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SPACES OF p-TENSOR INTEGRABLE FUNCTIONS AND RELATED BANACH SPACE PROPERTIES

Abstract

In [9] G. F. Stefansson has studied the Banach space $L_1(\nu, X, Y)$, the space of all tensor integrable functions $f : \Omega \to X$ with respect to a countably additive vector valued measure $\nu : \Sigma \to Y$ and also the tensor integral of weakly ν-measurable functions. In [1] we obtained some Banach space properties of $L_1(\nu, X, Y)$ and also of w-$L_1(\nu, X, Y)$, the space of all weakly tensor integrable functions. In the present paper, for $1 < p < \infty$, we define the spaces $L_p(\nu, X, Y)$ and w-$L_p(\nu, X, Y)$ of all $\check{\otimes}_p$-integrable functions and weakly $\check{\otimes}_p$-integrable functions respectively and discuss several basic properties of these spaces. We also study vector measure duality in $L_p(\nu, X, Y)$ for $1 < p < \infty$.

1 Introduction, Notations and Preliminaries.

This paper may be considered as a continuation of the paper of Stefansson [9] and our paper [1]. Throughout this paper, X and Y are two real Banach spaces with topological duals X^* and Y^* respectively. B_X (respectively B_{X^*}) denotes the closed unit ball of X (respectively X^*) and $X \check{\otimes} Y$ is the injective tensor product of X and Y (see [3, Chapter VIII]).

If X is a Banach lattice, then its dual X^* is also a Banach lattice where the positive cone is defined by $x^* \geq \theta$ in X^* if and only if $x^*(x) \geq 0$ for every $x \geq \theta$ in X (see [6, p.3]).

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If (Ω, Σ) is a measurable space, then the semivariation of a countably additive vector measure $\nu : \Sigma \to Y$ is defined by $\|\nu\|(A) = \sup\{|y^*\nu|(A) : y^* \in B_{Y^*}\}$ for $A \in \Sigma$, where $|y^*\nu|$ is the variation of the scalar measure $y^*\nu$.

For $1 \leq p < \infty$, let $L_p(\nu)$ and w-$L_p(\nu)$ denote the Banach spaces of all ($\|\nu\|$-equivalence classes of) p-integrable and weakly p-integrable functions $f : \Omega \to \mathbb{R}$ with respect to ν respectively equipped with the norm

$$\|f\|_{p,\nu} = \sup\left\{ \left(\int_\Omega |f|^p \, d|y^*\nu| \right)^{1/p} : y^* \in B_{Y^*} \right\}.$$

The space w-$L_p(\nu)$ is a Banach lattice with respect to the natural order $\|\nu\|$-a.e. containing $L_p(\nu)$ as a closed sublattice (see [2, p.319], [10, p.227], [4, p.7]).

Moreover, $L_p(\nu)$ is an order continuous Banach lattice with weak order unit (see [7, p.912]).

Also for $1 \leq p < \infty$, we have the following inclusions

$$L_p(\nu) \subset \text{w-}L_p(\nu) \subset \text{w-}L_1(\nu) \text{ and } L_p(\nu) \subset L_1(\nu) \subset \text{w-}L_1(\nu),$$

where the inclusion mappings are continuous. The space w-$L_p(\nu)$ has an order continuous norm if and only if w-$L_p(\nu) = L_p(\nu)$ (see [10, Theorem 10, p.228] and [4, Corollary 3.10, p.13]).

For $1 \leq p < \infty$, the symbol $L_p(\mu, X)$ denotes the Banach space of all (equivalence classes of) Bochner integrable functions $f : \Omega \to X$ with respect to the scalar measure μ, equipped with the norm

$$\|f\|_p = \left(\int_\Omega \|f\|^p \, d|\mu| \right)^{1/p}.$$

In [9] Stefansson defines a ν-measurable function $f : \Omega \to X$ to be $\check{\otimes}$-integrable with respect to ν if there exists a sequence of X-valued simple functions $\{\phi_n\}$ such that

$$\lim_n \sup\left\{ \int_\Omega \|f - \phi_n\| \, d|y^*\nu| : y^* \in B_{Y^*} \right\} = 0.$$

In this case, we have $\int_E f \, d\nu = \lim_n \int_E \phi_n \, d\nu$ for every $E \in \Sigma$ and $\int_E f \, d\nu$ is called the $\check{\otimes}$-integral of f over E with respect to ν and the value of the integral is an element of the injective tensor product $X \check{\otimes} Y$. The space of all $\check{\otimes}$-integrable functions is denoted by $L_1(\nu, X, Y)$.

If $N(f) = \sup\{\int_\Omega \|f\| \, d|y^*\nu| : y^* \in B_{Y^*}\}$, then $N(f) < \infty$ if f is $\check{\otimes}$-integrable.

It has been shown in [9, Theorem 4, p.932] that $L_1(\nu, X, Y)$ is a Banach space with respect to the norm $N(\cdot)$ and it is an order continuous Banach lattice with weak order unit if X is an order continuous Banach lattice (see [1, Theorem 1, p.5]).

Let $y_0^* \in B_{Y^*}$ such that $\|\nu\| \ll |y_0^*\nu|$, that is, $\lambda = |y_0^*\nu|$ is a Rybakov control measure for ν (see [3, Theorem 2, p.268]).

In [9] Stefansson also studies the integral of weakly ν-measurable functions $f : \Omega \to X$ and shows that if $x^*f \in L_1(y^*\nu)$ for $x^* \in X^*$, $y^* \in Y^*$, then for every $g \in L_\infty(|y_0^*\nu|)$, the map Ψ_g defined by

$$\Psi_g(x^*, y^*) = \int_\Omega g \cdot x^* f \, dy^* \nu$$

is an element of $B(X^*, Y^*)$, the space of all bounded bilinear functionals on $X^* \times Y^*$, and the generalized weak $\otimes$-integral of f over a set $E \in \Sigma$ is defined by the element Ψ_{χ_E}. Since $X \check{\otimes} Y \subset B(X^*, Y^*)$, he defines f to be weakly $\check{\otimes}$-integrable if $\Psi_{\chi_E} \in X \check{\otimes} Y$ and in this case Ψ_{χ_E} is the weak $\check{\otimes}$-integral of f over E and is denoted by $\int_E f \, d\nu$.

