RESEARCH

Real Analysis Exchange
Vol. 32(2), 2006/2007, pp. 429–454

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INFORMATION PARAMETERS AND LARGE DEVIATION SPECTRUM OF DISCONTINUOUS MEASURES

Abstract

Let ν be a finite Borel measure on $[0,1]^d$. Consider the L^q-spectrum of ν: $\tau_\nu(q) = \liminf_{n\to\infty} -n^{-1}\log_b \sum_{Q\in\mathcal{G}_n} \nu(Q)^q$ $(q \geq 0)$, where $\mathcal{G}_n$ is the set of b-adic cubes of generation n (b integer ≥ 2). Let $q_\tau = \inf\{q : \tau_\nu(q) = 0\}$ and $H_\tau = \tau'_\nu(q_\tau^-)$. When ν is a mono-dimensional continuous measure of information dimension D, $(q_\tau, H_\tau) = (1, D)$. When ν is purely discontinuous, its information dimension is $D = 0$, but the pair (q_τ, H_τ) may be non-trivial and contains relevant information on the distribution of ν. Intrinsic characterizations of (q_τ, H_τ) are found, as well as sharp estimates for the large deviation spectrum of ν on $[0, H_\tau]$. We exhibit the differences between the cases $q_\tau = 1$ and $q_\tau \in (0,1)$. We conclude that the large deviation spectrum's properties observed for specific classes of measures are true in general.

1 Introduction.

During the last fifteen years, the multifractal behavior of purely discontinuous measures, i.e., constituted only by Dirac masses, has been precisely described for several classes of well-structured objects. Equivalently, the behavior of non-decreasing functions whose derivative is a purely discontinuous measure

Key Words: Large deviations, fractals
Mathematical Reviews subject classification: 60F10, 28A80
Received by the editors July 5, 2006
Communicated by: Zoltán Buczolich

has also been widely investigated. Examples of such objects are Lévy subordinators [20] and homogeneous sums of Dirac masses studied in [1, 13, 19], Lévy subordinators in multifractal time [8] and more generally heterogeneous sums of Dirac masses governed by a self-similar measure [31, 4, 5]. Nevertheless, few general results are known about the fine structure of purely discontinuous measures. For instance, all the possible classical information dimensions vanish for such measures. One of the motivations of the present work is the need for other relevant parameters.

From the multifractal standpoint, a common feature between the above examples is that all their multifractal spectra are linear on a non trivial interval whose left-end point is 0. It is natural to ask whether this property is shared by all or at least by a large class of the purely discontinuous measures. This is particularly important for the large deviation spectrum since it is the most numerically tractable spectrum among the several multifractal spectra. Hence, a priori estimates are of great importance for practical purposes. We focus on this spectrum and find that under a weak assumption it is indeed linear on a non trivial interval I whose left-end point is 0 (see Theorem 1.3). The slope of this linear part and the right-end point of I are related to new information parameters deduced from the so-called L^q-spectrum of the measure.

Let us start by recalling the notions of information dimension, multifractal and large deviation spectra. We then expose the achievements of this paper.

1.1 Multifractal Spectra and Information Parameters.

Let $d \in \mathbb{N}$. In the context of fractal sets and dynamical systems, it is usual to describe the geometry and the distribution at small scales of a finite Borel measure ν on $[0,1]^d$ thanks to its (lower and upper) Hausdorff, packing and entropy dimensions. In general these dimensions differ from one another, but when they coincide, they determine without ambiguity the dimension of the measure. This situation arises when there exists $D \in [0,d]$ such that

$$\lim_{r\to 0^+} \frac{\log \nu(B(x,r))}{\log(r)} = D \quad \nu\text{-a.e.} \tag{1}$$

The dimension of ν is equal to D [32, 15], and ν is said to be mono-dimensional.

Let $b \geq 2$ be an integer and let $\mathcal{G}_n$ be the partition of $[0,1]^d$ into b-adic boxes of generation n written as $\prod_{i=1}^d [b^{-n}k_i, b^{-n}(k_i+1))$ with $(k_1,\dots,k_d) \in \{0,1,\dots,b^n-1\}^d$.

Let us introduce on $\mathbb{R}$ the L^q-spectrum of ν

$$\tau_\nu(q) = \liminf_{n\to\infty} -\frac{1}{n}\log_b s_n(q) \text{ where } s_n(q) = \sum_{Q\in\mathcal{G}_n,\, \nu(Q)\neq 0} \nu(Q)^q. \tag{2}$$

It is easy to see that τ_ν is a concave function which does not depend on $b \geq 2$ when $q \in \mathbb{R}_+$ or when $q \in \mathbb{R}$ and $\mathrm{Supp}\,(\nu) = [0,1]^d$ ($\mathrm{Supp}\,(\nu)$ stands for the support of the measure ν). Property (1) holds with $D = \tau_\nu'(1)$ for instance as soon as $\tau_\nu'(1)$ exists, and the dimensions mentioned above always lie in $[\tau_\nu'(1^+), \tau_\nu'(1^-)]$ (see Section 2 and [25, 27, 15, 9, 16]).

The behavior of ν at small scales may be more generally geometrically described by the *Hausdorff* and *packing singularity spectra* defined as follows (see [10, 26] and references therein). For $x \in \mathrm{Supp}\,(\nu)$, the pointwise Hölder exponent of ν at x is defined by

$$h_\nu(x) = \liminf_{r \to 0^+} \frac{\log \nu(B(x,r))}{\log r}. \tag{3}$$

Then, one considers the level sets of the pointwise Hölder exponent of ν

$$E_h^\nu = \{x \in \mathrm{Supp}\,(\nu) : h_\nu(x) = h\} \ (h \geq 0). \tag{4}$$

Finally, the Hausdorff and packing spectra of ν are respectively

$$d_\nu : h \geq 0 \mapsto \dim E_h^\nu \text{ and } D_\nu : h \geq 0 \mapsto \mathrm{Dim}\, E_h^\nu,$$

where dim and Dim stand for the Hausdorff and the packing dimension.

