notion of convexity is related to more classical geometric ideas. Applications are given both to analysis and geometry.

1 Introduction.

Convexity is an old subject in mathematics. Archimedes used convexity in his studies of area and arc length. The concept appeared intermittently in the work of Fermat, Cauchy, Minkowski, and others. Even Johannes Kepler treated convexity. But it can be said that the subject was not really formalized until the seminal tract of Bonneson and Fenchel [2]. See also [3] for the history. Modern treatments of convexity may be found in [8] and [11].

In what follows, we let the term "domain" denote a connected, open set. We usually denote a domain by Ω. If Ω is a domain and $P, Q \in \Omega$ then the *closed segment* determined by P and Q is the set

$$\overline{PQ} \equiv \{(1-t)P + tQ : 0 \leq t \leq 1\}.$$

Most of the classical treatments of convexity rely on the following synthetic definition of the concept:

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Definition 1. Let $\Omega \subseteq \mathbb{R}^N$ be a domain. We say that Ω is *convex* if, whenever $P, Q \in \Omega$, then the closed segment $\overline{PQ}$ from P to Q lies in Ω.

Works such as [8] and [11] treat theorems of Helly and Kirchberger—about configurations of convex sets in the plane, and points in those convex sets. However, studies in analysis and differential geometry (as opposed to synthetic geometry) require results—and definitions—of a different type. We need hard analytic facts about the shape of the boundary—formulated in differential-geometric language. We need invariants that we can calculate and estimate. That is the point of view that we wish to explore in the present paper.

2 The Concept of Defining Function.

Let $\Omega \subseteq \mathbb{R}^N$ be a domain with C^1 boundary. A C^1 function $\rho : \mathbb{R}^N \to \mathbb{R}$ is called a *defining function* for Ω if

1. $\Omega = \{x \in \mathbb{R}^N : \rho(x) < 0\}$;
2. ${}^c\overline{\Omega} = \{x \in \mathbb{R}^N : \rho(x) > 0\}$;
3. $\nabla \rho(x) \neq 0 \quad \forall x \in \partial\Omega$.

In case $k \geq 2$ and ρ is C^k then we say that the domain Ω has C^k boundary.

This last point merits some discussion. For the notion of a domain having C^k boundary has many different formulations. One may say that Ω has C^k boundary if $\partial\Omega$ is a regularly imbedded C^k manifold in $\mathbb{R}^N$. Or if $\partial\Omega$ is locally the graph of a C^k function. In the very classical setting of $\mathbb{R}^2$, it is common to say that the boundary of a domain or region (which of course is simply a *curve* $\gamma : S^1 \to \mathbb{R}^2$) is C^k if **(a)** γ is a C^k function and **(b)** $\gamma' \neq 0$.

We shall not take the time here to prove the equivalence of all the different formulations of C^k boundary for a domain (but see the rather thorough discussion in Appendix I of [5]). But we do discuss the equivalence of the "local graph" definition with the defining function definition.

First suppose that Ω is a domain with C^k defining function ρ as specified above, and let $P \in \partial\Omega$. Since $\nabla\rho(P) \neq 0$, the implicit function theorem (see [7]) guarantees that there is a a neighborhood V_P of P, a variable (which we may take to be x_N) and a C^k function φ_P defined on a small open set $U_P \subseteq \mathbb{R}^{N-1}$ so that

$$\partial\Omega \cap V_P = \{(x_1, x_2, \ldots, x_N) : x_N = \varphi_P(x_1, \ldots, x_{N-1}) \, , \quad (x_1, \ldots, x_{N-1}) \in U_P\} \, .$$

Thus $\partial\Omega$ is locally the graph of the function φ_P near P.

Conversely, assume that each point $P \in \partial\Omega$ has a neighborhood V_P and an associated $U_P \subseteq \mathbb{R}^{N-1}$ and function φ_P such that

$$\partial\Omega \cap V_P = \{(x_1, x_2, \ldots, x_N) : x_N = \varphi_P(x_1, \ldots, x_{N-1})\,, \quad (x_1, \ldots, x_{N-1}) \in U_P\}\,.$$

We may suppose that the positive x_N-axis points *out* of the domain, and set $\rho_P(x) = x_N - \varphi_P(x_1, \ldots, x_{N-1})$. Thus, on a small neighborhood of P, ρ_P behaves like a defining function. It is equal to 0 on the boundary, certainly has non-vanishing gradient, and is C^k.

Now $\partial\Omega$ is compact, so we may cover $\partial\Omega$ with finitely many $V_{P_1}, \ldots, V_{P_k}$. Let $\{\psi_j\}$ be a partition of unity subordinate to this finite cover, and set

$$\widetilde{\rho}(x) = \sum_{j=1}^{k} \psi_j(x) \cdot \rho_{P_j}(x)\,.$$

Then, in a neighborhood of $\partial\Omega$, $\widetilde{\rho}$ is a defining function. We may extend $\widetilde{\rho}$ to all of space as follows. Let V be a neighborhood of $\partial\Omega$ on which $\widetilde{\rho}$ is defined. Let V' be an open, relatively compact subset of Ω and V'' an open subset of ${}^c\overline{\Omega}$ so that V, V', V'' cover $\mathbb{C}^n$. Let η, η', η'' be a partition of unity subordinate to the cover V, V', V''. Now set

$$\rho(x) = \eta'(x) \cdot [-(C+10)^2] + \eta(x) \cdot \widetilde{\rho}(x) + \eta''(x) \cdot (C+10)^2\,.$$

Here C is a large positive constant that exceeds the diameter of Ω. Then ρ is a globally defined, C^k function that is a defining function for Ω.

Definition 2. Let $\Omega \subseteq \mathbb{R}^N$ have C^1 boundary and let ρ be a C^1 defining function. Let $P \in \partial\Omega$. An $N-$tuple $w = (w_1, \ldots, w_N)$ of real numbers is called a *tangent vector* to $\partial\Omega$ at P if

$$\sum_{j=1}^{N} (\partial\rho/\partial x_j)(P) \cdot w_j = 0.$$

We write $w \in T_P(\partial\Omega)$.

For Ω with C^1 boundary, we think of $\nu_P = \nu = \langle \partial\rho/\partial x_1(P), \ldots, \partial\rho/\partial x_N(P)\rangle$ as the outward-pointing normal vector to $\partial\Omega$ at P. Of course the union of all the tangent vectors to $\partial\Omega$ at a point $P \in \partial\Omega$ is the *tangent plane* or *tangent hyperplane*. The tangent hyperplane is defined by the condition

$$\nu_P \cdot w = 0\,.$$

This definition makes sense when ν_P is well defined, in particular when $\partial\Omega$ is C^1.

If Ω is convex and $\partial\Omega$ is not smooth—say that it is Lipschitz—then any point $P \in \partial\Omega$ will still have one (or many) hyperplanes $\mathcal{P}$ such that $\mathcal{P} \cap \overline{\Omega} = \{P\}$. We call such a hyperplane a *support hyperplane* for $\partial\Omega$ at P. As noted, such a support hyperplane need not be unique. For example, if $\Omega = \{(x_1, x_2) : |x_1| < 1, |x_2| < 1\}$ then the points of the form $(\pm 1, \pm 1)$ in the boundary do *not* have well-defined tangent planes, but they do have (uncountably) many support hyperplanes.

Of course the definition of the normal ν_P makes sense only if it is independent of the choice of ρ. We shall address that issue in a moment. It should be observed that the condition defining tangent vectors simply mandates that $w \perp \nu_P$ at P. And, after all, we know from calculus that $\nabla\rho$ is the normal ν_P and that the normal is uniquely determined and independent of the choice of ρ. In principle, this settles the well-definedness issue.

