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ON THE MAXIMAL ADDITIVE AND MULTIPLICATIVE FAMILIES FOR THE QUASICONTINUITIES OF PIOTROWSKI AND VALLIN

Abstract

In this article we investigate the maximal additive and maximal multiplicative families for the classes of quasicontinuous functions in the sense of Piotrowski and Vallin introduced in [8].

1 Introduction.

If (X, T_X) and (Y, T_Y) are topological spaces and (Z, ρ) is a metric space, then a function $f : X \to Z$ is said to be

1. quasicontinuous at a point $x \in X$ ([6, 7]) if for every set $U \in T_X$ containing x and for each positive real η, there is a nonempty set $U' \in T_X$ contained in U such that $f(U') \subset K(f(x), \eta) = \{t \in Z; \rho(t, f(x)) < \eta\}$.

A function $f : X \times Y \to Z$ is said to be

2. quasicontinuous at (x, y) with respect to x (alternatively y) if for every set $U \times V \in T_X \times T_Y$ containing (x, y) and for each positive real η, there are nonempty sets $U' \in T_X$ contained in U and $V' \in T_Y$ contained in V such that $x \in U'$ (alternatively $y \in V'$) and $f(U' \times V') \subset K(f(x, y), \eta)$ ([8]);
3. symmetrically quasicontinuous at (x, y) if it is quasicontinuous at (x, y) with respect to x and with respect to y ([8]);

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4. separately continuous if the sections $f_x(t) = f(x,t)$ and $f^y(u) = f(u,y)$, $x, u \in X$, $y, t \in Y$, are continuous.

Observe that if a function $f : X \times Y \to Z$ is quasicontinuous at (x,y) with respect to x (alternatively y), then the section f_x (alternatively f^y) is quasicontinuous at y (alternatively x).

Let $(\mathbb{R}, \rho)$ be the set of all reals with the natural metric $\rho(x_1, x_2) = |x_1 - x_2|$. For a family $\mathcal{A}$ of functions $f : X \times Y \to \mathbb{R}$, define the maximal additive family for $\mathcal{A}$ (see [1]) as

$$\mathrm{Max}_{\mathrm{ad}}(\mathcal{A}) = \{g : X \times Y \to \mathbb{R};\ \text{for all } f \in \mathcal{A} \text{ the sum } f + g \in \mathcal{A}\}.$$

Similarly, we define the maximal multiplicative family for $\mathcal{A}$ (see [1]) as

$$\mathrm{Max}_{\mathrm{mult}}(\mathcal{A}) = \{g : X \times Y \to \mathbb{R};\ \text{for all } f \in \mathcal{A} \text{ the product } fg \in \mathcal{A}\}.$$

If the function constant $0 \in \mathcal{A}$ (resp. $1 \in \mathcal{A}$), then evidently $\mathrm{Max}_{\mathrm{ad}}(\mathcal{A}) \subset \mathcal{A}$ (resp. $\mathrm{Max}_{\mathrm{mult}}(\mathcal{A}) \subset \mathcal{A}$).

The maximal additive and multiplicative families for the class Q of all quasicontinuous functions from X to $\mathbb{R}$ were investigated in [2] and [3]. In this article we investigate these families for the quasicontinuities of Piotrowski and Vallin.

2 Maximal Additive Families.

In [2] it is proved that the maximal family $\mathrm{Max}_{\mathrm{ad}}(Q)$ for the class Q of all quasicontinuous functions from X to $\mathbb{R}$ is the same as the family of all continuous functions from X to $\mathbb{R}$.

Denote by Q_1 (alternatively Q_2) the family of all functions $f : X \times Y \to \mathbb{R}$ which are quasicontinuous with respect to x (alternatively with respect to y) at each point. Moreover let $Q_3 = Q_1 \cap Q_2$ denote the family of all symmetrically quasicontinuous functions from $X \times Y$ to $\mathbb{R}$. Since the constant function $0 \in Q_3$, the inclusions $\mathrm{Max}_{\mathrm{ad}}(Q_i) \subset Q_i$ are true for $i = 1, 2, 3$.