Another description of the distribution of ν is given by the following statistical (rather than geometrical) approach. For $\varepsilon > 0$, $h \geq 0$, $n \in \mathbb{N}$, let

$$\mathcal{S}_n^\nu(h,\varepsilon) = \left\{ Q \in \mathcal{G}_n : b^{-n(h+\varepsilon)} \leq \nu(Q) \leq b^{-n(h-\varepsilon)} \right\}. \tag{5}$$

The *large deviation spectrum* f_ν of ν is the upper semi-continuous function

$$h \geq 0 \mapsto f_\nu(h) = \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log_b \#\mathcal{S}_n^\nu(h,\varepsilon).$$

The following Proposition, which follows from [10, 26, 29, 22, 30] and also Theorem 2.1 hereafter, gathers properties of these several spectra.

Recall that if g is a function from $\mathbb{R}$ to $\mathbb{R} \cup \{-\infty\}$, its Legendre transform is the mapping $g^* : h \mapsto \inf_{q \in \mathbb{R}} (hq - g(q)) \in \mathbb{R} \cup \{-\infty\}$.

The *Legendre spectrum* of ν is the concave function $h \geq 0 \mapsto \tau_\nu^*(h)$.

For a subset E of $[0,1]^d$, $\dim E < 0$ means that E is empty.

Proposition 1.1. *Let ν be a finite positive Borel measure on $[0,1]^d$.*

1. *For every $h \geq 0$, $d_\nu(h) \leq f_\nu(h) \leq f_\nu^{**}(h) = \tau_\nu^*(h)$ and $D_\nu(h) \leq \tau_\nu^*(h)$. The multifractal formalism is said to hold at h if $d_\nu(h) = \tau_\nu^*(h)$.*
2. *For every $h \in \{\tau_\nu'(q^+) : q \in \mathbb{R}\} \cup \{\tau_\nu'(q^-) : q \in \mathbb{R}\}$, $f_\nu(h) = f_\nu^{**}(h)$.*

When $D = \tau_\nu'(1)$ exists or (1) holds, the multifractal formalism holds at D and ν is carried by the set E_D^ν ([25]). Examples of continuous measures for which this arises are provided by classes of measures possessing scaling invariance properties (see [23, 24, 21, 28, 10, 14, 18, 16, 3] for a non-exhaustive list). For these measures, $D > 0$; hence D is relevant as information dimension since it takes in general different values for two distinct such measures.

On the other hand, it is proved in [11] that in the Baire generic sense, in the one-dimensional case ($d = 1$), quasi-all Borel continuous measures ν on $[0, 1]$ are concentrated on the set E_D^ν with $D = 0$, and neither $\tau_\nu'(1)$ exists nor (1) holds. This naturally leads to consider the extremal situation when ν is purely discontinuous. In this case, (1) holds with $D = 0$, and all the Hausdorff, packing and entropy dimensions equal 0, whatever the behavior of τ_ν at 1 is. Consequently, the classical dimension parameters are not relevant in this case.

We thus look for other natural information parameters related to the distribution of a measure. Of course, these parameters must coincide in some sense with the dimension $\tau_\nu'(1)$ when it is defined and positive. We consider

$$q_\tau(\nu) = \inf\{q : \tau_\nu(q) = 0\} \text{ and } H_\tau(\nu) = \tau_\nu'\left(q_\tau(\nu)^-\right).$$

Let us list some of the properties of these parameters (see also Theorem 2.1).

- If $\dim \operatorname{Supp}(\nu) > 0$ and if τ_ν is continuous at 0^+, then one always has $0 < q_\tau(\nu) \leq 1$ and $H_\tau(\nu) \leq d/q_\tau(\nu)$.
- If $\tau_\nu'(1)$ exists and is positive, then $q_\tau(\nu) = 1$ and $H_\tau(\nu) = \tau_\nu'(1)$. Moreover, this real number is in this case the only fixed point of f_ν.
- From Proposition 1.1, $H_\tau(\nu)$ is always the largest solution of the equation $f_\nu(h) = q_\tau(\nu)h$, while $\tau_\nu'(q_\tau(\nu)^+)$ is the smallest solution of the same equation. In particular, $f_\nu(H_\tau(\nu)) = \tau_\nu^*(H_\tau(\nu))$.

1.2 Sharper Estimates for the Large Deviation Spectrum of ν on $[0, H_\tau(\nu)]$ When ν is a Purely Discontinuous Measure.

In the sequel, we focus on purely discontinuous measures of the form

$$\nu = \sum_{k\geq 1} m_k \delta_{x_k} \tag{6}$$

for a sequence of masses $\widetilde{m} = (m_k)_{k\geq 1} \in (\mathbb{R}^+)^{\mathbb{N}^*}$ such that $\sum_k m_k < \infty$ and a sequence of pairwise distinct points $\widetilde{x} = (x_k)_{k\geq 1} \in ([0, 1]^d)^{\mathbb{N}^*}$.

Under weak assumptions on the sequences $\widetilde{m}$ and $\widetilde{x}$ (see assumption **(H)** in Definition 1.2 below), we have

$$0 = \tau_\nu'(q_\tau(\nu)^+) < \tau_\nu'(q_\tau(\nu)^-) = H_\tau(\nu).$$

As a consequence of Proposition 1.1 and the third property pointed out above, for such measures ν, $f_\nu(0)=0$, $f_\nu(H_\tau(\nu))=q_\tau(\nu)H_\tau(\nu)$ and for every $h\in(0,H_\tau(\nu))$, $f_\nu(h)\leq q_\tau h$.

Moreover, for all the purely discontinuous measures mentioned in the introduction, $f_\nu(h)=q_\tau(\nu)h$ on $[0,H_\tau(\nu)]$. The main purpose of the following work is to understand whether this equality holds true in general.

First, two intrinsic parameters $q_g(\nu)$ and $H_g(\nu)$, depending only on the geometrical repartition of $\widetilde{m}$ and $\widetilde{x}$, are proposed in (7) and (8). We investigate them in details. In particular, their relationships with $q_\tau(\nu)$ and $H_\tau(\nu)$ are of great interest and are the subject of a large part of the paper (Sections 3-4).