However this point is so important, and the point of view that we are considering so pervasive, that further discussion is warranted. The issue is this: if $\widehat{\rho}$ is another defining function for Ω then it should give the same tangent vectors as ρ at any point $P \in \partial\Omega$. The key to seeing that this is so is to write $\widehat{\rho}(x) = h(x) \cdot \rho(x)$, for h a function that is non-vanishing near $\partial\Omega$. Then, for $P \in \partial\Omega$,

$$\begin{aligned}\sum_{j=1}^{N} (\partial\widehat{\rho}/\partial x_j)(P) \cdot w_j &= h(P) \cdot \left(\sum_{j=1}^{N} (\partial\rho/\partial x_j)(P) \cdot w_j \right) \\ &\quad + \rho(P) \cdot \left(\sum_{j=1}^{N} (\partial h/\partial x_j)(P) \cdot w_j \right) \\ &= h(P) \cdot \left(\sum_{j=1}^{N} (\partial\rho/\partial x_j)(P) \cdot w_j \right) \\ &\qquad + 0, \end{aligned} \tag{1.1}$$

because $\rho(P) = 0$. Thus w is a tangent vector at P vis a vis ρ if and only if w is a tangent vector vis a vis $\widehat{\rho}$. But why does h exist?

After a change of coordinates, it is enough to assume that we are dealing with a piece of $\partial\Omega$ that is a piece of flat, $(N-1)-$dimensional real hypersurface (just use the implicit function theorem). Thus we may take $\rho(x) = x_N$ and $P = 0$. Then any other defining function $\widehat{\rho}$ for $\partial\Omega$ near P must have the Taylor expansion

$$\widehat{\rho}(x) = c \cdot x_N + \mathcal{R}(x) \tag{1.2}$$

about 0. Here $\mathcal{R}$ is a remainder term[1] satisfying $\mathcal{R}(x) = o(|x_N|)$. There is no loss of generality to take $c = 1$, and we do so in what follows. Thus we wish to define

$$h(x) = \frac{\widehat{\rho}(x)}{\rho(x)} = 1 + \mathcal{S}(x).$$

Here $\mathcal{S}(x) \equiv \mathcal{R}(x)/x_N$ and $\mathcal{S}(x) = o(1)$ as $x_N \to 0$. Since this remainder term involves a derivative of $\widehat{\rho}$, it is plain that h is not even differentiable. (An explicit counterexample is given by $\widehat{\rho}(x) = x_N \cdot (1 + |x_N|)$.) Thus the program that we attempted in equation (1.1) above is apparently flawed.

However an inspection of the explicit form of the remainder term $\mathcal{R}$ reveals that, because $\widehat{\rho}$ is constant on $\partial\Omega$, h as defined above *is* continuously differentiable *in tangential directions.* That is, for tangent vectors w (vectors that are orthogonal to ν_P), the derivative

$$\sum_j \frac{\partial h}{\partial x_j}(P) w_j$$

is defined. Thus it does indeed turn out that our definition of tangent vector is well-posed when it is applied to vectors *that are already known to be tangent vectors* by the geometric definition $w \cdot \nu_P = 0$. For vectors that are *not* geometric tangent vectors, an even simpler argument shows that

$$\sum_j \frac{\partial \widehat{\rho}}{\partial x_j}(P) w_j \neq 0$$

if and only if

$$\sum_j \frac{\partial \rho}{\partial x_j}(P) w_j \neq 0.$$

Thus Definition 2 is well-posed. Questions similar to the one just discussed will come up below when we define convexity using C^2 defining functions. They are resolved in just the same way and we shall leave details to the reader.

The reader should check that the discussion above proves the following: if $\rho, \widetilde{\rho}$ are C^k defining functions for a domain Ω, with $k \geq 2$, then there is a C^{k-1}, nonvanishing function h defined near $\partial\Omega$ such that $\rho = h \cdot \widetilde{\rho}$.

[1]We may think of (1.2) as proved by integration by parts in the x_N variable only, and that gives this favorable estimate on the error terms $\mathcal{R}(x)$.

3 The Analytic Definition of Convexity.

For convenience, we restrict attention for this section to *bounded* domains. Many of our definitions would need to be modified, and extra arguments given in proofs, were we to consider unbounded domains as well.

Definition 3. Let $\Omega \subset \subset \mathbb{R}^N$ be a domain with C^2 boundary and ρ a defining function for Ω. Fix a point $P \in \partial\Omega$. We say that $\partial\Omega$ is analytically (weakly) *convex* at P if

$$\sum_{j,k=1}^{N} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) w_j w_k \geq 0, \quad \forall w \in T_P(\partial\Omega).$$

We say that $\partial\Omega$ is analytically *strongly (strictly) convex* at P if the inequality is strict whenever $w \neq 0$.

If $\partial\Omega$ is convex (resp. strongly convex) at each boundary point then we say that Ω is convex (resp. strongly convex).

One interesting and useful feature of this new definition of convexity is that it treats the concept point-by-point. The classical, synthetic definition specifies convexity for the whole domain at once.

It is natural to ask whether the new definition of convexity is independent of the choice of defining function. We have the following result:

Proposition 1. *Let $\Omega \subseteq \mathbb{R}^N$ be a domain with C^2 boundary. Let ρ and ρ' be C^2 defining functions for Ω, and assume that, at points x near $\partial\Omega$,*

$$\rho(x) = h(x) \cdot \rho'(x)$$

for some non-vanishing, C^2 function h. Let $P \in \partial\Omega$. Then Ω is convex at P when measured with the defining function ρ if and only if Ω is convex at P when measured with the defining function ρ'.

PROOF. We calculate that

$$\begin{aligned}
\frac{\partial^2}{\partial x_j \partial x_k}\rho(P) &= h(P) \cdot \frac{\partial^2 \rho'}{\partial x_j \partial x_k}(P) + \rho'(P) \cdot \frac{\partial^2 h}{\partial x_j \partial x_k}(P) \\
&\quad + \frac{\partial \rho'}{\partial x_j}(P)\frac{\partial h}{\partial x_k}(P) + \frac{\partial \rho'}{\partial x_k}(P)\frac{\partial h}{\partial x_j}(P) \\
&= h(P) \cdot \frac{\partial^2 \rho'}{\partial x_j \partial x_k}(P) + \frac{\partial \rho'}{\partial x_j}(P)\frac{\partial h}{\partial x_k}(P) + \frac{\partial \rho'}{\partial x_k}(P)\frac{\partial h}{\partial x_j}(P)
\end{aligned}$$

because $\rho'(P) = 0$. But then, if w is a tangent vector to $\partial\Omega$ at P, we see that

$$\begin{aligned}\sum_{j,k} \frac{\partial^2}{\partial x_j \partial x_k} \rho(P) w_j w_k &= h(P) \sum_{j,k} \frac{\partial^2 \rho'}{\partial x_j \partial x_k}(P) w_j w_k \\ &\quad + \left[\sum_j \frac{\partial \rho'}{\partial x_j}(P) w_j \right] \left[\sum_k \frac{\partial h}{\partial x_k}(P) w_k \right] \\ &\quad + \left[\sum_k \frac{\partial \rho'}{\partial x_k}(P) w_k \right] \left[\sum_j \frac{\partial h}{\partial x_j}(P) w_j \right] .\end{aligned}$$

If we suppose that P is a point of convexity relative to the defining function ρ', then the first sum is nonnegative. Of course h is positive, so the first expression is then ≥ 0. Since w is a tangent vector, the sum in j in the second expression vanishes. Likewise the sum in k in the third expression vanishes.