Theorem 1. *A function $g : X \times Y \to \mathbb{R}$ belongs to $\mathrm{Max}_{\mathrm{ad}}(Q_1)$ (alternatively to $\mathrm{Max}_{\mathrm{ad}}(Q_2)$) if and only if $g \in Q_1$ (alternatively $g \in Q_2$) and the sections g_x, $x \in X$, (alternatively g^y, $y \in Y$), are continuous.*

PROOF. Let $f, g : X \times Y \to \mathbb{R}$ be quasicontinuous functions with respect to x. Assume that the sections g_x, $x \in X$, are continuous. For the proof of the quasicontinuity with respect to x of the sum $f + g$, fix a point $(a,b) \in X \times Y$, a real $\eta > 0$ and sets $U \in T_X$ and $V \in T_Y$ with $(a,b) \in U \times V$. Since the section g_a is continuous at the point b, there is a set $V_1 \in T_Y$ such that

$$b \in V_1 \subset V \text{ and } |g(a,v) - g(a,b)| < \frac{\eta}{3} \text{ for } v \in V_1.$$

From the quasicontinuity of f at (a,b) with respect to x it follows that there are nonempty sets $U_2 \in T_X$ and $V_2 \in T_Y$ such that

$$a \in U_2 \subset U,\ V_2 \subset V_1 \text{ and } |f(u,v) - f(a,b)| < \frac{\eta}{3} \text{ for } (u,v) \in U_2 \times V_2.$$

Fix a point $c \in V_2$. Since g is quasicontinuous at (a,c) with respect to x, there are nonempty sets $U_3 \in T_X$ and $V_3 \in T_Y$ such that

$$a \in U_3 \subset U_2,\ V_3 \subset V_2 \text{ and } |g(u,v) - g(a,c)| < \frac{\eta}{3} \text{ for } (u,v) \in U_3 \times V_3.$$

So, $a \in U_3 \subset U,\ V_3 \subset V$ and for $(u,v) \in U_3 \times V_3$ we have

$$|f(u,v) + g(u,v) - f(a,b) - g(a,b)| \leq |f(u,v) - f(a,b)| + |g(u,v) - g(a,b)|$$
$$\leq |f(u,v) - f(a,b)| + |g(u,v) - g(a,c)| + |g(a,c) - g(a,b)| < \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta.$$

This finishes the proof of the quasicontinuity of the sum $f+g$ with respect to x. If $f,g \in Q_2$, then the proof of the quasicontinuity of $f+g$ with respect to y is analogous.

For the proof of the inverse implication, suppose that g is a function in the class $\mathrm{Max}_{\mathrm{ad}}(Q_1) \subset Q_1$. Assume, to a contradiction, that there is a point $(u,v) \in X \times Y$ such that the section $g_u : Y \to \mathbb{R}$ is discontinuous at the point $v \in Y$. Since $g \in Q_1$, the section g_u is quasicontinuous. But the section g_u is discontinuous at v, so by [2] there is a quasicontinuous function $h : Y \to \mathbb{R}$ such that the sum $g_u + h$ is not quasicontinuous. Put

$$f(x,y) = h(y) \text{ for } (x,y) \in X \times Y,$$

and observe that $f \in Q_1$. Since the section $(f+g)_u = f_u + g_u = h + g_u$ is not quasicontinuous, the function $f + g \notin Q_1$ and we obtain a contradiction to $g \in \mathrm{Max}_{\mathrm{ad}}(Q_1)$.

The proof of the continuity of the sections g^y $(y \in Y)$ for a function g in $\mathrm{Max}_{\mathrm{ad}}(Q_2)$ is analogous. □

Corollary 1. *If a function $g : X \times Y \to \mathbb{R}$ belongs to Q_3 and is separately continuous, then $g \in \mathrm{Max}_{\mathrm{ad}}(Q_3)$.*

Recall that there are topological spaces (X, T_X) and (Y, T_Y) and separately continuous functions $g : X \times Y \to \mathbb{R}$ which are not symmetrically quasicontinuous ([4, 5]).

For a set $A \subset X \times Y$, denote by $A_x = \{v \in Y; (x,v) \in A\}$, $x \in X$, the vertical section of A and by $A^y = \{u \in X; (u,y) \in A\}$, $y \in Y$, the horizontal section of A.

For the investigation of the family $\mathrm{Max}_{\mathrm{ad}}(Q_3)$, we will use the following notation.