For every $n\geq 1$, let

$$K_n=\left\{k: m_k\in[b^{-n},b^{-(n-1)})\right\} \text{ and } X_n=\{x_k: k\in K_n\}.$$

The set X_n contains the locations of the Dirac masses of size approximately equal to b^{-n}. When $\#K_n=0$, we set $q(n)=0$ and $\mathcal{J}(n)=1$, otherwise when $\#K_n>0$, we define the quantities

$$q(n)=\frac{\log_b \#K_n}{n} \quad \text{and} \quad \mathcal{J}(n)=\min\left\{n': \sup_{Q\in\mathcal{G}_{n'}}\#(Q\cap X_n)\leq 1\right\}.$$

Thus, provided that X_n is not empty, we have $\#K_n=b^{nq(n)}$, and $\mathcal{J}(n)$ is the first generation which "separates" the elements of X_n. Then let

$$q_g(\nu)=\limsup_{n\to\infty}\ q(n) \text{ and } \underline{H}_g(\nu)=\liminf_{n\to\infty}\frac{n}{\mathcal{J}(n)}. \tag{7}$$

Let $\alpha>0$ and $n\geq 1$. When $\#K_n=0$, we set $\mathcal{J}(n,\alpha)=1$. Otherwise, when $\#K_n>0$, we set

$$\mathcal{J}(n,\alpha)=\min\left\{n'\in\mathbb{N}:\begin{cases}\text{there is a set } X_n'\subset X_n \text{ of cardinality} \geq b^{n(q(n)-\alpha)}\\ \text{such that } \sup_{Q\in\mathcal{G}_{n'}}\#(Q\cap X_n')\leq 1\end{cases}\right\}.$$

Provided that $X_n\neq\emptyset$ and α small enough, $\mathcal{J}(n,\alpha)$ is the first generation which separates a large proportion of the elements of X_n. Let us denote by $\mathcal{U}$ the set of positive sequences of real numbers converging to 0. When $q_g(\nu)>0$, let

$$\forall\varepsilon>0,\ H_{g,\varepsilon}(\nu)=\sup_{(\alpha_n)_n\in\mathcal{U}}\ \limsup_{\substack{n\to\infty,\\ q(n)\geq q_g(\nu)-\varepsilon}}\frac{n}{\mathcal{J}(n,\alpha_n)}$$

and

$$H_g(\nu)=\lim_{\varepsilon\to 0^+}\ H_{g,\varepsilon}(\nu). \tag{8}$$

Heuristically, asymptotically when $n \to +\infty$, there is no more than $b^{q_g(\nu)n}$ Dirac masses with a weight $\sim b^{-n}$ involved in the sum (6), while $H_g(\nu)$ depends on the proximity between Dirac masses of same order.

Definition 1.2. A measure ν of the form (6) is said to satisfy assumption **(H)** when $q_g(\nu) > 0$ and $H_g(\nu) > 0$.

Theorem 1.3. *Let* $\widetilde{m} = (m_k)_{k\geq 1} \in (\mathbb{R}^+)^{\mathbb{N}^*}$ *such that* $\sum_k m_k < \infty$ *and consider a sequence of pairwise distinct points* $\widetilde{x} = (x_k)_{k\geq 1} \in ([0,1]^d)^{\mathbb{N}^*}$. *Assume that the purely discontinuous measure* ν *defined by (6) satisfies* **(H)**. *Then:*

1. *If* $q_\tau(\nu) \in (0,1)$, *then* $q_g(\nu) = q_\tau(\nu)$, $H_\tau(\nu) = H_g(\nu)$ *and* $f_\nu(h) = q_\tau(\nu)h$ *for every* $h \in [0, H_\tau(\nu)]$.
2. *If* $q_\tau(\nu) = 1$, *then* $q_g(\nu) = q_\tau(\nu)$ *and* $H_g(\nu) \leq H_\tau(\nu)$. *Moreover,* $f_\nu(h) = q_\tau(\nu)h$ *for every* $h \in [0, H_g(\nu)]$ *and* $f_\nu(H_\tau(\nu)) = q_\tau(\nu)H_\tau(\nu)$ *(i.e., the large deviation spectrum may exhibit a gap between* $H_g(\nu)$ *and* $H_\tau(\nu)$*).*
3. *When* $q_\tau(\nu) = 1$, *the exponent* $H_g(\nu)$ *is optimal in the following sense:*
   (a) *For every exponents* $0 < h_0 \leq h_1 \leq d$, *there exists a measure* ν *of the form (6) such that* $q_\tau(\nu) = 1$, $H_g(\nu) = h_0$, $H_\tau(\nu) = h_1$ *and* $f_\nu(h) = h$ *for every* $h \in [0, h_1]$.
   (b) *For every exponents* $0 < h_0 < h_1 \leq d$, *there is a measure* ν *of the form (6) such that* $q_\tau(\nu) = 1$, $H_g(\nu) = h_0$, $H_\tau(\nu) = h_1$ *and* $f_\nu(h) < h$ *on* (h_0, h_1).
4. *Assume that for all* $\varepsilon > 0$ *there is an increasing sequence of integers* $(n_j)_{j\geq 1}$ *such that* $q(n_j)$ *converges to 1 and*

$$\dim \Big(\limsup_{j\to\infty} \bigcup_{k\in K_{n_j}} B(x_k, b^{-n_j(H_\tau(\nu)^{-1}-\varepsilon)}) \Big) \geq H_\tau(\nu).$$

   Then $H_g(\nu) = H_\tau(\nu)$.

When **(H)** is satisfied, Theorem 1.3 yields an intrinsic and geometrical interpretation of $(q_\tau(\nu), H_\tau(\nu))$. This theorem also implies that the large deviation spectrum of any measure ν of the form (6) satisfying **(H)** always starts with a straight line, of slope $q_\tau(\nu)$.