Let w-$L_1(\nu, X, Y)$ be the space of all weakly $\check{\otimes}$-integrable functions with respect to the semivariation norm

$$\|f\|_\nu = \sup\left\{ \int_\Omega |x^* f| \, d|y^*\nu| : x^* \in B_{X^*}, y^* \in B_{Y^*} \right\}.$$

It has been shown in [1, Theorem 7, p.15] that w-$L_1(\nu, X, Y)$ is an incomplete normed linear space which is barrelled if ν is nonatomic.

Let $1 < p < \infty$. The main object of our paper is to extend the definition of $L_1(\nu, X, Y)$ and w-$L_1(\nu, X, Y)$ to $L_p(\nu, X, Y)$ and w-$L_p(\nu, X, Y)$ respectively and study some basic properties of these spaces. We also study vector measure duality in $L_p(\nu, X, Y)$ for $1 < p < \infty$, which is a generalization of the idea of vector measure duality in $L_p(\nu)$ as introduced by Sánchez Pérez in [7].

2 The Spaces $L_p(\nu, X, Y)$ and w-$L_p(\nu, X, Y)$.

Definition 1. Let $1 < p < \infty$. A ν-measurable function $f : \Omega \to X$ is called $\check{\otimes}_p$-integrable, if there exists a sequence $\{\phi_n\}$ of X-valued simple functions such that $\lim_n N_p(f - \phi_n) = 0$, where

$$N_p(f) = \sup\left\{ \left(\int_\Omega \|f\|^p \, d|y^*\nu| \right)^{1/p} : y^* \in B_{Y^*} \right\}.$$

It is easy to prove that for $1 < p < \infty$, if a ν-measurable function f is $\check{\otimes}_p$-integrable then $N_p(f) < \infty$ and if f and g are two $\check{\otimes}_p$-integrable functions, then $(f+g)$ is $\check{\otimes}_p$-integrable and $N_p(f+g) \le N_p(f) + N_p(g)$.

Theorem 1. *Let $1 < p < \infty$. A ν-measurable function f is $\check{\otimes}_p$-integrable if and only if $\|f\|^p$ is ν-integrable.*

PROOF. The proof is similar to that of Theorem 1 of [9]. So we give a sketch of the proof.

Let f be $\check{\otimes}_p$-integrable. Then $N_p(f) < \infty$ and so it follows, by definition, that $\|f\| \in$ w-$L_p(\nu)$. Since $L_p(\nu)$ is a closed subspace of w-$L_p(\nu)$, we have by a similar argument as given in ([9, Theorem 1]) that $\|f\| \in L_p(\nu)$, that is, $\|f\|^p$ is ν-integrable.

Conversely, let $\|f\|^p$ be ν-integrable. By [5, Theorem 2.2], the indefinite integral of $\|f\|^p$ with respect to ν is a countably additive Y-valued measure and $\lim\limits_{\|\nu\|(E)\to 0} N_p(f\chi_E) = 0$.

Again, following the arguments as given in the sufficiency part of [9, Theorem 1], we have that

$$\lim_{\|\nu\|(E)\to 0} N_p(f_n\chi_E) = 0 \tag{1}$$

where $\{f_n\}$ is a sequence of countably valued functions converging $\|\nu\|$-a.e. uniformly to f. Let us represent f_n by

$$f_n = \sum_{k=1}^{\infty} x_{n,k}\chi_{E_{n,k}}$$

with $E_{n,i} \cap E_{n,j} = \emptyset$ if $i \ne j$, $E_{n,k} \in \Sigma$ and $x_{n,k} \in X$.

Applying equation (1), for each n we can choose p_n so large that

$$\sup\left\{ \int_{\substack{\cup E_{n,k} \\ k > p_n}} \|f_n\|^p \, d|y^*\nu| : y^* \in B_{Y^*} \right\} < \frac{\|\nu\|(\Omega)}{n}.$$

If we take $\phi_n = \sum\limits_{k \le p_n} x_{n,k}\chi_{E_{n,k}}$, then an easy calculation shows that

$$N_p(f - \phi_n) \to 0 \text{ as } n \to \infty,$$

which implies that f is $\check{\otimes}_p$-integrable and the proof is complete. $\square$

We denote the space of all $\check{\otimes}_p$-integrable functions by $L_p(\nu, X, Y)$.

Remark 1. If $X = \mathbb{R}$, then $L_p(\nu, X, Y) = L_p(\nu)$. So by the above theorem, $f \in L_p(\nu)$ if and only if $|f|^p$ is ν-integrable which coincides with the Definition 1 of $L_p(\nu)$ as given in Sánchez Pérez (see [7, p.909]).

Remark 2. $L_p(\nu, X, Y) \subset L_1(\nu, X, Y)$, $1 \le p < \infty$. For, let $f \in L_p(\nu, X, Y)$. Then $\|f\| \in L_p(\nu)$. Since $L_p(\nu)$ is a subset of $L_1(\nu)$ by [7, Remark 3, p.909], we have $\|f\| \in L_1(\nu)$ and hence f is $\check{\otimes}$-integrable by [9, Theorem 1, p.928], which implies that $f \in L_1(\nu, X, Y)$.

Corollary 1. *If f is ν-measurable and bounded, then f is $\check{\otimes}_p$-integrable.*

Corollary 2. *Let f and g be two ν-measurable functions. If g is $\check{\otimes}_p$-integrable and $\|f\| \le \|g\|$ $\|\nu\|$-a.e., then f is $\check{\otimes}_p$-integrable.*

For, since g is $\check{\otimes}_p$-integrable, it follows that $\|g\|^p \in L_1(\nu)$. Now $\|f\| \le \|g\|$ $\|\nu\|$-a.e. implies that $\|f\|^p \le \|g\|^p$ $\|\nu\|$-a.e., for $1 \le p < \infty$. Therefore, by [10, p.225], $\|f\|^p \in L_1(\nu)$ which implies that f is $\check{\otimes}_p$-integrable.