Let $(x,y) \in X \times Y$ be a point. We will say that a closed set $A \subset X \times Y$ belongs to the family $S(x,y)$ (resp. $P(x,y)$) if and only if it fulfils the following conditions:

- $A_x = \{y\}$ (resp. $A^y = \{x\}$);
- for each point $(u,v) \in A \setminus \{(x,y)\}$, $u \in \mathrm{cl}(\mathrm{int}(A))^v)$ and $v \in \mathrm{cl}((\mathrm{int}(A))_u)$ (int and cl denote the interior and the closure operations, respectively);
- $x \in \mathrm{cl}((\mathrm{int}(A))^y)$ (resp. $y \in \mathrm{cl}(\mathrm{int}(A))_x)$).

Observe that if a point (x,y) is isolated in $X \times Y$ and the singleton $\{(x,y)\}$ is closed, then $\{(x,y)\} \in S(x,y) \cap P(x,y)$.

Moreover, if $X = Y = \mathbb{R}$ and $T_X = T_Y$ is the natural topology, then for all topologies $T_1, T_2 \supset T_X$ in $\mathbb{R}$ such that for each open interval (a,b) the closure (in T_1 and in T_2) of (a,b) is the closed interval $[a,b]$, the product topology $T_1 \times T_2$ in $\mathbb{R}^2$ is such that for each point $(x,y) \in \mathbb{R}^2$ the set

$$\{(u,v) \in \mathbb{R}^2; u \geq x \text{ and } -u + (x-y) \leq v \leq u + (y-x)\} \in S(x,y),$$

and the set

$$\{(u,v) \in \mathbb{R}^2; v \geq y \text{ and } -v + (x-y) \leq u \leq v + (y-x)\} \in P(x,y).$$

Theorem 2. *Let (X,T_X) and (Y,T_Y) be topological spaces such that for each point $(x,y) \in T_X \times T_Y$ the families $S(x,y)$ and $P(x,y)$ are nonempty. If a function $g : X \times Y \to \mathbb{R}$ belongs to $\mathrm{Max}_{\mathrm{ad}}(Q_3)$, then g is separately continuous.*

Proof. Assume to the contrary that there is a function $g : X \times Y \to \mathbb{R}$ belonging to $\mathrm{Max}_{\mathrm{ad}}(Q_3)$ which is not separately continuous. So there is a point $(a,b) \in \mathbb{R}^2$ such that either the section g_a is discontinuous at b or the section g^b is discontinuous at a. Suppose that g_a is discontinuous at b. Then there is an open bounded interval $(c_1, d_1) = I \ni g(a,b)$ such that $b \in \mathrm{cl}((g_a)^{-1}(\mathbb{R} \setminus I))$. Without loss of generality we can assume that $b \in \mathrm{cl}((g_a)^{-1}([d_1, \infty))$. Fix a real $d \in (g(a,b), d_1)$ and a set $A \in S(a,b)$. Observe that the function

$$f(x,y) = \begin{cases} -d & \text{if} (x,y) \in A; \\ -d & \text{if} (x,y) \notin A \text{ and } g(x,y) > d; \\ -g(x,y) & \text{otherwise on } X \times Y \end{cases}$$

is symmetrically quasicontinuous. Consider the section $(g+f)_a$ of the sum $g+f$. Since

$$g(a,b) + f(a,b) = g(a,b) - d < 0 \text{ and } g(a,y) + f(a,y) = 0 \text{ for } y \neq b,$$

the section $(g+f)_a$ is not quasicontinuous, and consequently the sum $f+g$ is not symmetrically quasicontinuous. So, $g \notin \mathrm{Max}_{\mathrm{ad}}(Q_3)$ and we obtain a contradiction. In the other cases the reasoning is analogous. $\square$

From Theorems 1 and 2 we deduce

Theorem 3. *Let (X, T_X) and (Y, T_Y) be topological spaces such that for each point $(x, y) \in T_X \times T_Y$ the families $S(x, y)$ and $P(x, y)$ are nonempty. A function $g : X \times Y \to \mathbb{R}$ belongs to $\mathrm{Max}_{\mathrm{ad}}(Q_3)$ if and only if it belongs to Q_3 and is separately continuous.*

Problem. Let (X, T_X) and (Y, T_Y) be arbitrary topological spaces and let $g : X \times Y \to \mathbb{R}$ be a function belonging to $\mathrm{Max}_{\mathrm{ad}}(Q_3)$. Is the function g separately continuous?

3 Maximal Multiplicative Families.

Since the constant function $1 \in Q_1 \cap Q_2$, evidently $\mathrm{Max}_{\mathrm{mult}}(Q_i) \subset Q_i$ for $i = 1, 2, 3$.