The reader can check that for most of the measures mentioned in the very beginning of this section, assumption **(H)** holds true and $H_g(\nu) = H_\tau(\nu)$ (see the comments in Section 1.3). Consequently, Theorem 1.3 allows us to recover the linear increasing part of the large deviation spectrum of these measures.

The case where $q_g(\nu) > 0$ but $H_g(\nu) = 0$ is also interesting (though exceptional) and arises for instance if $q_\tau(\nu) > 0$ and $H_\tau(\nu) = 0$. Examples of such measures are constructed in [4]. We do not consider this situation hereafter.

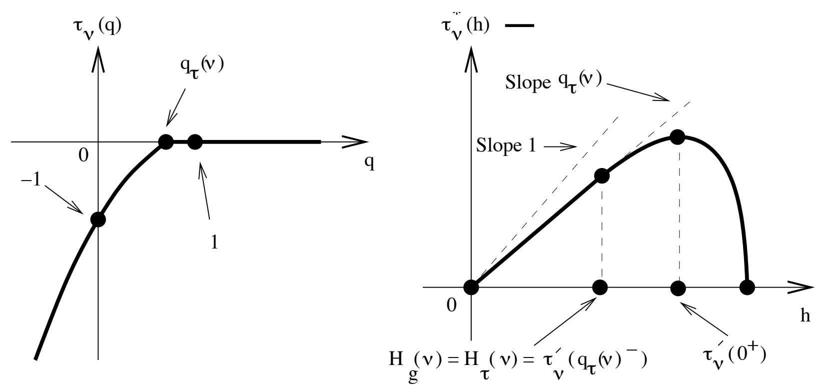

Figure 1: For a discontinuous measure ν with $q_\tau(\nu) < 1$: **Left:** scaling function τ_ν and **Right:** typical Legendre spectrum. The Legendre and large deviation spectra coincide for every $h \leq H_\tau(\nu) = H_g(\nu)$.

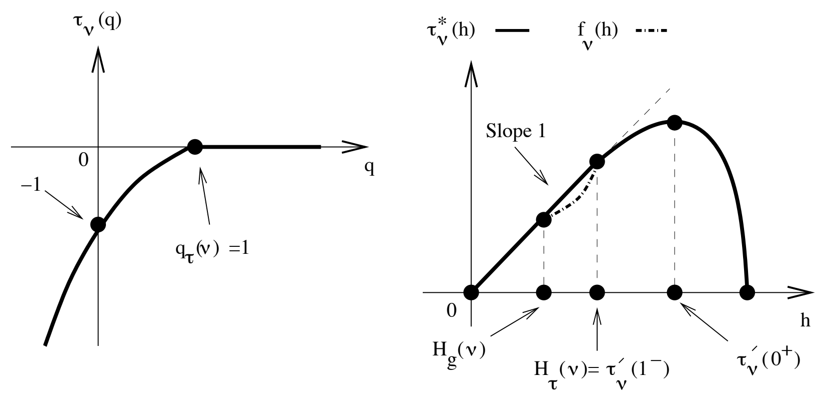

Figure 2: For a discontinuous measure ν with $q_\tau(\nu) = 1$: **Left:** scaling function τ_ν and **Right:** typical Legendre and large deviation spectra. They coincide when $h \leq H_g(\nu)$ and when $h = H_\tau(\nu)$.

1.3 Remarks and Related Works.

• It is worth noticing that $q_g(\nu)$, $H_g(\nu)$, $q_\tau(\nu)$ and $H_\tau(\nu)$ do not depend on the choice of the integer basis b (while f_ν clearly depends on b in general). We have thus established that, under **(H)**, f_ν is always linear with slope $q_\tau(\nu)$ on the range of exponents $h \in [0, H_g(\nu)]$ for every choice of $b \geq 2$.

• There is a strong asymmetry between the cases $q_g(\nu) = 1$ and $q_g(\nu) \in (0,1)$. In this latter case, the large deviation spectrum is totally determined by $H_g(\nu) = H_\tau(\nu)$ on its linear part starting at $(0,0)$. In the case $q_g(\nu) = q_\tau(\nu) = 1$, it is only known for $h \in [0, H_g(\nu)]$ and at $H_\tau(\nu)$ (which may be strictly greater than $H_g(\nu)$). Nevertheless, when $q_g(\nu) = 1$, item *4* of Theorem 1.3 gives a sufficient condition to have $H_\tau(\nu) = H_g(\nu)$, and the large deviation spectrum increases along a straight line with slope 1 until it reaches $H_\tau(\nu)$. This condition imposes a homogeneous repartition of the Dirac masses of the same intensity.

• The reader should keep in mind that no comparable result can hold if the large deviation spectrum is replaced by the Hausdorff multifractal spectrum.

• For the examples of homogeneous and heterogeneous sums of discontinuous measures studied in [1, 20, 13, 31, 30, 4, 5, 8], it is not difficult to verify, thanks to item *4* of Theorem 1.3, that when $H_\tau(\nu) > 0$, $H_g(\nu) = H_\tau(\nu)$. For instance, the derivative ν_β of a stable Lévy subordinator L_β of index $\beta \in (0,1)$ satisfies $H_g(\nu) = H_\tau(\nu) = 1/\beta$ and $q_\tau(\nu_\beta) = \beta$ (see [20]).

• Finally, in the companion paper [6], we illustrate the important role played by the information parameters $(q_\tau(\nu), H_\tau(\nu))$ for the Hausdorff spectrum. An interesting example is provided by a class of discontinuous measures ν_b introduced in [6], whose atoms are located at b-adic numbers of $[0,1]$. For such measures ν_b, $H_g(\nu_b) = H_\tau(\nu_b)$ even when $q_\tau(\nu_b) = 1$. Inspired by Proposition 3.3 of the present paper, a natural procedure is to apply a threshold to ν_b by keeping only the masses which contribute to the fact that $f_{\nu_b}(H_\tau(\nu_b)) = q_\tau(\nu_b)H_\tau(\nu_b)$. This yields a second measure ν_b^t. It is shown in [4] that ν_b and ν_b^t have the same multifractal properties on $[0, H_\tau(\nu_b)]$ in the sense that their Hausdorff, large deviation and Legendre spectra coincide. Moreover, when $q_\tau(\nu_b) = 1$, these spectra also coincide at the exponents $h > H_\tau(\nu_b)$. This striking result confirms that valuable information on the local behavior of ν are stored in the masses which are detected by $(q_\tau(\nu), H_\tau(\nu))$.