Theorem 2. *Let $1 \le p < \infty$. Then $L_p(\nu, X, Y)$ is a Banach space with respect to the norm $N_p(\cdot)$.*

PROOF. For $p = 1$, it has been shown in [9, Theorem 4, p.932] that $L_1(\nu, X, Y)$ is a Banach space. A similar proof applies for $1 < p < \infty$ and is therefore omitted. $\square$

Theorem 3. *Let $1 \le p < \infty$. If X is an order continuous Banach lattice, then $L_p(\nu, X, Y)$ is an order continuous Banach lattice with weak order unit.*

PROOF. The following proof is similar to the proof of Theorem 1 in [1, p.5] but we include it for the sake of completeness. It is easy to see that $L_p(\nu, X, Y)$ is a Banach lattice with respect to the norm $N_p(\cdot)$ and usual order relation where $f_1 \le f_2$ means $f_1(\omega) \le f_2(\omega)$ $\|\nu\|$-a.e., for $\omega \in \Omega$.

In order to show that $L_p(\nu, X, Y)$ is order continuous, we shall use the following characterization:

A Banach lattice is order continuous if and only if every order bounded increasing sequence is norm convergent (see [6, p.7]).

Let $\{f_n\}$ be an order bounded increasing sequence in $L_p(\nu, X, Y)$. We can assume that $0 \le f_n \le f_{n+1} \le g$ where $g \in L_p(\nu, X, Y)$. Set $f(\omega) = \sup_n f_n(\omega)$. Since X is order complete and $\{f_n\}$ is increasing, we have $f(\omega) = \lim_n f_n(\omega)$ and hence f is ν-measurable and $\|f\| \le \|g\|$ $\|\nu\|$-a.e.. As $g \in L_p(\nu, X, Y)$ we have by Corollary 2 that $f \in L_p(\nu, X, Y)$.

Let $\varepsilon > 0$. Since $(f_1 - f)$ is $\check{\otimes}_p$-integrable, $\|f_1 - f\|^p$ is ν-integrable. If

$$\phi(B) = \int_B \|f_1 - f\|^p \, d\nu \text{ for } B \in \Sigma,$$

then $\phi \ll \|\nu\|$, by [5, Theorem 2.2] and so there exists a $\delta > 0$ such that $\|\nu\|(B) < \delta$ implies that $\|\phi\|(B) < \varepsilon/2$; that is, $\sup\{\int_B \|f_1(\omega) - f(\omega)\|^p \, d|y^*\nu| : y^* \in B_{Y^*}\} < \varepsilon/2$, which implies that

$$\int_B \|f_1(\omega) - f(\omega)\|^p \, d|y^*\nu| < \varepsilon/2$$

for each $y^* \in B_{Y^*}$ and so

$$\sup\left\{ \left(\int_B \|f_1(\omega) - f(\omega)\|^p \, d|y^*\nu| \right)^{1/p} : y^* \in B_{Y^*} \right\} < \varepsilon^{1/p} \tag{2}$$

for each $B \in \Sigma$. Since $f_n(\omega) \to f(\omega)$ pointwise, by Egoroff's theorem, there exists a set $A \in \Sigma$ such that $\|\nu\|(A) < \delta$ and $f_n \to f$ uniformly on $\Omega \setminus A$. So there exists a positive integer n_0 such that $\|f_n(\omega) - f(\omega)\| < \varepsilon$ for all $\omega \in \Omega \setminus A$ and for all $n \geq n_0$. Therefore

$$\begin{aligned} N_p(f_n - f) &= \sup\left\{ \left(\int_\Omega \|f_n(\omega) - f(\omega)\|^p \, d|y^*\nu| \right)^{1/p} : y^* \in B_{Y^*} \right\} \\ &\leq \sup\left\{ \left(\int_{\Omega \setminus A} \|f_n(\omega) - f(\omega)\|^p \, d|y^*\nu| \right)^{1/p} : y^* \in B_{Y^*} \right\} \\ &\quad + \sup\left\{ \left(\int_A \|f_n(\omega) - f(\omega)\|^p \, d|y^*\nu| \right)^{1/p} : y^* \in B_{Y^*} \right\}. \end{aligned}$$

Now,

$$\begin{aligned} &\sup\left\{ \left(\int_{\Omega \setminus A} \|f_n(\omega) - f(\omega)\|^p \, d|y^*\nu| \right)^{1/p} : y^* \in B_{Y^*} \right\} \\ &\qquad < \varepsilon \sup\{(|y^*\nu|(\Omega \setminus A))^{1/p} : y^* \in B_{Y^*}\} \\ &\qquad \leq \varepsilon \{\|\nu\|(\Omega \setminus A)\}^{1/p} < \varepsilon \{\|\nu\|(\Omega)\}^{1/p} \end{aligned}$$

for all $n \geq n_0$. Also, $\sup\{(\int_A \|f_1(\omega) - f(\omega)\|^p \, d|y^*\nu|)^{1/p} : y^* \in B_{Y^*}\} < \varepsilon^{1/p}$, by (2), therefore $N_p(f_n - f) < \varepsilon\{\|\nu\|(\Omega)\}^{1/p} + \varepsilon^{1/p}$ for $n \geq n_0$. This implies that $\{f_n\}$ converges to f in $L_p(\nu, X, Y)$ and so $L_p(\nu, X, Y)$ is order continuous.

Finally, let us show that for any $x \in X$ such that $x > \theta$, $x\chi_\Omega$ is a weak order unit in $L_p(\nu, X, Y)$.

Note that an element $e \geq \theta$ of a Banach lattice L is said to be a weak order unit of L if $e \wedge x = \theta$ for $x \in L$ implies $x = \theta$, where $y \wedge z$ denotes the greatest lower bound for $y, z \in L$ (see [6, p.9]).