A function $h : X \to \mathbb{R}$ satisfies Foran's condition (F) (compare [3]) if for each discontinuity point x of h the value $h(x) = 0$ and $x \in \mathrm{cl}(C(h) \cap h^{-1}(0))$, where $C(h)$ denotes the set of all continuity points of h.

Let $\mathcal{F}$ denote the class of all functions $h : X \to \mathbb{R}$ satisfying Foran's condition (F). In [3] it is proved that $\mathcal{F} \subset \mathrm{Max}_{\mathrm{mult}}(Q)$. Moreover, in [3] the following theorem is proved.

Theorem 4. *Let $f : X \to \mathbb{R}$ be a function. If there is a nonempty set $U \in T_X$ such that $A = \{u \in U; f(u) = 0\} \neq \emptyset$ and $f(u) \neq 0$ for every point $u \in C(f) \cap \mathrm{cl}(U)$, then there exists a function $g \in Q$ such that $fg \notin Q$.*

In this article we will prove the following theorems.

Theorem 5. *If a function $g : X \times Y \to \mathbb{R}$ belongs to Q_1 (alternatively to Q_2) and if the sections g_u, $u \in X$ (alternatively the sections g^v, $v \in Y$), satisfy condition (F), then $g \in \mathrm{Max}_{\mathrm{mult}}(Q_1)$ (alternatively $g \in \mathrm{Max}_{\mathrm{mult}}(Q_2)$).*

PROOF. Let $f, g : X \times Y \to \mathbb{R}$ be quasicontinuous functions with respect to x. Assume that the sections g_u, $u \in X$, satisfy condition (F). For the proof of the quasicontinuity with respect to x of the product fg, fix a point $(a, b) \in X \times Y$, a real $\eta > 0$ and sets $U \in T_X$ and $V \in T_Y$ with $(a, b) \in U \times V$.

At the start we suppose that the section g_a is continuous at b. Fix a positive real number

$$M > \max(|f(a,b) - \eta|, |f(a,b) + \eta|, |g(a,b) - \eta|, |g(a,b) + \eta|).$$

Since the section g_a is continuous at the point b, there is a set $V_1 \subset V$ belonging to T_Y and such that

$$b \in V_1 \text{ and } |g(a,v) - g(a,b)| < \frac{\eta}{3M} \text{ for } v \in V_1.$$

From the quasicontinuity of f with respect to x at the point (a,b), it follows that there are nonempty sets $U_2 \in T_X$ and $V_2 \in T_Y$ such that

$$a \in U_2,\ U_2 \times V_2 \subset U \times V_1 \text{ and } |f(u,v) - f(a,b)| < \frac{\eta}{3M} \text{ for } (u,v) \in U_2 \times V_2.$$

Fix a point $c \in V_2$. Since the function g is quasicontinuous with respect to x at the point (a,c), there are nonempty sets $U_3 \in T_X$ and $V_3 \in T_Y$ such that

$$a \in U_3 \subset U_2,\ V_3 \subset V_2 \text{ and } |g(u,v) - g(a,c)| < \frac{\eta}{3M} \text{ for } (u,v) \in U_3 \times V_3.$$

Observe, for $(u,v) \in U_3 \times V_3$, the inequalities

$$|g(u,v) - g(a,b)| \le |g(u,v) - g(a,c)| + |g(a,c) - g(a,b)| < \frac{\eta}{3M} + \frac{\eta}{3M} = \frac{2\eta}{3M}$$

and

$$|f(u,v)g(u,v) - f(a,b)g(a,b)| =$$
$$|f(u,v)g(u,v) - f(a,b)g(u,v) + f(a,b)g(u,v) - f(a,b)g(a,b)|$$
$$\le |g(u,v)||f(u,v) - f(a,b)| + |f(a,b)||g(u,v) - g(a,b)| < M\frac{\eta}{3M} + M\frac{2\eta}{3M} = \eta.$$

This finishes the proof of the quasicontinuity with respect to x of the product fg at (a,b) in the considered case.