2 Universal Bounds for the Large Deviation Spectrum.

For $j \geq 1$ and $x \in (0,1)^d$, $Q_j(x)$ is the unique b-adic cube of scale $j \geq 1$ containing x, and for every $\eta \in \{-1,0,1\}^d$, $Q_j^{(\eta)}(x) = Q_j(x) + b^{-j}\eta$. In the following, $|B|$ always denotes the diameter of the set B. Eventually, for the rest of the paper, the convention $\log(0) = -\infty$ is adopted.

2.1 Links Between $(q_\tau(\nu), H_\tau(\nu))$ and the Large Deviation Spectrum.

The next result goes slightly beyond item *1* of Proposition 1.1 and it also resumes some comments made at the end of Section 1.1.

Theorem 2.1. *Let ν be a finite Borel measure on $[0,1]^d$. Suppose that τ_ν is continuous at 0^+ and $\dim(\mathrm{Supp}(\nu)) > 0$. Set $H_\tau^+(\nu) = \tau_\nu'(q_\tau(\nu)^+)$. We have:*

1. *$q_\tau(\nu) > 0$.*
2. *For every $h \geq 0$, $d_\nu(h) \leq f_\nu(h) \leq \tau_\nu^*(h)$. Moreover, $\tau_\nu^*(h) = q_\tau(\nu)h$ if $h \in [\tau_\nu'(q_\tau(\nu)^+), H_\tau(\nu)]$ and $\tau_\nu^*(h) < q_\tau(\nu)h$ otherwise.*
3. *$H_\tau^+(\nu) = \min\{h \geq 0 : f_\nu(h) = q_\tau(\nu)h\} = \min\{h \geq 0 : \tau_\nu^*(h) = q_\tau(\nu)h\}$.*
4. *$H_\tau(\nu) = \max\{h \geq 0 : f_\nu(h) = q_\tau(\nu)h\} = \max\{h \geq 0 : \tau_\nu^*(h) = q_\tau(\nu)h\}$.*
5. *If $E_0^\nu \neq \emptyset$, then $\tau_\nu'(q_\tau(\nu)^+) = 0$.*

Remark 2.2. *Notice that, if $E_0^\nu \neq \emptyset$ and $q_\tau(\nu) > 0$, then for every $h \in [0, H_\tau(\nu)]$, $\tau_\nu^*(h) = q_\tau h$, while this may not be the case for f_ν. There may exist $0 < h < H_\tau(\nu)$ such that $f_\nu(h) < q_\tau h$ (see for instance Theorem 1.3, item 3.(b)).*

Remark 2.3. *Let $\widehat{E}_h^\nu = \left\{x : \lim_{j\to\infty} \frac{\log \nu(Q_j(x))}{-j \log b} = h\right\}$. In item 2 of Theorem 2.1, the inequality $d_\nu(h) \leq f_\nu(h)$ is a refinement of the well-known inequality $\dim \widehat{E}_h^\nu \leq f_\nu(h)$ [10, 29]. For classes of continuous measures possessing some self-similarity property, one often has $\dim \widehat{E}_h^\nu = f_\nu(h)$ for all h such that $f_\nu(h) \geq 0$.*

We include the inequality $d_\nu(h) \leq f_\nu(h)$ in the statement because for purely discontinuous measures studied in [31, 20, 13, 4], or for the derivative of a generic increasing continuous function [11], the set $\widehat{E}_h^\nu$ is empty for $h \in (0, H_\tau(\nu))$, while $\dim E_h^\nu = \tau_\nu^*(h)$ and thus $d_\nu(h) = f_\nu(h)$. This emphasizes the fact that, in general, sets like E_h^ν or $\left\{x : \liminf_{j\to\infty} \frac{\log \nu(Q_j(x))}{-j \log b} = h\right\}$ must be used rather than $\widehat{E}_h^\nu$ to describe the local behavior of ν. Then $f_\nu(h)$ provides a convenient upper bound estimate for $\dim E_h^\nu$ rather than for $\dim \widehat{E}_h^\nu$.

Proof. We simply write $(q_\tau, H_\tau) = (q_\tau(\nu), H_\tau(\nu))$.

1. Assume that $\dim(\mathrm{Supp}(\nu)) > 0$ and τ_ν is continuous at 0^+. We clearly see that $\dim \mathrm{Supp}(\nu) \leq -\tau_\nu(0)$, thus $0 < -\tau_\nu(0)$. The continuity and monotonicity of τ_ν at 0^+ and the fact that $\tau_\nu(1)$ always equals 0 yield the result.

2. We shall need the following simple lemma, whose proof is easy and left to the reader.

Lemma 2.4. *Let ν be a positive Borel measure on $[0,1]^d$. For every $x \in (0,1)^d$, we have*

$$h_\nu(x) = \min_{\eta\in\{-1,0,1\}^d} h_\nu^{(\eta)}(x), \quad \textit{where} \quad h_\nu^{(\eta)}(x) = \liminf_{j\to\infty} \frac{\log \nu(Q_j^{(\eta)}(x))}{-j\log b}.$$

For every $h \geq 0$, the fact that $f_\nu(h) \leq \tau_\nu^*(h) \leq q_\tau h$ is a well-known property already mentioned in the introduction, and the comparison between $\tau_\nu^*(h)$ and $q_\tau(\nu)h$ follows from the definition of τ_ν^*. We only prove the inequality $d_\nu(h) \leq f_\nu(h)$ for $h \geq 0$ such that $d_\nu(h) > 0$. Indeed, the proof of this inequality (using the liminf in the definition of the exponent), though very fast to obtain, has never been written entirely, according to our best knowledge.