In the end, we see that the Hessian of ρ is positive semi-definite on the tangent space if the Hessian of ρ' is. The reasoning also works if the roles of ρ and ρ' are reversed. The result is thus proved. $\square$

The quadratic form

$$\left(\frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) \right)_{j,k=1}^{N}$$

is frequently called the "real Hessian" of the function ρ. This form carries considerable geometric information about the boundary of Ω. It is of course closely related to the second fundamental form of Riemannian geometry (see B. O'Neill [10]).

There is a technical difference between "strong" and "strict" convexity that we shall not discuss here (see L. Lempert [9] for details). It is common to use either of the words "strong" or "strict" to mean that the inequality in the last definition is strict when $w \neq 0$. The reader may wish to verify for himself that, at a strongly convex boundary point, all curvatures are positive (in fact one may, by the positive definiteness of the matrix $\left(\partial^2 \rho / \partial x_j \partial x_k\right)$, impose a change of coordinates at P so that the boundary of Ω agrees with a ball up to second order at P).

Now we explore our analytic notions of convexity. The first lemma is a technical one:

Lemma 2. *Let $\Omega \subseteq \mathbb{R}^N$ be strongly convex. Then there is a constant $C > 0$ and a defining function $\widetilde{\rho}$ for Ω such that*

$$\sum_{j,k=1}^{N} \frac{\partial^2 \widetilde{\rho}}{\partial x_j \partial x_k}(P) w_j w_k \geq C|w|^2, \quad \forall P \in \partial\Omega, w \in \mathbb{R}^N. \tag{2.1}$$

PROOF. Let ρ be some fixed C^2 defining function for Ω. For $\lambda > 0$ define

$$\rho_\lambda(x) = \frac{\exp(\lambda\rho(x)) - 1}{\lambda}.$$

We shall select λ large in a moment. Let $P \in \partial\Omega$ and set

$$X = X_P = \left\{ w \in \mathbb{R}^N : |w| = 1 \text{ and } \sum_{j,k} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) w_j w_k \leq 0 \right\}.$$

Then no element of X could be a tangent vector at P, hence $X \subseteq \{w : |w| = 1 \text{ and } \sum_j \partial\rho/\partial x_j(P) w_j \neq 0\}$. Since X is defined by a non-strict inequality, it is closed; it is of course also bounded. Hence X is compact and

$$\mu \equiv \min \left\{ \left| \sum_j \partial\rho/\partial x_j(P) w_j \right| : w \in X \right\}$$

is attained and is non-zero. Define

$$\lambda = \frac{-\min_{w \in X} \sum_{j,k} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) w_j w_k}{\mu^2} + 1.$$

Set $\widetilde{\rho} = \rho_\lambda$. Then for any $w \in \mathbb{R}^N$ with $|w| = 1$ we have (since $\exp(\rho_(P)) = 1$) that

$$\begin{aligned}
\sum_{j,k} \frac{\partial^2 \widetilde{\rho}}{\partial x_j \partial x_k}(P) w_j w_k &= \sum_{j,k} \left\{ \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) + \lambda \frac{\partial \rho}{\partial x_j}(P) \frac{\partial \rho}{\partial x_k}(P) \right\} w_j w_k \\
&= \sum_{j,k} \left\{ \frac{\partial^2 \rho}{\partial x_j \partial x_k} \right\} (P) w_j w_k + \lambda \left| \sum_j \frac{\partial \rho}{\partial x_j}(P) w_j \right|^2
\end{aligned}$$

If $w \notin X$ then this expression is positive by definition. If $w \in X$ then the expression is positive by the choice of λ. Since $\{w \in \mathbb{R}^N : |w| = 1\}$ is compact, there is thus a $C > 0$ such that

$$\sum_{j,k} \left\{ \frac{\partial^2 \widetilde{\rho}}{\partial x_j \partial x_k} \right\} (P) w_j w_k \geq C, \quad \forall w \in \mathbb{R}^N \text{ such that } |w| = 1.$$

This establishes our inequality (2.1) for $P \in \partial\Omega$ fixed and w in the unit sphere of $\mathbb{R}^N$. For arbitrary w, we set $w = |w|\widehat{w}$, with $\widehat{w}$ in the unit sphere. Then (2.1) holds for $\widehat{w}$. Multiplying both sides of the inequality for $\widehat{w}$ by $|w|^2$ and performing some algebraic manipulations gives the result for fixed P and all $w \in \mathbb{R}^N$.

Finally, notice that our estimates—in particular the existence of C, hold uniformly over points in $\partial\Omega$ near P. Since $\partial\Omega$ is compact, we see that the constant C may be chosen uniformly over all boundary points of Ω. □

Notice that the statement of the lemma has two important features: **(i)** that the constant C may be selected uniformly over the boundary and **(ii)** that the inequality (2.1) holds for all $w \in \mathbb{R}^N$ (not just tangent vectors). In fact it is impossible to arrange for anything like (2.1) to be true at a weakly convex point.

Our proof shows in fact that (2.1) is true not just for $P \in \partial\Omega$ but for P in a neighborhood of $\partial\Omega$. It is this sort of stability of the notion of strong convexity that makes it a more useful device than ordinary (weak) convexity.

Proposition 3. *If Ω is strongly convex then Ω is geometrically convex.*

Proof. We use a connectedness argument.

Clearly $\Omega \times \Omega$ is connected. Set $S = \{(P_1, P_2) \in \Omega \times \Omega : (1-\lambda)P_1 + \lambda P_2 \in \Omega, \text{ all } 0 < \lambda < 1\}$. Then S is plainly open and non-empty.

To see that S is closed, fix a defining function $\widetilde{\rho}$ for Ω as in the Lemma. If S is not closed in $\Omega \times \Omega$ then there exist $P_1, P_2 \in \Omega$ such that the function

$$t \mapsto \widetilde{\rho}((1-t)P_1 + tP_2)$$

assumes an interior maximum value of 0 on $[0, 1]$. But the positive definiteness of the real Hessian of $\widetilde{\rho}$ contradicts that assertion. The proof is complete. □

We gave a special proof that strong convexity implies geometric convexity simply to illustrate the utility of the strong convexity concept. It is possible to prove that an arbitrary (weakly) convex domain is geometrically convex by showing that such a domain can be written as the increasing union of strongly convex domains. However the proof is difficult and technical. We thus give another proof of this fact:

Proposition 4. *If Ω is (weakly) convex then Ω is geometrically convex.*

Proof. To simplify the proof we shall assume that Ω has at least C^3 boundary.

Assume without loss of generality that $N \geq 2$ and $0 \in \Omega$. For $\epsilon > 0$, let $\rho_\epsilon(x) = \rho(x) + \epsilon|x|^{2M}/M$ and $\Omega_\epsilon = \{x : \rho_\epsilon(x) < 0\}$. Then $\Omega_\epsilon \subseteq \Omega_{\epsilon'}$ if $\epsilon' < \epsilon$

and $\cup_{\epsilon>0}\Omega_\epsilon = \Omega$. If $M \in \mathbb{N}$ is large and ϵ is small, then Ω_ϵ is strongly convex. By Proposition 4, each Ω_ϵ is geometrically convex, so Ω is convex. □

We mention in passing that a nice treatment of convexity, from roughly the point of view presented here, appears in V. Vladimirov [12].