For any $x(> \theta) \in X$, $x\chi_\Omega$ is a weak order unit, for if $\inf\{f(\omega), x\chi_\Omega\} = \theta$ for any $f \in L_p(\nu, X, Y)$, then $f(\omega) = \theta$ for all $\omega \in \Omega$, which implies that $f \equiv 0$. Thus $\{x\chi_\Omega : x(> \theta) \in X\}$ is a family of weak order units in $L_p(\nu, X, Y)$ and the proof is complete. □

Theorem 4 (**Dominated Convergence Theorem**). *Let* $1 \leq p < \infty$. *Let* $\{f_n\}$ *be a sequence of* $\check{\otimes}_p$*-integrable functions which converges* $\|\nu\|$*-a.e. to a function* f *and* g *be a* $\check{\otimes}_p$*-integrable function such that* $\|f_n\| \leq \|g\|$ $\|\nu\|$*-a.e. for each* n. *Then* f *is* $\check{\otimes}_p$*-integrable and* $\lim\limits_n N_p(f_n - f) = 0$ *and hence* $\lim_n \int_E f_n \, d\nu = \int_E f \, d\nu$ *for all* $E \in \Sigma$.

PROOF. Since $\|f_n\| \leq \|g\|$ $\|\nu\|$-a.e., it follows that $\|f\| \leq \|g\|$ $\|\nu\|$-a.e. and hence by Corollary 2, f is $\check{\otimes}_p$-integrable. That $\lim_n N_p(f_n - f) = 0$ follows from the arguments as given in the proof of Theorem 3. By an application of Hölder's inequality, it follows by an easy calculation that $\lim_n \int_E f_n \, d\nu = \int_E f \, d\nu$ for all $E \in \Sigma$. □

Recall that a bounded set K of a Banach lattice X is L-weakly compact if every disjoint sequence of the solid hull of K converges to zero in norm. An operator T from a Banach space Z to X is L-weakly compact if $T(B_Z)$ is L-weakly compact in X. As L-weakly compact sets are relatively weakly compact, every L-weakly compact operator is weakly compact (see [4, p.9]).

The following theorem is a generalization of Proposition 3.3 of [4].

Theorem 5. *If* $1 < p < \infty$ *and* X *is a Banach lattice, then the inclusion map* $L_p(\nu, X, Y) \subset L_1(\nu, X, Y)$ *is a* L*-weakly compact operator. In particular, it is a weakly compact operator.*

PROOF. We note that the unit ball $B_{L_p(\nu,X,Y)}$ of $L_p(\nu, X, Y)$ is a norm bounded and solid subset of $L_1(\nu, X, Y)$. So it is enough to prove that every disjoint sequence of $B_{L_p(\nu,X,Y)}$ converges to zero in the norm of $L_1(\nu, X, Y)$.

Let $\{f_n\}$ be a disjoint sequence in $B_{L_p(\nu,X,Y)}$ and put $A_n = \{\omega \in \Omega : f_n(\omega) \neq \theta\}$ for all n. Then $\{A_n\}$ is a disjoint sequence of measurable sets and therefore $\|\nu\|(A_n) \to 0$ as $n \to \infty$ (see [3, Corollary 18, p.9]).

By applying Hölder's inequality we get that

$$\begin{aligned}
N(f_n) = N(f_n\chi_{A_n}) &= \sup_{\|y^*\|\le 1}\left\{\int_\Omega \|f_n\chi_{A_n}\|\,d|y^*\nu|\right\}\\
&= \sup_{\|y^*\|\le 1}\left\{\int_\Omega \|f_n(\omega)\||\chi_{A_n}(\omega)|\,d|y^*\nu|\right\}\\
&\le \left\{\sup_{\|y^*\|\le 1}\left(\int_\Omega \|f_n(\omega)\|^p\,d|y^*\nu|\right)^{\frac{1}{p}}\right\}\left\{\sup_{\|y^*\|\le 1}\left(\int_\Omega |\chi_{A_n}(\omega)|^q\,d|y^*\nu|\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

(where $\frac{1}{p}+\frac{1}{q}=1$),

$$\begin{aligned}
&= N_p(f_n)\left\{\sup_{\|y^*\|\le 1}\left(\int_{A_n} d|y^*\nu|\right)^{\frac{1}{q}}\right\}\\
&\le N_p(f_n)(\|\nu\|(A_n))^{1/q} \le (\|\nu\|(A_n))^{1/q} \to 0 \text{ as } n\to\infty.
\end{aligned}$$

So the inclusion mapping $T_{p,\nu} : L_p(\nu,X,Y) \to L_1(\nu,X,Y)$ is a L-weakly compact operator. In particular, it is a weakly compact operator for $1<p<\infty$. □

Corollary 3. *If $1<p<\infty$, then the integration map*

$$I_{p,\nu} : L_p(\nu,X,Y) \to X\check{\otimes}Y$$

is weakly compact.

PROOF. First we show that the integration map $I_\nu : L_1(\nu,X,Y) \to X\check{\otimes}Y$ defined by

$$I_\nu(f) = \int_\Omega f\,d\nu$$

is bounded. Now

$$\begin{aligned}
\|I_\nu(f)\| = \left\|\int_\Omega f\,d\nu\right\| &\le \sup_{\substack{\|x^*\|\le 1\\ \|y^*\|\le 1}}\left(\int_\Omega |x^*f|\,d|y^*\nu|\right)\\
&\le \sup_{\substack{\|x^*\|\le 1\\ \|y^*\|\le 1}}\left(\int_\Omega \|x^*\|\|f\|\,d|y^*\nu|\right)\\
&\le \sup_{\|y^*\|\le 1}\left(\int_\Omega \|f\|\,d|y^*\nu|\right) = N(f)
\end{aligned}$$

which implies that I_ν is bounded and $\|I_\nu\| \leq 1$.

So $I_{p,\nu} = I_\nu \circ T_{p,\nu}$ from $L_p(\nu, X, Y)$ to $X\check{\otimes}Y$ is weakly compact, where $T_{p,\nu}$ is defined as in the proof of the previous theorem. □

Definition 2. Let $1 < p < \infty$. A weakly $\|\nu\|$-measurable function $f : \Omega \to X$ is said to have a generalized weak $\otimes_p$-integral with respect to $\nu : \Sigma \to Y$ if $|x^*f|^p$ is $|y^*\nu|$-integrable for all $(x^*, y^*) \in X^* \times Y^*$, that is, $|x^*f| \in$ w-$L_p(\nu)$. Since w-$L_p(\nu) \subset$ w-$L_1(\nu)$, the generalized weak $\otimes_p$-integral of f over $E \in \Sigma$ is defined by the element Ψ_{χ_E} which is an element of $B(X^*, Y^*)$.