Let $\varepsilon > 0$. By definition of $f_\nu(h)$, for every n large enough, $\#\mathcal{S}_n^\nu(h,\varepsilon) \leq b^{n(f_\nu(h)+\varepsilon)}$ (see (5)). Let us consider the limsup set

$$K_\nu(h,\varepsilon) = \bigcap_{N\geq 1} \bigcup_{n\geq N} \bigcup_{Q\in\mathcal{S}_n^\nu(h,\varepsilon)} \bigcup_{\eta\in\{-1,0,1\}^d} Q^{(\eta)}.$$

(Recall that when Q is a b-adic cube of generation n, $Q^{(\eta)} = Q+\eta b^{-j}$.) Let now $x \in E_h^\nu$. By Lemma 2.4, there exists a sequence $(n_j^x)_{j\geq 1}$ such that for every $j \geq 1$, there is $\eta \in \{-1,0,1\}^d$ such that $b^{-n(h+\varepsilon)} \leq \nu(Q_{n_j^x}^{(\eta)}(x)) \leq b^{-n(h-\varepsilon)}$. As a consequence, $x \in K_\nu(h,\varepsilon)$, thus $E_h^\nu \subset K_\nu(h,\varepsilon)$.

We now find an upper bound for $\dim K_\nu(h,\varepsilon)$. Let $t > f_\nu(h)+\varepsilon$, and let us estimate the t-Hausdorff measure of the set $K_\nu(h,\varepsilon)$. For $N \geq 1$, the union $\bigcup_{n\geq N} \bigcup_{Q\in\mathcal{S}_n^\nu(h,\varepsilon)} \bigcup_{\eta\in\{-1,0,1\}^d} Q^{(\eta)}$ forms a covering of $K_\nu(h,\varepsilon)$. Then,

$$\begin{aligned}\sum_{n\geq N} \sum_{Q\in\mathcal{S}_n^\nu(h,\varepsilon)} \sum_{\eta\in\{-1,0,1\}^d} |Q^{(\eta)}|^t &\leq \sum_{n\geq N} \sum_{Q\in\mathcal{S}_n^\nu(h,\varepsilon)} 3^d b^{-nt} \\ &\leq 3^d \sum_{n\geq N} b^{-nt} b^{n(f_\nu(h)+\varepsilon)}.\end{aligned}$$

This last sum converges, since $t > f_\nu(h)+\varepsilon$, and its value goes to zero when N goes to infinity. As a consequence, the t-Hausdorff measure of $K_\nu(h,\varepsilon)$ equals zero, and $d_\nu(h) \leq \dim K_\nu(h,\varepsilon) \leq f_\nu(h)+\varepsilon$. Letting $\varepsilon \to 0$ yields the result.

3. and 4. As we already noticed, these properties are consequences of Proposition 1.1 and the definition of the Legendre transform.

5. We always have $\tau_\nu(q_\tau) = 0$. If $q_\tau < 1$, since τ_ν is concave and positive when $q \geq 1$, then $\tau_\nu(q) = 0$ for every $q \geq q_\tau$, and in particular it is zero at q_τ^+.

Assume that $q_\tau = 1$. Let $x \in E_0^\nu$. Then fix $\varepsilon > 0$ and $(r_j)_{j\geq 1}$ a sequence decreasing to 0 such that $\nu(B(x,r_j)) \geq (r_j)^\varepsilon$ for all $j \geq 1$. Let n_j be the unique integer such that $b^{-n_j} \leq 2r_j \leq b^{-n_j+1}$. One of the (at most) b^d b-adic cubes of generation n_j intersecting $B(x,r_j)$, say Q_j, satisfies $\nu(Q_j) \geq r_j^\varepsilon/b^d \geq b^{-n_j\varepsilon}b^{-d}$. Let $q > 1$. Remembering (2), we obtain $s_{n_j}(q) \geq b^{-n_j\varepsilon}b^{-d}$. Taking the liminf yields $\tau_\nu(q) \leq \varepsilon$, which holds $\forall q \geq 1$ and $\varepsilon > 0$. Hence the result. $\square$

2.2 Additional Definitions and Large Deviation Bounds.

Let ν be a positive Borel measure on $[0,1]^d$. Before establishing Theorem 1.3, some definitions and estimates for f_ν and related quantities are needed. For $x \in (0,1)^d$, recall the definitions (3) and (4) of the Hölder exponent at x and of the corresponding level sets E_h^ν, for any $h \geq 0$.

Definition 2.5. Let ν be a positive Borel measure on $[0,1]^d$. For $h \geq 0$ and $n \geq 1$, let us introduce the quantities

$$\underline{N}_\nu(h,n) = \#\left\{Q \in \mathcal{G}_n : b^{-nh} \leq \nu(Q)\right\},\ \underline{f}_\nu(h) = \limsup_{n\to\infty} \ n^{-1}\log_b \underline{N}_\nu(h,n).$$

Hence, $\underline{f}_\nu(h)$ is related to the asymptotic number of b-adic cubes Q of generation n such $\nu(Q) \geq b^{-nh}$.

Using [10] and the definition of the Legendre transform, we deduce the following useful properties (some of them were recalled in Proposition 1.1).

Proposition 2.6. *Let ν be a positive Borel measure on $[0,1]^d$ with $q_\tau(\nu) > 0$.*

1. *For every exponent $h > \tau_\nu'(0^+)$, $f_\nu(h) \leq \underline{f}_\nu(h) \leq \tau_\nu^*(h)$ and $\underline{f}_\nu(h) \leq q_\tau(\nu)\tau_\nu'(0^+) < q_\tau(\nu)h$.*
2. *If $h \in [0,\tau_\nu'(0^+)]$, then $f_\nu(h) \leq \underline{f}_\nu(h) \leq \tau_\nu^*(h) \leq q_\tau(\nu)h$.*
   Moreover, if $\underline{f}_\nu(h) = q_\tau(\nu)h$, then $f_\nu(h) = q_\tau(\nu)h$.