Proposition 5. *Let $\Omega \subset\subset \mathbb{R}^N$ have C^2 boundary and be geometrically convex. Then Ω is (weakly) convex.*

PROOF. Seeking a contradiction, we suppose that for some $P \in \partial\Omega$ and some $w \in T_P(\partial\Omega)$ we have

$$\sum_{j,k} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) w_j w_k = -2K < 0. \tag{2.2}$$

Suppose without loss of generality that coordinates have been selected in $\mathbb{R}^N$ so that $P = 0$ and $(0, 0, \ldots, 0, 1)$ is the unit outward normal vector to $\partial\Omega$ at P. We may further normalize the defining function ρ so that $\partial\rho/\partial x_N(0) = 1$. Let $Q = Q^t = tw + \epsilon \cdot (0, 0, \ldots, 0, 1)$, where $\epsilon > 0$ and $t \in \mathbb{R}$. Then, by Taylor's expansion,

$$\begin{aligned}
\rho(Q) &= \rho(0) + \sum_{j=1}^{N} \frac{\partial \rho}{\partial x_j}(0) Q_j + \frac{1}{2} \sum_{j,k=1}^{N} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(0) Q_j Q_k + o(|Q|^2) \\
&= \epsilon \frac{\partial \rho}{\partial x_N}(0) + \frac{t^2}{2} \sum_{j,k=1}^{N} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(0) w_j w_k + \mathcal{O}(\epsilon^2) + o(t^2) \\
&= \epsilon - Kt^2 + \mathcal{O}(\epsilon^2) + o(t^2).
\end{aligned}$$

Thus, if $t = 0$ and $\epsilon > 0$ is small enough, then $\rho(Q) > 0$. However, for that same value of ϵ, if $|t| > \sqrt{2\epsilon/K}$ then $\rho(Q) < 0$. This contradicts the definition of geometric convexity. □

Remark: The reader can already see in the proof of the proposition how useful the quantitative version of convexity can be.

The assumption that $\partial\Omega$ be C^2 is not very restrictive, for convex functions of one variable are twice differentiable almost everywhere (see A. Zygmund [13]). On the other hand, C^2 smoothness of the boundary is essential for our approach to the subject. □

Exercises for the Reader: If $\Omega \subseteq \mathbb{R}^N$ is a domain then the *closed convex hull* of Ω is defined to be the closure of the set $\left\{\sum_{j=1}^m \lambda_j s_j : s_j \in \Omega, m \in \mathbb{N}, \lambda_j \geq 0, \sum \lambda_j = 1\right\}$. Equivalently, the closed convex hull of Ω is the intersection of all closed, convex sets that contain Ω.

Assume in the following problems that $\overline{\Omega} \subseteq \mathbb{R}^N$ is closed, bounded, and convex. Assume that Ω has C^2 boundary.

(**a**) We shall say more about extreme points in the penultimate section. For now, a point $P \in \partial\Omega$ is extreme (for Ω convex) if, whenever $P = (1-\lambda)x + \lambda y$ and $0 \leq \lambda \leq 1$, $x, y \in \overline{\Omega}$, then $x = y = P$. Prove that $\overline{\Omega}$ is the closed convex hull of its extreme points (this result is usually referred to as the *Krein-Milman theorem* and is true in much greater generality).

(**b**) Let $P \in \partial\Omega$ be extreme. Let $\mathbf{p} = P + T_P(\partial\Omega)$ be the geometric tangent affine hyperplane to the boundary of Ω that passes through P. Show by an example that it is not necessarily the case that $\mathbf{p} \cap \overline{\Omega} = \{P\}$.

(**c**) Prove that if Ω_0 is *any* bounded domain with C^2 boundary then there is a relatively open subset U of $\partial\Omega_0$ such that U is strongly convex. (Hint: Fix $x_0 \in \Omega_0$ and choose $P \in \partial\Omega_0$ that is as far as possible from x_0).

(**d**) If Ω is a convex domain then the Minkowski functional[2] (see [8]) less 1 gives a convex defining function for Ω.

4 Convex Functions and Exhaustion Functions.

Let $F : \mathbb{R}^N \to \mathbb{R}$ be a function. We say that F is *convex* if, for any $P, Q \in \mathbb{R}^N$ and any $0 \leq t \leq 1$, it hold that

$$F((1-t)P + tQ) \leq (1-t)f(P) + tf(Q) \,.$$

In the case that F is C^2, we may restrict F to the line passing through P and Q and differentiate the function

$$\varphi_{P,Q} : t \longmapsto F((1-t)P + tQ)$$

twice to see (from calculus—reference [BLK]) that

$$\frac{d^2}{dt^2}\varphi_{P,Q} \geq 0 \,.$$

[2] A simple instance of the Minkowski functional is this. Let $K \subseteq \mathbb{R}^N$ be convex. For $x \in \mathbb{R}^N$, define

$$p(x) = \inf\{r > 0 : x \in rK\} \,.$$

Then p is a Minkowski functional for K.

If we set $\alpha = Q - P = (\alpha_1, \alpha_2, \ldots, \alpha_N)$, then this last result may be written as

$$\frac{\partial^2}{\partial \alpha^2} F \geq 0\,.$$

This in turn may be rewritten as

$$\sum_{j,k} \frac{\partial^2}{\partial x_j \partial x_k} \alpha_j \alpha_k \geq 0\,.$$

In other words, the Hessian of F is positive semi-definite.

In the case that a C^2 function F has positive definite Hessian at each point then we say that F is *strictly convex* or *strongly convex*.

The reasoning in the penultimate paragraph can easily be reversed to see that the following is true:

Proposition 6. *A C^2 function on $\mathbb{R}$ is convex if and only if it has positive semi-definite Hessian at each point of its domain.*

Of course it is also useful to consider convex functions on a domain. Certainly we may say that $F : \Omega \to \mathbb{R}$ is convex (with Ω a convex domain) if

$$\sum_{j,k} \frac{\partial^2}{\partial x_j \partial x_k}(x) \alpha_j \alpha_k \geq 0$$

for all $(\alpha_1, \ldots, \alpha_N)$ and all points $x \in \Omega$. Equivalently, F is convex on a convex domain Ω if, whenever $P, Q \in \Omega$ and $0 \leq \lambda \leq 1$ we have

$$F((1-t)P + tQ) \leq (1-t)F(P) + tF(Q)\,.$$

It is straightforward to prove that any convex function is continuous. See [13] or [12, p. 85]. Other properties of convex functions are worth noting. For example, if $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $\varphi : \mathbb{R} \to \mathbb{R}$ is convex and increasing then $\varphi \circ f$ is convex. Certainly the sum of any two convex functions is convex. If $\{f_\alpha\}_{\alpha \in A}$ is any family of convex functions then

$$f(x) \equiv \sup_{\alpha \in A} f_\alpha(x)$$

is convex. The proof of this latter assertion is straightforward: If P, Q lie in the common domain of the f_α and $0 \leq \lambda \leq 1$ and $\alpha \in A$ then

$$f_\alpha((1-\lambda)P + \lambda Q) \leq (1-\lambda) f_\alpha(P) + \lambda f_\alpha(Q)\,.$$

Then certainly

$$f_\alpha((1-\lambda)P + \lambda Q) \leq (1-\lambda)f(P) + \lambda f(Q)\,.$$

Now take the supremum over α on the lefthand side to obtain the result.

It is always useful to be able to characterize geometric properties of domains in terms of functions. For functions are more flexible objects than domains: one can do more with functions. With this thought in mind we make the following definition:

Definition 4. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. We call a function

$$\lambda : \Omega \to \mathbb{R}$$

an *exhaustion function* if, for each $c \in \mathbb{R}$, the set

$$\lambda^{-1}((-\infty, c]) = \{x \in \Omega : \lambda(x) \leq c\}$$

is a compact subset of Ω.

The key idea here is that the function λ is real-valued and blows up at $\partial\Omega$.