Now $X\check{\otimes}Y \subset B(X^*, Y^*)$ and if $\Psi_{\chi_E} \in X\check{\otimes}Y$ for all $E \in \Sigma$, then f is said to be weakly $\check{\otimes}_p$-integrable and the weak $\check{\otimes}_p$-integral of f over E, which is an element of $X\check{\otimes}Y$, is denoted by w-$\int_E f\, d\nu$.

For convenience, we write w-$\int_E f\, d\nu$ as $\int_E f\, d\nu$ when no confusion arises.

The set of all weakly $\check{\otimes}_p$-integrable functions is denoted by w-$L_p(\nu, X, Y)$.

For $f \in$ w-$L_p(\nu, X, Y)$, we define the norm of f as

$$N_{p,w}(f) = \sup_{\substack{\|x^*\| \leq 1 \\ \|y^*\| \leq 1}} \left(\int_\Omega |x^*f|^p \, d|y^*\nu| \right)^{1/p}.$$

Following the arguments as in the proof of Theorem 5 and Theorem 6 of [1] we can show that if ν is non-atomic, then w-$L_p(\nu, X, Y)$ is a normed linear space which is not complete with respect to the above norm $N_{p,w}(\cdot)$ but barrelled.

It follows easily from the definitions that

$$L_p(\nu, X, Y) \subset \text{w-}L_p(\nu, X, Y) \subset \text{w-}L_1(\nu, X, Y) \text{ and}$$
$$L_p(\nu, X, Y) \subset L_1(\nu, X, Y) \subset \text{w-}L_1(\nu, X, Y),$$

where the inclusion mappings are continuous.

Definition 3. Let $1 < p < \infty$. Let us define a family of seminorms $\{p_{x^*,y^*}\}_{\substack{x^* \in X^* \\ y^* \in Y^*}}$ on w-$L_p(\nu, X, Y)$ by

$$p_{x^*,y^*}(f) = \left(\int_\Omega |x^*f|^p \, d|y^*\nu| \right)^{1/p}, \qquad f \in \text{w-}L_p(\nu, X, Y).$$

Let τ be the locally convex topology on w-$L_p(\nu, X, Y)$ generated by the above family of seminorms.

The following two theorems are generalization of Proposition 2.7 and Lemma 3.8 of [4] to w-$L_p(\nu, X, Y)$ respectively.

Theorem 6. *If X is weakly sequentially complete, then w-$L_p(\nu, X, Y)$ endowed with the topology τ is sequentially complete.*

PROOF. Let $\{f_n\}$ be a τ-Cauchy sequence in w-$L_p(\nu, X, Y)$. If $x^* \in X^*$, $y^* \in Y^*$ are arbitrary, then $\{x^* f_n\}$ is a Cauchy sequence in $L_p(|y^*\nu|)$. So it is convergent to some element of $L_p(|y^*\nu|)$. Now select $y_0^* \in B_{Y^*}$ such that $|y_0^*\nu|$ is a Rybakov control measure for ν.

We extract a subsequence $\{x^* f_{n_i}\}$ of $\{x^* f_n\}$ which is pointwise convergent except for a set $E_{y_0^*} \in \Sigma$, with $|y_0^*\nu|(E_{y_0^*}) = 0$. So, for each $x^* \in X^*$, $\{x^* f_{n_i}(\omega)\}$ is a Cauchy sequence of scalars which implies that $\{f_{n_i}(\omega)\}$ is a weak Cauchy sequence in X. Since X is weakly sequentially complete, there exists an $f_{y_0^*}(\omega) \in X$ such that $x^* f_{n_i}(\omega) \to x^* f_{y_0^*}(\omega)$ for all $\omega \notin E_{y_0^*}$. Fix any $y^* \in Y^*$ and observe that $\{x^* f_{n_i}(\omega)\}$ converges to $x^* f_{y^*}(\omega)$ $|y^*\nu|$-a.e. Now, since $\{x^* f_{n_i}\}$ is a Cauchy sequence in $L_p(|y^*\nu|)$, it is bounded in $L_p(|y^*\nu|)$ and since $x^* f_{n_i} \to x^* f_{y^*}$ pointwise a.e., it follows by bounded convergence theorem that $x^* f_{y^*} \in L_p(|y^*\nu|)$ and $x^* f_{n_i} \to x^* f_{y^*}$ in $L_p(|y^*\nu|)$.

We can extract a subsequence $\{x^* f_{n_{i_j}}\}$ of $\{x^* f_{n_i}\}$ which is pointwise convergent to $x^* f_{y^*}$ except for a set $E_{y^*} \in \Sigma$ with $|y^*\nu|(E_{y^*}) = 0$.

Thus $\{x^* f_{n_{i_j}}(\omega)\}$ converges to $x^* f_{y^*}(\omega)$ and $\{x^* f_{n_{i_j}}(\omega)\}$ converges to $x^* f_{y_0^*}(\omega)$ for every $\omega \notin E_{y^*} \cup E_{y_0^*}$ with $|y^*\nu|(E_{y^*} \cup E_{y_0^*}) = 0$ and for each $x^* \in X^*$. Therefore it follows that $x^* f_{y^*} = x^* f_{y_0^*}$ a.e. for each $x^* \in X^*$.