3 Theorem 1.3(1-2): Characterization of $(q_\tau(\nu), H_\tau(\nu))$.

Let $\widetilde{m} = (m_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{k\in\mathbb{N}} m_k < \infty$ and $\widetilde{x} = (x_k)_{k\in\mathbb{N}} \in \left([0,1]^d\right)^{\mathbb{N}}$ a sequence of pairwise distinct points, and consider the purely discontinuous measure ν defined by (6).

Let us begin with a proposition relating the quantities $q_g(\nu)$ and $H_g(\nu)$.

Proposition 3.1. *If $q_g(\nu) > 0$, then $0 \leq H_g(\nu) \leq d/q_g(\nu)$.*

PROOF. By definition, there exists an increasing sequence of integers $(n_j)_{j\geq 1}$ and a non-increasing positive sequence going down to 0 $(\alpha_j)_{j\geq 1}$ such that

$$\lim_{j\to+\infty} \frac{\log_b \# X_{n_j}(\alpha_j)}{n_j} = q_g(\nu) \text{ and } \lim_{j\to+\infty} \frac{n_j}{\mathcal{J}(n_j,\alpha_j)} = H_g(\nu).$$

Let $\varepsilon > 0$, small enough so that $q_g(\nu) - 2\varepsilon > 0$. There is $j_\varepsilon \geq 0$ such that $j \geq j_\varepsilon$ implies $|\frac{\log_b \# X_{n_j}(\alpha_j)}{n_j} - q_g(\nu)| \leq \varepsilon$ and $|n_j/\mathcal{J}(n_j,\alpha_j) - H_g(\nu)| \leq \varepsilon$. At scale n_j (where $j \geq j_\varepsilon$), we have

$$b^{n_j(q_g(\nu)-\varepsilon)} \leq \# X_{n_j}(\alpha_j) \leq b^{n_j(q_g(\nu)+\varepsilon)}.$$

Let $n \leq n_j((q_g(\nu)-2\varepsilon)/d)$. The cardinality of $\mathcal{G}_n$ is $b^{nd} \leq b^{n_j(q_g(\nu)-2\varepsilon)}$. Hence, at least one b-adic cube of $\mathcal{G}_n$ contains two points of $X_{n_j}(\alpha_j)$. Consequently, we get $\mathcal{J}(n_j,\alpha_j) \geq n_j((q_g(\nu)-2\varepsilon)/d)$, and $n_j/\mathcal{J}(n_j,\alpha_j) \leq d/(q_g(\nu)-2\varepsilon)$. By letting j go to infinity, we finally obtain that $H_g(\nu) \leq d/(q_g(\nu)-4\varepsilon)$, and the result follows by letting ε go to zero. □

3.1 An Intrinsic Characterization of $q_\tau(\nu)$.

Theorem 3.2. *Let $\widetilde{m} = (m_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{k\in\mathbb{N}} m_k < \infty$ and $\widetilde{x} = (x_k)_{k\in\mathbb{N}}$ a sequence of pairwise distinct points in $[0,1]^d$. Let $\nu = \sum_{k\in\mathbb{N}} m_k\ \delta_{x_k}$. Then:*

1. *$q_\tau(\nu) \leq q_g(\nu) \leq 1$.*
2. *If **(H)** is satisfied by ν, then $q_g(\nu) = q_\tau(\nu)$.*

PROOF. **1.** We indeed have $q_g(\nu) \leq 1$, because $\widetilde{m}$ is summable. If $q_g(\nu) = 1$, then $q_\tau(\nu) \leq 1 = q_g(\nu)$ and the result is proved. We thus assume that $q_g(\nu) < 1$, and we prove that $q_\tau(\nu) \leq q_g(\nu)$.

Let $\varepsilon > 0$ be such that $q_g(\nu) + 2\varepsilon \leq 1$ and $n_\varepsilon \geq 1$ such that $\# K_n \leq b^{n(q_g(\nu)+\varepsilon)}$ for $n \geq n_\varepsilon$. Then, using the sub-additivity of the mapping $t \mapsto t^{q_g(\nu)+2\varepsilon}$ on $\mathbb{R}_+$, we see that for all $n \geq 1$

$$\begin{aligned}
s_n(q_g(\nu)+2\varepsilon) &\leq \sum_{k\in\mathbb{N}} m_k^{q_g(\nu)+2\varepsilon} \\
&\leq \sum_{1\leq n'<n_\varepsilon} \sum_{k\in K_{n'}} m_k^{q_g(\nu)+2\varepsilon} + \sum_{n'\geq n_\varepsilon} \sum_{k\in K_{n'}} m_k^{q_g(\nu)+2\varepsilon} \\
&\leq \sum_{1\leq n'<n_\varepsilon} \sum_{k\in K_{n'}} m_k^{q_g(\nu)+2\varepsilon} + \sum_{n'\geq n_\varepsilon} (\# K_{n'}) b^{-(n'-1)(q_g(\nu)+2\varepsilon)} \\
&\leq \sum_{1\leq n'<n_\varepsilon} \sum_{k\in K_{n'}} m_k^{q_g(\nu)+2\varepsilon} + \sum_{n'\geq n_\varepsilon} b^{n'(q_g(\nu)+\varepsilon)} b^{-(n'-1)(q_g(\nu)+2\varepsilon)}.
\end{aligned}$$

The first term of the right hand side of the last inequality does not depend on n, and the second one converges since it is upper bounded by the sum $b^{q_g(\nu)+\varepsilon}\sum_{n'\geq n_\varepsilon} b^{-n'\varepsilon}$. Hence, $s_n(q_g(\nu)+2\varepsilon)$ is bounded independently of n. This yields $\tau_\nu(q_g(\nu)+2\varepsilon)=0$, and so $q_\tau(\nu)\leq q_g(\nu)+2\varepsilon$. This is true for all $\varepsilon>0$; hence the result.