Theorem 7. *A domain $\Omega \subseteq \mathbb{R}^N$ is convex if and only if it has a continuous, convex exhaustion function.*

Proof. If Ω possesses such an exhaustion function λ, then the domains

$$\Omega_k \equiv \{x \in \Omega : \lambda < k\}$$

are convex. And Ω itself is the increasing union of the Ω_k. It follows immediately, from the synthetic definition of convexity, that Ω is convex.

For the converse, observe that if Ω is convex and $P \in \partial\Omega$, then the tangent hyperplane at P has the form $a \cdot (x - P) = 0$. Here a is a Euclidean unit vector. It then follows that the quantity $a \cdot (x - P)$ is the distance from $x \in \Omega$ to this hyperplane. Now the function

$$\mu_{a,P}(x) \equiv -\ln a \cdot (x - P)$$

is convex since one may calculate the Hessian $\mathcal{H}$ directly. Its value at a point x equals

$$\mathcal{H}(b, b) = \frac{(a \cdot b)^2}{[a \cdot (x - P)]^2} \geq 0\,.$$

If $\delta_\Omega(x)$ is the Euclidean distance of x to $\partial\Omega$, then

$$\tau_\Omega(x) \equiv -\log \delta_\Omega(x) = \sup_{P \in \partial\Omega} \left[-\log a \cdot (x - P)\right]\,.$$

Thus $-\log \delta_\Omega$ is a convex function that blows up at $\partial\Omega$. Now set

$$\lambda(x) = \max\left\{\tau_\Omega(x), |x|^2\right\}.$$

This is a continuous, convex function that blows up at the boundary. So it is the convex exhaustion function that we seek. □

Lemma 8. *Let F be a convex function on $\mathbb{R}^N$. Then there is a sequences $f_1 \geq f_2 \geq \cdots$ of C^∞, strongly convex functions such that $f_j \to F$ pointwise.*

PROOF. Let φ be a C_c^∞ function which is nonnegative and has integral 1. We may also take φ to be supported in the unit ball, and be radial. For $\epsilon > 0$ we set

$$\varphi_\epsilon(X) = \epsilon^{-N}\varphi(x/\epsilon).$$

We define

$$F_\epsilon(x) = F * \varphi_\epsilon(x) = \int F(x-t)\varphi_\epsilon(t)\,dt.$$

We assert that each F_ϵ is convex. For let $P, Q \in \mathbb{R}^N$ and $0 \leq \lambda \leq 1$. Then

$$\begin{aligned}
F_\epsilon((1-\lambda)P + \lambda Q) &= \int F((1-\lambda)P + \lambda Q - t)\varphi_\epsilon(t)\,dt \\
&= \int F((1-\lambda)(P-t) + \lambda(Q-t))\varphi_\epsilon(t)\,dt \\
&\leq \int \big[(1-\lambda)F(P-t) + \lambda F(Q-t)\big]\varphi_\epsilon(t)\,dt \\
&= (1-\lambda)F_\epsilon(P) + \lambda F_\epsilon(Q).
\end{aligned}$$

So F_ϵ is convex.

Now set

$$f_j(x) = F_{\epsilon_j} + \delta_j|x|^2.$$

Certainly f_j is strongly convex because F_ϵ is convex and $|x|^2$ strongly convex. If $\epsilon_j > 0$, $\delta_j > 0$ are chosen appropriately, then we will have

$$f_1 \geq f_2 \geq \ldots$$

and $f_j \to F$ pointwise. That is the desired conclusion. □

Proposition 9. *Let $F : \mathbb{R}^N \to \mathbb{R}$ be a continuous function. Then F is convex if and only if, for any $\varphi \in C_c^\infty(\mathbb{R}^N)$ with $\varphi \geq 0$, $\int \varphi\,dx = 1$, and any $w = (w_1, w_2, \ldots, w_N) \in \mathbb{R}^N$ it holds that*

$$\int_{\mathbb{R}^N} F(x)\left[\sum_{j,k} \frac{\partial^2\varphi}{\partial x_j \partial x_k}(x) w_j w_k\right] dx \geq 0.$$

Proof. Assume that F is convex. In the special case that $F \in C^\infty$, we certainly know that

$$\sum_{j,k} \frac{\partial^2 F}{\partial x_j \partial x_k}(x) w_j w_k \geq 0 .$$

Hence it follows that

$$\int_{\mathbb{R}^N} \sum_{j,k} \frac{\partial^2 F}{\partial x_j \partial x_k}(x) w_j w_k \cdot \varphi(x)\, dx \geq 0 .$$

Now the result follows from integrating by parts twice (the boundary terms vanish since φ is compactly supported).

Now the general case follows by approximating F as in the preceding lemma.

For the converse direction, we again first treat the case when $F \in C^\infty$. Assume that

$$\int_{\mathbb{R}^N} \sum_{j,k} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(x) w_j w_k \cdot F(x)\, dx \geq 0$$

for all suitable φ. Then integration by parts twice gives us the inequality we want.

For general F, let ψ be a nonnegative C_c^∞ function, supported in the unit ball, with integral 1. Set $\psi_\epsilon(x) = \epsilon^{-N}\psi(x/\epsilon)$. Let $K = \operatorname{supp} \varphi$ and let $\overline{U}$ be the closure of an open set U that contains K. Define $F_\epsilon(x) = F * \psi_\epsilon(x) + c_\epsilon$, where the constant c_ϵ is chosen so that $F_\epsilon \geq F$ on $\overline{U}$. Then $F_\epsilon \to F$ pointwise and

$$\int_{\mathbb{R}^N} \sum_{j,k} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(x) w_j w_k \cdot F(x)\, dx \geq 0$$

certainly implies that

$$\int_{\mathbb{R}^N} \sum_{j,k} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(x) w_j w_k \cdot F_\epsilon(x)\, dx \geq 0 .$$

We may integrate by parts twice in this last expression to obtain

$$\int_{\mathbb{R}^N} \varphi(x) \sum_{j,k} \frac{\partial^2 F_\epsilon}{\partial x_j \partial x_k}(x) w_j w_k \, dx \geq 0 .$$

It follows that each F_ϵ is convex. Thus

$$F_\epsilon((1-\lambda)P + \lambda Q) \leq (1-\lambda)F_\epsilon(P) + \lambda F_\epsilon(Q)$$

for every P, Q, λ. Letting $\epsilon \to 0^+$ yields that

$$F((1-\lambda)P + \lambda Q) \leq (1-\lambda)F(P) + \lambda F(Q)$$

hence F is convex. That completes the proof. $\square$

For applications in the next theorem, it is useful to note the following:

Proposition 10. *Any convex function f is subharmonic.*

PROOF. To see this, let P and P' be distinct points in the domain of f and let X be their midpoint. Then certainly

$$2f(X) \leq f(P) + f(P')\,.$$

Let η be any special orthogonal rotation centered at X. We may write

$$2f(X) \leq f(\eta(P)) + f(\eta(P'))\,.$$

Now integrate out over the special orthogonal group to derive the usual sub-mean-value property for subharmonic functions. $\square$

The last topic is also treated quite elegantly in Chapter 3 of [4]. One may note that the condition that the Hessian be positive semi-definite is stronger than the condition that the Laplacen be nonnegative. That gives another proof of our result.

Theorem 11. *A domain $\Omega \subseteq \mathbb{R}^N$ is convex if and only if it has a C^∞, strictly convex exhaustion function.*

PROOF. Only the forward direction need be proved (as the converse direction is contained in the last theorem).