Hence $x^* f_{y_0^*} \in L_p(|y^*\nu|)$ for each $x^* \in X^*$. Since y^* is arbitrary, it follows that $x^* f_{y_0^*} \in L_p(|y^*\nu|)$ for each $y^* \in Y^*$ and for each $x^* \in X^*$ and hence $f_{y_0^*} \in$ w-$L_p(\nu, X, Y)$. Since $\{x^* f_n\}$ is a Cauchy sequence in $L_p(|y^*\nu|)$ and since its subsequence $\{x^* f_{n_i}\}$ converges to $x^* f_{y_0^*}$ in $L_p(|y^*\nu|)$, it follows that $\{x^* f_n\}$ converges to $x^* f_{y_0^*}$ in $L_p(|y^*\nu|)$. This means that

$$\left(\int_\Omega |x^* f_n - x^* f_{y_0^*}|^p \, d|y^*\nu| \right)^{1/p} \to 0 \text{ as } n \to \infty$$

for each $x^* \in X^*$ and $y^* \in Y^*$, which implies that $p_{x^*, y^*}(f_n - f_{y_0^*}) \to 0$ as $n \to \infty$; that is, $f_n \to f_{y_0^*}$ in the τ-topology of w-$L_p(\nu, X, Y)$ and this shows that w-$L_p(\nu, X, Y)$ is sequentially complete with respect to the τ-topology. $\square$

Theorem 7. *Let $1 \leq p < \infty$ and let X be a weakly sequentially complete Banach lattice with X^* as its dual Banach lattice. Let $\{f_n\}$ be a norm bounded, positive, increasing sequence in w-$L_p(\nu, X, Y)$. Then $f = \sup_n f_n$ exists weakly in X, that is, $x^* f = \sup_n x^* f_n$ for each $x^* \in X^*$ and $f \in$ w-$L_p(\nu, X, Y)$.*

PROOF. Let $y_0^* \in B_{Y^*}$ be such that $|y_0^*\nu|$ is a Rybakov control measure for ν. Since $\{f_n\}$ is a norm bounded, positive, increasing sequence in w-$L_p(\nu, X, Y)$

and X^* is a Banach lattice, for each $x^* \in X^*$, $\{x^* f_n\}$ is a norm bounded, positive, increasing sequence in $L_p(|y_0^*\nu|)$. Since $L_p(|y_0^*\nu|)$ is weakly sequentially complete and $\{x^* f_n\}$ is a norm bounded, increasing sequence in $L_p(|y_0^*\nu|)$, the sequence $\{x^* f_n\}$ converges in norm to an element of $L_p(|y_0^*\nu|)$. So, there exists a subsequence $\{x^* f_{n_k}\}$ of $\{x^* f_n\}$ which is pointwise convergent except for a set $E_{y_0^*} \in \Sigma$ with $|y_0^*\nu|(E_{y_0^*}) = 0$. So, for each $x^* \in X^*$, $\{x^* f_{n_k}(\omega)\}$ is a Cauchy sequence of scalars $\|\nu\|$-a.e. which implies that $\{f_{n_k}(\omega)\}$ is a weak Cauchy sequence in X $\|\nu\|$-a.e. Since X is weakly sequentially complete, there exist $f(\omega) \in X$ such that $x^* f_{n_k}(\omega) \to x^* f(\omega)$ for all $\omega \notin E_{y_0^*}$.

Since $\{x^* f_{n_k}\}$ is norm bounded in $L_p(|y_0^*\nu|)$ and $x^* f_{n_k} \to x^* f$ pointwise a.e., it follows by bounded convergence theorem that $x^* f \in L_p(|y_0^*\nu|)$.

Again, since $\{x^* f_n\}$ is a positive increasing sequence, it follows that the sequence $\{x^* f_n\}$ converges pointwise to $x^* f$ and so $\sup_n x^* f_n = x^* f$ for each $x^* \in X^*$; that is, $\sup_n f_n = f$ exists weakly in X.

For an arbitrary $y^* \in Y^*$ we can apply the same argument as above to obtain a function $x^* f_{y^*}$ in $L_p(|y^*\nu|)$ such that $\{x^* f_n\}$ converges to f_{y^*} in $L_p(|y^*\nu|)$ and hence also pointwise except for a set E_{y^*} for which $|y^*\nu|(E_{y^*}) = 0$. Therefore it follows that $x^* f_{y^*}(\omega) = x^* f(\omega)$ for every $\omega \notin E_{y^*} \cup E_{y_0^*}$ with $|y^*\nu|(E_{y^*} \cup E_{y_0^*}) = 0$ and for each $x^* \in X^*$.

So $x^* f_{y^*} = x^* f$ a.e. for each $x^* \in X^*$. Then $x^* f_{y^*} \in L_p(|y^*\nu|)$ for each $x^* \in X^*$.

Since $y^* \in Y^*$ is arbitrary, it follows that $x^* f \in L_p(|y^*\nu|)$ for each $y^* \in Y^*$ and $x^* \in X^*$ and hence $f \in$ w-$L_p(\nu, X, Y)$. $\square$

3 Vector Measure Duality.

Let $1 < p < \infty$ and q is the real number that satisfies $\frac{1}{p} + \frac{1}{q} = 1$. It is well known that if (Ω, Σ, μ) is a finite measure space, then $L_p(\mu, X)^* = L_q(\mu, X^*)$ if and only if X^* has the Radon-Nikodym property (RNP) with respect to μ (see [3, Theorem 1, p.98]). For example, reflexive Banach spaces and separable dual spaces have the RNP.

In [7, p.915] Sánchez Pérez has shown by a counter example that the dual of $L_p(\nu)$ is different from $L_q(\nu)$ even for reflexive Banach spaces. He has, however, introduced a new concept known as vector measure duality in $L_p(\nu)$ and has shown that $(L_p(\nu))^\nu = L_q(\nu)$ (see [7, Proposition 8, p.914]).

In [8] Sánchez Pérez has applied this vector measure duality theory for tensor product representations of L_p-spaces of vector measures.