Now suppose that the section g_a is discontinuous at b. Then, by condition (F) of g_a, the value $g(a,b) = 0$ and there is a continuity point $c \in V$ of g_a with $g(a,c) = 0$. By the previous part of the proof there is a nonempty set $U_3 \times V_3 \subset U \times V$ such that $U_3 \times V_3 \in T_X \times T_Y$, $a \in U_3$ and

$$\begin{aligned} &|f(u,v)g(u,v) - f(a,c)g(a,c)| = |f(u,v)g(u,v)| \\ &= |f(u,v)g(u,v) - f(a,b)g(a,b)| < \eta \end{aligned}$$

for $(u,v) \in U_3 \times V_3$. This finishes the proof of the quasicontinuity with respect to x of the product of fg. The proof of the quasicontinuity of fg with respect to y is analogous. □

As an immediate consequence we obtain the following.

Corollary 2. *If the sections g_u and g^v, $u \in X$, $v \in Y$, of a symmetrically quasicontinuous function $g : X \times Y \to \mathbb{R}$ satisfy condition (F), then $g \in \mathrm{Max}_{\mathrm{mult}}(Q_3)$.*

Theorem 6. *Let $g : X \times Y \to \mathbb{R}$ be a function belonging to Q_1 (alternatively to Q_2). If there is a point $a \in X$ (alternatively $b \in Y$) such that the section g_a (alternatively g^b) does not belong to $\mathrm{Max}_{\mathrm{mult}}(Q)$, then $g \notin \mathrm{Max}_{\mathrm{mult}}(Q_1)$ (alternatively $g \notin \mathrm{Max}_{\mathrm{mult}}(Q_2)$).*

Proof. The section $g_a : Y \to \mathbb{R}$ is quasicontinuous everywhere on Y and does not belong to $\mathrm{Max}_{\mathrm{mult}}(Q)$, so there is a quasicontinuous function $h : Y \to \mathbb{R}$ such that the product $g_a h : Y \to \mathbb{R}$ is not quasicontinuous. For $(u, v) \in X \times Y$, let $f(u, v) = h(v)$. Then the function f is quasicontinuous with respect to x, but the product gf is not quasicontinuous with respect to x, because its section $(gf)_a = g_a h$ is not quasicontinuous. So $g \notin \mathrm{Max}_{\mathrm{mult}}(Q_1)$. Similarly, we can prove in the alternative case that $g \notin \mathrm{Max}_{\mathrm{mult}}(Q_2)$. □

Theorem 7. *Let $g : X \times Y \to \mathbb{R}$ be a function belonging to Q_3. If there is a point $(a, b) \in X \times Y$ such that*

(i) the section g_a is discontinuous at b and there is a set $A \in S(a, b)$ with $g^{-1}(0) \subset A$

or

(ii) the section g^b is not continuous at a and there is a set $B \in P(a, b)$ with $g^{-1}(0) \subset B$,

then $g \notin \mathrm{Max}_{\mathrm{mult}}(Q_3)$.

Proof. Assume that (i) holds. Since the section g_a is not continuous at b, there is a real $r > 0$ such that $b \in \mathrm{cl}((g_a)^{-1}(\mathbb{R} \setminus (g(a, b) - r, g(a, b) + r)))$ and $(g(a, b) - r)(g(a, b) + r) \neq 0$. Suppose that $b \in \mathrm{cl}((g_a)^{-1}([g(a, b) + r, \infty)))$. Observe that the function $h(x, y) = \min(g(x, y), g(a, b) + r)$ is symmetrically quasicontinuous. So the function

$$f(x,y) = \begin{cases} \frac{1}{g(x,y)} & \text{for} \quad (x,y) \in (X \times Y) \setminus A; \\ \frac{1}{g(a,b)+r} & \text{for} \quad (x,y) \in A \end{cases}$$

is also symmetrically quasicontinuous. Since

$$f(a,y)g(a,y) = g(a,y)\frac{1}{g(a,y)} = 1 \text{ for } y \neq b,$$

and

$$f(a,b)g(a,b) = g(a,b)\frac{1}{g(a,b)+r} \neq 1,$$

the section $(fg)_a$ of the product fg is not quasicontinuous, and consequently $fg \notin Q_3$. So $g \notin \mathrm{Max}_{\mathrm{mult}}(Q_3)$ in the considered case. In the other cases the reasoning is similar. □

If, for each point $(x, y) \in X \times Y$, the classes $S(x, y)$ and $P(x, y)$ are nonempty and

$$Q_4 = \{f \in Q_3; f^{-1}(0) = \emptyset\},$$

then, from Theorems 5 and 6, it follows that

$$\mathrm{Max}_{\mathrm{mult}}(Q_4) = \{f \in Q_4; f \text{ is separately continuous}\}.$$

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