We build the function up iteratively. We know by the preceding theorem that there is a continuous exhaustion function λ. Let

$$\Omega_c = \{x \in \Omega : \lambda(x) + |x|^2 < c\}$$

for $c \in \mathbb{R}$. Then each $\Omega_c \subset\subset \Omega$ and $c' > c$ implies that $\Omega_c \subset\subset \Omega_c'$. Now let $0 \leq \varphi \in C_c^\infty(\mathbb{R}^N)$ with $\int \varphi\, dx = 1$, φ radial. We may take φ to be supported in $B(0,1)$. Let $0 < \epsilon_j < \mathrm{dist}(\Omega_{j+1}, \partial\Omega)$. If $x \in \Omega_{j+1}$, set

$$\lambda_j(x) = \int_\Omega [\lambda(t) + |t|^2]\epsilon_j^{-N}\varphi((x-t)/\epsilon_j)\, dV(t) + |x|^2 + 1\,.$$

Then each λ_j is C^∞ and strictly convex on Ω_{j+1}. Moreover, by the previously noted subharmonicity of λ, we may be sure that $\lambda_j(x) > \lambda(x) + |x|^2$ on $\overline{\Omega}_j$.

Now let $\chi \in C^\infty(\mathbb{R})$ be a convex function with $\chi(t) = 0$ for $t \le 0$ and $\chi'(t), \chi''(t) > 0$ when $t > 0$. Note that, $\Psi_j(x) \equiv \chi(\lambda_j(x) - (j-1))$ is positive and convex on $\Omega_j \setminus \overline{\Omega}_{j-1}$ and is, of course, C^∞. Notice now that $\lambda_0 > \lambda$ on Ω_0. If a_1 is large and positive, then $\lambda_1' \equiv \lambda_0 + a_1\Psi_1 > \lambda$ on Ω_1. Inductively, if $a_1, a_2, \dots a_{\ell-1}$ have been chosen, select $a_\ell > 0$ such that $\lambda_\ell' \equiv \lambda_0 + \sum_{j=1}^{\ell} a_j \Psi_j > \lambda$ on Ω_ℓ.

Since $\Psi_{\ell+k} = 0$ on Ω_ℓ, $k > 0$, we see that $\lambda_{\ell+k}' = \lambda_{\ell+k'}'$ on Ω_ℓ for any $k, k' > 0$. So the sequence λ_ℓ' stabilizes on compacta and $\lambda' \equiv \lim_{\ell\to\infty} \lambda_\ell'$ is a C^∞ strictly convex function that majorizes λ. Hence λ' is the smooth, strictly convex exhaustion function that we seek. $\square$

Corollary 12. *Let $\Omega \subseteq \mathbb{R}^N$ be any convex domain. Then we may write*

$$\Omega = \bigcup_{j=1}^{\infty} \Omega_j \,,$$

where this is an increasing union and each Ω_j is strongly convex with C^∞ boundary.

PROOF. Let λ be a smooth, strictly convex exhaustion function for Ω. By Sard's theorem (see [6]), there is a strictly increasing sequence of values $c_j \to +\infty$ so that

$$\Omega_{c_j} = \{x \in \Omega : \lambda(x) < c_j\}$$

has smooth boundary. Then of course each Ω_{c_j} is strongly convex. And the Ω_{c_j} form an increasing sequence of domains whose union is Ω. $\square$

5 Other Characterizations of Convexity.

Let $\Omega \subseteq \mathbb{R}^N$ be a domain and let $\mathcal{F}$ be a family of real-valued functions on Ω (we do not assume in advance that $\mathcal{F}$ is closed under any algebraic operations, although often in practice it will be). Let K be a compact subset of Ω. Then the *convex hull of K in Ω with respect to $\mathcal{F}$* is defined to be

$$\widehat{K}_{\mathcal{F}} \equiv \left\{ x \in \Omega : f(x) \le \sup_{t \in K} f(t) \text{ for all } f \in \mathcal{F} \right\}.$$

We sometimes denote this hull by $\widehat{K}$ when the family $\mathcal{F}$ is understood or when no confusion is possible. We say that Ω is *convex* with respect to $\mathcal{F}$ provided $\widehat{K}_{\mathcal{F}}$ is compact in Ω whenever K is. When the functions in $\mathcal{F}$ are complex-valued then $|f|$ replaces f in the definition of $\widehat{K}_{\mathcal{F}}$.

Proposition 13. *Let $\Omega \subset\subset \mathbb{R}^N$ and let $\mathcal{F}$ be the family of real linear functions. Then Ω is convex with respect to $\mathcal{F}$ if and only if Ω is geometrically convex.*

PROOF. Exercise. Use the classical definition of convexity at the beginning of the paper. □

Proposition 14. *Let $\Omega \subset\subset \mathbb{R}^N$ be any domain. Let $\mathcal{F}$ be the family of continuous functions on Ω. Then Ω is convex with respect to $\mathcal{F}$.*

PROOF. If $K \subset\subset \Omega$ and $x \notin K$ then the function $F(t) = 1/(1 + |x - t|)$ is continuous on Ω. Notice that $f(x) = 1$ and $|f(k)| < 1$ for all $k \in K$. Thus $x \notin \widehat{K}_{\mathcal{F}}$. Therefore $\widehat{K}_{\mathcal{F}} = K$ and Ω is convex with respect to $\mathcal{F}$. □

We close this discussion of convexity with a geometric characterization of the property. We shall, later in the paper, refer to this as the "segment characterization". First, if $\Omega \subseteq \mathbb{R}^N$ is a domain and I is a closed one-dimensional segment lying in Ω then the boundary ∂I is the set consisting of the two endpoints of I. Now the domain Ω is convex if and only if, whenever $\{I_j\}_{j=1}^{\infty}$ is a collection of closed segments in Ω and $\{\partial I_j\}$ is relatively compact in Ω, then so is $\{I_j\}$. This is little more than a restatement of the classical definition of geometric convexity. We invite the reader to supply the details.

In fact the formulation in the last paragraph admits of many variants. One of these is the following: If $\{I_j\}$ is a collection of closed segments in Ω then

$$\mathrm{dist}\,(\partial I_j, \partial\Omega)$$

is bounded from 0 if and only if

$$\mathrm{dist}\,(I_j, \partial\Omega)$$

is bounded from 0. The following example puts these ideas in perspective.

Example 5. Let $\Omega \subseteq \mathbb{R}^2$ be

$$\Omega = B((0,0),2) \setminus \overline{B((1,0),1)}\,.$$

Let

$$I_j = \{(-1/j, t) : -1/2 \leq t \leq 1/2\}\,.$$

Then it is clear that

$$\{\partial I_j\}$$

is relatively compact in Ω while

$$\{I_j\}$$

is not. And of course Ω is not convex.

6 Convexity of Finite Order.

There is a fundamental difference between the domains

$$B = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$$

and

$$E = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^4 < 1\} .$$

Both of these domains are convex. The first of these is strongly convex and the second is not. More generally, each of the domains

$$E_m = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^{2m} < 1\}$$

is, for $m = 2, 3, \ldots$, weakly (not strongly) convex. Somehow the intuition is that, as m increases, the domain E_m becomes *more* weakly convex. Put differently, the boundary points $(\pm 1, 0)$ are becoming flatter and flatter as m increases.

We would like to have a way of quantifying, indeed of measuring, the indicated flatness. These considerations lead to a new definition. We first need a bit of terminology.

Let f be a function on an open set $U \subseteq \mathbb{R}^N$ and let $P \in U$. We say that *f vanishes to order k at P* if any derivative of f, up to and including order k, vanishes at P. Thus, if $f(P) = 0$ but $\nabla f(P) \neq 0$, then we say that f vanishes to order 0. If $f(P) = 0$, $\nabla f(P) = 0$, $\nabla^2 f(P) = 0$, and $\nabla^3 f(P) \neq 0$, then we say that f vanishes to order 2.