In this section we generalize the idea of vector measure duality to the space $L_p(\nu, X, Y)$. We proceed as follows :

Let (Ω, Σ, μ) be a complete finite measure space and let $(E, \|\cdot\|_E)$ be a Köthe function space (Banach function space) over (Ω, Σ, μ) such that $L_\infty \subset E \subset L_1$, where the inclusion maps are continuous. Let L_0 denote the space of all μ-equivalence classes of Σ-measurable real valued functions. Let E' be the Köthe dual of E where E' is defined by

$$E' = \left\{ v \in L_0 : \int_\Omega |u(\omega)v(\omega)|\, d\mu < \infty, \text{ for all } u \in E \right\}.$$

Then the associated norm $\|\cdot\|_{E'}$ on E' is defined by

$$\|v\|_{E'} = \sup\left\{ \int_\Omega |u(\omega)v(\omega)|\, d\mu : u \in E. \|u\| \leq 1 \right\}.$$

Let X be an order continuous Banach lattice. By $L_0(X)$ we denote the set of equivalence classes of strongly Σ-measurable functions $f : \Omega \to X$. For $f \in L_0(X)$, let $\tilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$. So $\tilde{f} \in L_0$. The space $E(X) = \{f \in L_0(X) : \tilde{f} \in E\}$ equipped with the norm $\|f\|_{E(X)} = \|\tilde{f}\|_E$ is called a Köthe-Bochner space.

Definition 4. Let μ be a control measure for the vector measure $\nu : \Sigma \to Y$.

Let $E(X)$ be a Köthe-Bochner space on (Ω, Σ, μ). Consider the linear space $L_0(\mu, X)$ of μ-a.e. equivalence classes of simple functions $f : \Omega \to X$ that satisfy:

1. The function $f\tilde{g} \in L_1(\nu, X, Y)$ where $\tilde{g}(\omega) = \|g(\omega)\|_X$, $g \in E(X)$.

2. The norm $\|f\|_{(E(X))^\nu} = \sup\limits_{\|\tilde{g}\|_E \leq 1} N(f\tilde{g})$ is finite.

We define the Banach space $(E(X))^\nu$ of all X-valued μ-measurable functions as the completion of the space $L_0(\mu, X)$ with respect to the norm given in (2). The same expression can be used for every $f \in (E(X))^\nu$.

Theorem 8. *Let $1 < p < \infty$. If $f \in L_q(\nu, X, Y)$ and $g \in L_p(\nu, X, Y)$, then $f\tilde{g} \in L_1(\nu, X, Y)$ and $N(f\tilde{g})$ is finite.*

Proof. Since $f \in L_q(\nu, X, Y)$, $\|f\| \in L_q(\nu)$ and since $g \in L_p(\nu, X, Y)$, $\tilde{g} \in L_p(\nu)$.

Now $\|f\| \in L_q(\nu)$ and $\tilde{g} \in L_p(\nu)$ implies that $\|f\tilde{g}\| \in L_1(\nu)$, that is, $f\tilde{g} \in L_1(\nu, X, Y)$.

Also

$$\begin{aligned} N(f\tilde{g}) &= \sup_{\|y^*\|\le 1} \int_\Omega \|f\tilde{g}\| \, d|y^*\nu| \\ &\le \left\{ \sup_{\|y^*\|\le 1} \left(\int_\Omega |\tilde{g}|^p \, d|y^*\nu| \right)^{1/p} \right\} \left\{ \sup_{\|y^*\|\le 1} \left(\int_\Omega \|f\|^q \, d|y^*\nu| \right)^{1/q} \right\} \\ &= \|\tilde{g}\|_{p,\nu} N_q(f) < \infty. \end{aligned}$$

□

We are now in a position to extend Proposition 8 of [7] to $L_p(\nu, X, Y)$.

Theorem 9. *Let $1 < p < \infty$. Then $(L_p(\nu, X, Y))^\nu = L_q(\nu, X, Y)$.*

Proof. Let $f \in L_0(\mu, X)$. Then for all $g \in L_p(\nu, X, Y)$ we have, by Theorem 8, that

$$\|f\|_{(L_p(\nu,X,Y))^\nu} = \sup_{\|\tilde{g}\|_{p,\nu}\le 1} N(f\tilde{g}) \le \sup_{\|\tilde{g}\|_{p,\nu}\le 1} \|\tilde{g}\|_{p,\nu} N_q(f) \le N_q(f).$$

Next, let $f \in L_q(\nu, X, Y)$. Then, by Definition 1, there exists a sequence of X-valued simple functions $\{\phi_n\}$ such that $\lim_n N_q(f - \phi_n) = 0$ as $n \to \infty$.

Since $(L_p(\nu, X, Y))^\nu$ is the completion of $L_0(\mu, X)$ with respect to the norm given in Definition 4, it follows that

$$\|f - \phi_n\|_{(L_p(\nu,X,Y))^\nu} \le N_q(f - \phi_n) \to 0 \text{ as } n \to \infty.$$

Therefore,

$$\|f\|_{(L_p(\nu,X,Y))^\nu} = \lim_n \|\phi_n\|_{(L_p(\nu,X,Y))^\nu} \le \lim_n N_q(\phi_n) = N_q(f). \tag{3}$$

On the other hand, let $f \in L_q(\nu, X, Y)$. Then, $\tilde{f} \in L_q(\nu)$. Define the function $g = \dfrac{\tilde{f}^{q-1}}{(\|\tilde{f}\|_{q,\nu})^{q/p}} x$, where $\|x\| = 1$. Then

$$\|g\|^p = \frac{\tilde{f}^{(q-1)p}}{(\|\tilde{f}\|_{q,\nu})^q} \|x\|^p = \frac{\tilde{f}^q}{(\|\tilde{f}\|_{q,\nu})^q}.$$

Since $\tilde{f} \in L_q(\nu)$, it follows that $\|g\|^p \in L_1(\nu)$, which implies that $g \in$

$L_p(\nu, X, Y)$, by Theorem 1. Hence, we have

$$\begin{aligned} N_p(g) &= \sup_{\|y^*\|\le 1}\left(\int_\Omega \|g\|^p \, d|y^*\nu|\right)^{1/p} \\ &= \sup_{\|y^*\|\le 1}\left(\int_\Omega \frac{\tilde{f}^q}{(\|\tilde{f}\|_{q,\nu})^q} \, d|y^*\nu|\right)^{1/p} \\ &= \frac{\sup_{\|y^*\|\le 1}\left(\int_\Omega \tilde{f}^q \, d|y^*\nu|\right)^{1/p}}{\sup_{\|y^*\|\le 1}\left(\int_\Omega \tilde{f}^q \, d|y^*\nu|\right)^{1/p}} = 1. \end{aligned}$$