Let Ω be a domain and $P \in \partial\Omega$. Suppose that $\partial\Omega$ is smooth near P. We say that the tangent plane $T_P(\partial\Omega)$ has order of contact k with $\partial\Omega$ at P if the defining function ρ for Ω satisfies

$$|\rho(x)| \leq C|x - P|^k \qquad \text{for all } x \in T_P(\partial\Omega) ,$$

and this same inequality does *not* hold with k replaced by $k + 1$.

Definition 6. *Let $\Omega \subseteq \mathbb{R}^N$ be a domain and $P \in \partial\Omega$ a point at which the boundary is at least C^k for k a positive integer. We say that P is convex of order k if*

- *The point P is convex;*
- *The tangent plane to $\partial\Omega$ at P has order of contact k with the boundary at P.*

Example 7. Notice that a point of strong convexity will be convex of order 2. The boundary point $(1, 0)$ of the domain

$$E_{2k} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^{2k} < 1\}$$

is convex of order $2k$.

Proposition 15. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, and let $P \in \partial\Omega$ be convex of finite order. Then that order is an even number.*

PROOF. Let m be the order of the point P.

We may assume that P is the origin and that the outward normal direction at P is the x_1 direction. If ρ is a defining function for Ω near P then we may use the Taylor expansion about P to write

$$\rho(x) = 2x_1 + \varphi(x)\,,$$

and φ will vanish to order m. If m is odd, then the domain will not lie on one side of the tangent hyperplane

$$T_P(\partial\Omega) = \{x : x_1 = 0\}\,.$$

So Ω cannot be convex. □

A very important feature of convexity of finite order is its stability. We formulate that property as follows:

Proposition 16. *Let $\Omega \subseteq \mathbb{R}^N$ be a smoothly bounded domain and let $P \in \partial\Omega$ be a point that is convex of finite order m. Then points in $\partial\Omega$ that are sufficiently near P are also convex of finite order at most m.*

PROOF. Let $\Omega = \{x \in \mathbb{R}^N : \rho(x) < 0\}$, where ρ is a defining function for Ω. Then the "finite order" condition is given by the nonvanishing of a derivative of ρ at P. Of course that same derivative will be nonvanishing at nearby points, and that proves the result. □

Proposition 17. *Let $\Omega \subseteq \mathbb{R}^N$ be a smoothly bounded domain. Then there will be a point $P \in \partial\Omega$ and a neighborhood U of P so that each point of $U \cap \partial\Omega$ will be convex of order 2 (i.e., strongly convex).*

PROOF. Let D be the diameter of Ω. We may assume that $\overline{\Omega}$ is distance at least $10D + 10$ from the origin 0. Let P be the point of $\partial\Omega$ which is furthest (in the Euclidean metric) from 0. Then P is the point that we seek.

Let L be the distance of 0 to P. Then we see that the sphere with center 0 and radius L externally osculates $\partial\Omega$ at P. Of course the sphere is strongly convex at the point of contact. Hence so is $\partial\Omega$. By the continuity of second derivatives of the defining function for Ω, the same property holds for nearby points in the boundary. That completes the proof. $\square$

Example 8. Consider the domain

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^4 < 1\}\,.$$

The boundary points of the form $(a, b, 0)$ are convex of order 4. All others are convex of order 2 (i.e., strongly convex).

It is straightforward to check that Euclidean isometries preserve convexity, preserve strong convexity, and preserve convexity of finite order. Diffeomorphisms do not. In fact we have:

Proposition 18. *Let Ω_1, Ω_2 be smoothly bounded domains in $\mathbb{R}^N$, let $P_1 \in \partial\Omega_1$ and $P_2 \in \partial\Omega_2$. Let Φ be a diffeomorphism from $\overline{\Omega_1}$ to $\overline{\Omega_2}$ and assume that $\Phi(P_1) = P_2$. Further suppose that the Jacobian matrix of Φ at P_1 is an orthogonal linear mapping. Then we have:*

- *If P_1 is a convex boundary point then P_2 is a convex boundary point;*
- *If P_1 is a strongly convex boundary point then P_2 is a strongly convex boundary point;*
- *If P_1 is a boundary point that is convex of order $2k$ then P_2 is a boundary point that is convex of order $2k$.*

Proof. We consider the first assertion. Let ρ be a defining function for Ω_1. Then $\rho \circ \Phi^{-1}$ will be a defining function for Ω_2. Of course we know that the Hessian of ρ at P_1 is positive semi-definite. It is straightforward to calculate the Hessian of $\rho' \equiv \rho \circ \Phi^{-1}$ and see that it is just the Hessian of ρ composed with Φ applied to the vectors transformed under Φ^{-1}. So of course ρ' will have positive semi-definite Hessian.

The other two results are verified using the same calculation. $\square$

Proposition 19. *Let Ω be a smoothly bounded domain in $\mathbb{R}^N$. Let $\mathcal{L}$ be an invertible linear map on $\mathbb{R}^N$. Define $\Omega' = \mathcal{L}(\Omega)$. Then*

- *Each convex boundary point of Ω is mapped to a convex boundary point of Ω'.*
- *Each strongly convex boundary point of Ω is mapped to a strongly convex boundary point of Ω'.*

- *Each boundary point of Ω that is convex of order $2k$ is mapped to a boundary point of Ω' that is convex of order $2k$.*

PROOF. Obvious. $\square$

Maps which are not invertible tend to decrease the order of a convex point. An example will illustrate this idea:

Example 9. Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ be the unit ball and $\Omega' = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^4 + x_2^4 < 1\}$. We see that

$$\Phi(x_1, x_2) = (x_1^2, x_2^2)$$

maps Ω' onto Ω. And we see that Ω is strongly convex (i.e., convex of order 2 at each boundary point) while Ω' has boundary points that are convex of order 4. The points of order 4 are mapped by Φ to points of order 2.

7 Extreme Points.

A point $P \in \partial\Omega$ is called an *extreme point* if, whenever $a, b \in \partial\Omega$ and $P = (1-\lambda)a + \lambda b$ for some $0 \le \lambda \le 1$ then $a = b = P$.

It is easy to see that, on a convex domain, a point of strong convexity must be extreme, and a point that is convex of order $2k$ must be extreme. But convex points in general are *not* extreme.

Example 10. Let

$$(\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < 1, |x_2| < 1\}\,.$$

Then Ω is clearly convex. But any boundary point with x_1, x_2 not both 1 is not extreme.

For example, consider the boundary point $(1/2, 1)$. Then

$$(1/2, 1) = \frac{1}{2}(1/4, 1) + \frac{1}{2}(3/4, 1)\,.$$

Example 11. Let $\Omega \subseteq \mathbb{R}^2$ be the domain with boundary consisting of

- The segments from $(-3/4, 1)$ to $(3/4, 1)$, from $(1, 3/4)$ to $(1, -3/4)$, from $(3/4, -1)$ to $(-3/4, -1)$, and from $(-1, 3/4)$ to $(-1, -3/4)$.

- The four circular arcs

$$(x + 3/4)^2 + (y - 3/4)^2 = \frac{1}{16}\,, \quad y \ge 0\,,\ x \le 0\,;$$

$$(x-3/4)^2 + (y-3/4)^2 = \frac{1}{16}\,, \quad y \geq 0\,,\ x \geq 0\,;$$
$$(x-3/4)^2 + (y+3/4)^2 = \frac{1}{16}\,, \quad y \leq 0\,,\ x \geq 0\,;$$
$$(x+3/4)^2 + (y+3/4)^2 = \frac{1}{16}\,, \quad y \leq 0\,,\ x \leq 0\,;$$

Then any point on any of the circular arcs is extreme. But no other boundary point is extreme. Note, however, that the extreme points $(-3/4, 1)$, $(3/4, 1)$, $(1, -3/4)$, $(1, 3/4)$, $(-3/4, -1)$, $(3/4, -1)$, $(-1, -3, 4)$, and $(-1, 3/4)$ are *not* convex of finite order.