Since $\tilde{g} \in L_p(\nu)$ and $\|\tilde{g}\|_{p,\nu} = N_p(g) = 1$, we have, by Definition 4, that

$$\begin{aligned} \|f\|_{(L_p(\nu,X,Y))^\nu} \ge N(f\tilde{g}) &= \sup_{\|y^*\|\le 1}\int_\Omega \|f\tilde{g}\| \, d|y^*\nu| \\ &= \sup_{\|y^*\|\le 1}\int_\Omega \|f(\omega)\| \frac{\|f(\omega)\|^{q-1}}{\sup_{\|y^*\|\le 1}\left(\int_\Omega \|f(\omega)\|^q \, d|y^*\nu|\right)^{1/p}} \, d|y^*\nu| \\ &= \sup_{\|y^*\|\le 1}\left(\int_\Omega \|f\|^q \, d|y^*\nu|\right)^{1/q} = N_q(f), \end{aligned}$$

that is,

$$N_q(f) \le \|f\|_{(L_p(\nu,X,Y))^\nu}. \tag{4}$$

Thus it follows from (3) and (4) that

$$\|f\|_{(L_p(\nu,X,Y))^\nu} = N_q(f)$$

and consequently we have $(L_p(\nu, X, Y))^\nu = L_q(\nu, X, Y)$ and the theorem is proved. □

Definition 5. Let $f \in L_1(\nu, X, Y)$. We define other norm $M(\cdot)$ on $L_1(\nu, X, Y)$ as

$$M(f) = \sup_{A\in\Sigma}\left\|\int_A f \, d\nu\right\|.$$

We show that $M(f) \le N(f) \le 2M(f)$ and so these two norms on $L_1(\nu, X, Y)$ are equivalent.

It follows easily by an elementary calculation that $M(f) \le N(f)$.

On the other hand, let $F : \Sigma \to X\check{\otimes}Y$ be defined by

$$F(A) = \int_A f\, d\nu$$

for $A \in \Sigma$. Let π be a partition of Ω. For $x^* \otimes y^* \in X^* \otimes Y^*$, the algebraic tensor product of X^* and Y^*, such that $\|x^*\| \leq 1$, $\|y^*\| \leq 1$, we have

$$\sum_{A\in\pi} |(x^* \otimes y^*)F(A)| \leq 2 \sup_{H\subseteq\Omega}\{\|F(H)\|_{X\check{\otimes}Y}\},$$

by [3, p.5], which implies that $\|F\|(\Omega) \leq 2 \sup\limits_{H\subseteq\Omega}\{\|F(H)\|_{X\check{\otimes}Y}\}$ and so by [9, Theorem 2, p.929], we have

$$\sup_{\substack{\|x^*\|\leq 1\\ \|y^*\|\leq 1}} \left(\int_\Omega |x^* f|\, d|y^*\nu|\right) \leq 2M(f)$$

and from this it follows that $N(f) \leq 2M(f)$ and so

$$M(f) \leq N(f) \leq 2M(f).$$

Therefore, we see that the norm $M(\cdot)$ defined above is equivalent to the original norm $N(\cdot)$ of $L_1(\nu, X, Y)$.

Now, the norm $\|\cdot\|_{(E(X))^\nu}$ defined earlier on the Köthe-Bochner space $E(X)$ is given by

$$\|f\|_{(E(X))^\nu} = \sup_{\|\tilde{g}\|_E\leq 1} N(f\tilde{g}).$$

Using the equivalent formula $M(\cdot)$ for the norm of $L_1(\nu, X, Y)$ we see that the following norm is equivalent to the norm of $(E(X))^\nu$ defined earlier:

$$|||f|||_{(E(X))^\nu} = \sup_{\|\tilde{g}\|_E\leq 1} M(f\tilde{g}) = \sup_{\|\tilde{g}\|_E\leq 1}\sup_{A\in\Sigma}\left\|\int_A f\tilde{g}\, d\nu\right\|_{X\check{\otimes}Y}.$$

Now putting $L_q(\nu, X, Y)$ in place of $E(X)$ we have the following Lemma:

Lemma. *Let $1 < p < \infty$. Then*

$$N_p(g) = |||g|||_{(L_q(\nu,X,Y))^\nu} = \sup_{\|\tilde{f}\|_{q,\nu}\leq 1}\left\|\int_\Omega \tilde{f}g\, d\nu\right\|_{X\check{\otimes}Y}$$

for every $g \in L_p(\nu, X, Y)$.

The result is a direct consequence of Theorem 9 and the definition of the equivalent norm for the space $L_q(\nu, X, Y)^\nu$.

Theorem 10. *Let* $1 < p < \infty$ *and* $f \in L_q(\nu, X, Y)$. *Then the operator* $T_f : L_p(\nu, X, Y) \to X\check{\otimes}Y$ *defined by* $T_f(g) = \int_\Omega f\tilde{g}\, d\nu$ *is well defined and* $\|T_f\| = N_q(f)$, *where* $\tilde{g}(\omega) = \|g(\omega)\|_X$.

Proof. Let $f \in L_q(\nu, X, Y)$ and $g \in L_p(\nu, X, Y)$. Since $f \in L_q(\nu, X, Y)$ we have $\|f\| \in L_q(\nu)$ and $g \in L_p(\nu, X, Y)$ implies $\tilde{g} \in L_p(\nu)$ and so $\|f\tilde{g}\| \in L_1(\nu)$. Therefore $f\tilde{g} \in L_1(\nu, X, Y)$ and we have $\int_\Omega f\tilde{g}\, d\nu \in X\check{\otimes}Y$. Now

$$\begin{aligned} \|T_f\| &= \sup_{N_p(g)\leq 1} \|T_f(g)\|_{X\check{\otimes}Y} = \sup_{N_p(g)\leq 1} \Big\| \int_\Omega f\tilde{g}\, d\nu \Big\|_{X\check{\otimes}Y} \\ &= \sup_{\|\tilde{g}\|_{p,\nu}\leq 1} \Big\| \int_\Omega f\tilde{g}\, d\nu \Big\|_{X\check{\otimes}Y} = N_q(f), \end{aligned}$$

by the above lemma. $\square$

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