8 Support Functions.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded, convex domain with C^2 boundary. If $P \in \partial\Omega$ then let $T_P(\partial\Omega)$ be the tangent hyperplane to $\partial\Omega$ at P. We may take the outward unit normal at P to be the positive x_1 direction. Then the function

$$L(x) = x_1$$

is a linear function that is negative on Ω and positive on the other side of $T_P(\Omega)$. The function L is called a *support function* for Ω at P. Note that if we take the supremum of all support functions for all $P \in \partial\Omega$ then we obtain a defining function for Ω.

The support function of course takes the value 0 at P. It may take the value 0 at other boundary points—for instance in the case of the domain $\{(x_1, x_2) : |x_1| < 1, |x_2| < 1\}$. But if Ω is convex and $P \in \partial\Omega$ is a point of convexity of finite order $2k$ then the support function will vanish on $\partial\Omega$ only at the point P. The same assertion holds when P is an extreme point of the boundary.

9 Bumping.

One of the features that distinguishes a convex point of finite order from a convex point of infinite order is stability. The next example illustrates the point.

Example 12. Let

$$\Omega = \{(x, y) \in \mathbb{R}^2 : |x| < 1, |y| < 1\}\,.$$

Let P be the boundary point $(1/2, 1)$. Let U be a small open disc about P. Then there is no open domain $\widehat{\Omega}$ such that

(a) $\widehat{\Omega} \supseteq \Omega$ and $\widehat{\Omega} \ni P$;

(b) $\widehat{\Omega} \setminus \Omega \subseteq U$;

(b) $\widehat{\Omega}$ is convex.

To see this assertion, assume not. Let x be a point of $\widehat{\Omega} \setminus \Omega$. Let y be the point $(0.9, 0.9) \in \Omega$. Then the segment connecting x with y will *not* lie in $\widehat{\Omega}$.

The example shows that a flat point in the boundary of a convex domain cannot be perturbed while preserving convexity. But a point of finite order *can* be perturbed:

Proposition 20. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded, convex domain with C^k boundary. Let $P \in \partial\Omega$ be a convex point of finite order m. Write $\Omega = \{x \in \mathbb{R}^N : \rho(x) < 0\}$. Let $\epsilon > 0$. Then there is a perturbed domain $\widehat{\Omega} = \{x \in \mathbb{R}^N : \widehat{\rho}(x) < 0\}$ with C^k boundary such that*

(a) *$\widehat{\Omega} \supseteq \Omega$;*

(b) *$\widehat{\Omega} \ni P$;*

(c) *$\partial\widehat{\Omega} \setminus \overline{\Omega}$ consists of points of finite order m;*

(d) *The Hausdorff distance of $\partial\widehat{\Omega}$ and $\partial\Omega$ is less than ϵ.*

Before we begin the proof, we provide a useful technical lemma:

Lemma 21. *Let a be a fixed, positive number. Let $\alpha_0, \alpha_1, \ldots, \alpha_k$ and $\beta_0, \beta_1, \ldots, \beta_k$ and γ_0 be real numbers. Then there is a concave-down polynomial polynomial function $y = p(x)$ so that*

- *$p(0) = \gamma_0$;*
- *$p(-a) = \alpha_0$, $p(a) = \beta_0$;*
- *$p^{(j)}(-a) = \alpha_j$ for $j = 1, \ldots, k$;*
- *$p^{(j)}(a) = \beta_j$ for $j = 1, \ldots, k$.*

Here the exponents in parentheses are derivatives.

Proof of the Lemma. Define

$$g_j(x) = x^{2j}$$

and let $h_j^{\theta_j}$ be the function obtained from g_j by rotating the coordinates (x, y) through an angle of θ_j. Define

$$p(x) = c_0 - (c_1)^2 h_1^{\theta_1}(x) - (c_2)^2 h_2^{\theta_2}(x) - \cdots - (c_p)^2 h_p^{\theta_p}(x) ,$$

some positive integer p. If p is large enough (at least $k+1$), then there will be more free parameters in the definition of p than there are constants α_j, β_j, and γ_0. So we may solve for the c_j and θ_j and thereby define p. $\square$

Proof of the Proposition. First let us consider the case $N = 2$. Fix $P \in \partial\Omega$ as given in the statement of the proposition. We may assume without loss of generality that P is the origin and the tangent line to $\partial\Omega$ at P is the x-axis. We may further assume that Ω is so oriented that the boundary $\partial\Omega$ of Ω near P is the graph of a concave-down function φ.

Let $\delta > 0$ be small and let x and y be the two boundary points that are horizontally distant δ from P (situated, respectively, to the left and to the right of P). If δ is sufficiently small, then the angle between the tangent lines at x and at y will be less than $\pi/6$.

Now we think of $P = (0, 0)$, $\gamma_0 = \epsilon > 0$, of $x = (-a, \alpha_0)$, and of $y = (a, \beta_0)$. Further, we set

$$\alpha_j = \varphi^{(j)}(-a) \ , \quad j = 1, \ldots, k$$

and

$$\beta_j = \varphi^{(j)}(a) \ , \quad j = 1, \ldots, k \, .$$

Then we may apply the lemma to obtain a concave-down polynomial p which agrees with φ to order k at the points of contact x and y.

Thus the domain $\widehat{\Omega}$ which has boundary given by $y = p(x)$ for $x \in [-a, a]$ and boundary coinciding with $\partial\Omega$ elsewhere (we are simply replacing the portion of $\partial\Omega$ which lies between x and y by the graph of p) will be a convex domain that bumps Ω *provided that* the degree of p does not exceed the finite order of convexity m of $\partial\Omega$ near P. When the degree of p exceeds m, then the graph $y = p(x)$ may intersect $\partial\Omega$ between x and y, and therefore not provide a geometrically valid bump.

For higher dimensions, we proceed by slicing. Let $P \in \partial\Omega$ be of finite order m. Let $T_P(\partial\Omega)$ be the tangent hyperplane to $\partial\Omega$ at P as usual. If $\mathbf{v}$ is a unit vector in $T_P(\partial\Omega)$ and ν_P the unit outward normal vector to $\partial\Omega$ at P, then consider the 2-dimensional plane $\mathcal{P}_{\mathbf{v}}$ spanned by $\mathbf{v}$ and ν_P. Then $\Omega_{\mathbf{v}} \equiv \mathcal{P}_{\mathbf{v}} \cap \Omega$ is a 2-dimensional convex domain which is convex of order m at $P \in \partial\Omega_{\mathbf{v}}$. We may apply the two-dimensional perturbation result to this domain. We do so for each unit tangent vector $\mathbf{v} \in T_P(\partial\Omega)$, noting that the construction varies smoothly with the data vector $\mathbf{v}$. The result is a smooth, perturbed domain $\widehat{\Omega}$ as desired. $\square$

It is worth noting that the proof shows that, when we bump a piece of boundary that is convex of order m, then we may take the bump to be convex of order 2 or 4 or any degree up to and including m (which of course is even).

It is fortunate that the matter of bumping may be treated more or less heuristically in the present context. In several complex variables, bumping is a more profound and considerably more complicated matter (see, for instance [1]).

10 Concluding Remarks.

We have attempted here to provide the analytic tools so that convexity can be used in works of geometric analysis. There are many other byways to be explored in this vein, and we hope to treat them at another time.

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