2. It is enough to prove that $q_g(\nu)\leq q_\tau(\nu)$. The fact that this inequality holds when **(H)** is satisfied follows from the proof of Proposition 3.3. We prove it here under the stronger assumption that $q_g(\nu)>0$ and $\underline{H}_g(\nu)>0$. (This exponent is defined by (7).)

Let $\varepsilon\in(0,q_g(\nu))$, and let $(n_j)_{j\geq1}$ be an increasing sequence of integers converging to ∞, and let $(\varepsilon_j)_{j\geq1}$ be a positive sequence converging to 0 such that $\forall j\geq 1$, $b^{n_j(q_g(\nu)-\varepsilon_j)}\leq \#K_{n_j}$. For every $j\geq1$, recall that $p_j=\mathcal{J}(n_j)$ is the first scale which separates the elements of K_{n_j}. When j is large enough so that $\varepsilon_j\leq\varepsilon/4$, we have

$$
\begin{aligned}
s_{p_j}(q_g(\nu)-\varepsilon) &\geq \sum_{k\in K_{n_j}} m_k^{q_g(\nu)-\varepsilon}\\
&\geq (\#K_{n_j})b^{-n_j(q_g(\nu)-\varepsilon)} \geq b^{n_j(q_g(\nu)-\varepsilon_j)}b^{-n_j(q_g(\nu)-\varepsilon)},
\end{aligned}
$$

which equals $b^{n_j(\varepsilon-\varepsilon_j)}$. Thus, $-\log(s_{p_j}(q_g(\nu)-\varepsilon))/p_j\leq-(\varepsilon-\varepsilon_j)n_j/p_j$. By assumption, $\liminf_{j\to+\infty} n_j/p_j\geq \underline{H}_g(\nu)>0$. Taking the liminf yields $\tau_\nu(q_g(\nu)-\varepsilon)\leq-\varepsilon\underline{H}_g(\nu)<0$. Hence, $q_\tau(\nu)>q_g(\nu)-\varepsilon$, for every $\varepsilon>0$. □

3.2 Preliminary Work for the Large Deviation Spectrum.

Proposition 3.3. *Let $\widetilde{m}=(m_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{k\in\mathbb{N}}m_k<1$ and $\widetilde{x}=(x_k)_{k\in\mathbb{N}}$ a sequence of points in $[0,1]^d$. Let $\nu=\sum_{k\in\mathbb{N}}m_k\,\delta_{x_k}$. Suppose that* **(H)** *is satisfied (this implies that $q_\tau(\nu)=q_g(\nu)$). Let $h_0=H_\tau(\nu)$ if $q_\tau(\nu)\in(0,1)$ and $h_0=H_g(\nu)$ if $q_\tau(\nu)=1$. There exist a sequence $(p_j)_{j\geq1}$ of integers going to ∞, a positive sequence $(\varepsilon_j)_{j\geq1}$ going to 0 , and a sequence of sets of b-adic boxes $(B_j)_{j\geq1}$ such that*

1. $\lim_{j\to+\infty}\frac{\log_b \#\mathcal{S}^\nu_{p_j}(h_0,\varepsilon_j)}{p_j}=q_\tau(\nu)h_0$;
2. *For every $j\geq1$, $B_j\subset\mathcal{S}^\nu_{p_j}(h_0,\varepsilon_j)$;*
3. $\lim_{j\to\infty}\frac{\log_b(\#B_j)}{p_j}=q_\tau(\nu)h_0$;
4. *For every $Q\in B_j$, there exists $k\in\mathbb{N}$ such that if $x_Q:=x_k\in Q$, $\lim_{j\to\infty}\sup_{Q\in B_j}\left|\frac{\log_b m_Q}{-p_j}-h_0\right|=0$, where $m_Q=m_k$ if $x_Q=x_k$.*

Proposition 3.3 plays a crucial role in proving Theorem 1.3. It asserts that, when $q_\tau(\nu)<1$, there exists a sequence ε_j going to 0 and infinitely many integers j such that

- $\mathcal{S}_j^\nu(H_\tau(\nu),\varepsilon_j)\approx b^{jq_\tau(\nu)H_\tau(\nu)}$,
- a substantial proportion of the cubes Q of generation j such that $\nu(Q)\approx b^{-jH_\tau(\nu)}$ contains a point x_k such that its associated mass satisfies $m_k\approx b^{-jH_\tau(\nu)}$ (i.e., the ν-mass of these cubes is essentially concentrated on one single Dirac mass).

The same general fact holds true only with $H_g(\nu)$ instead of $H_\tau(\nu)$ if $q_\tau(\nu)=1$.

PROOF. **(i)** We first prove that the result holds true with $h_0=H_g(\nu)$ and with $q_g(\nu)$ instead of $q_\tau(\nu)$ (whatever the value of $q_\tau(\nu)$ is). By definition of $H_g(\nu)$, there is an increasing sequence of integers $(n_j)_{j\geq 1}$ and a positive sequence $(\alpha_j)_{j\geq 1}$ going to 0 such that $\lim_{j\to+\infty}\frac{\log_b \# X_{n_j}(\alpha_j)}{n_j}=q_g(\nu)$ and simultaneously $\lim_{j\to+\infty}\frac{n_j}{\mathcal{J}(n_j,\alpha_j)}=H_g(\nu)$. Let $\varepsilon_0\in\big(0,\min\big(q_g(\nu),H_g(\nu)\big)\big)$, and consider four positive real numbers $\varepsilon_1,\varepsilon_2,\varepsilon_3$ and ε_4 that all are in $(0,\varepsilon_0)$. For some integer j_1 large enough, we have:

- for every $j\geq j_1$,
$$\left|\frac{\log_b \# X_{n_j}(\alpha_j)}{n_j}-q_g(\nu)\right|\leq\varepsilon_1 \text{ and } \left|\frac{n_j}{\mathcal{J}(n_j,\alpha_j)}-H_g(\nu)\right|\leq\varepsilon_2,$$
- for every $n\geq n_{j_1}$,
$$\frac{\log_b \#\underline{N}_\nu(H_g(\nu)-\varepsilon_4,n)}{n}\leq q_\tau(\nu)(H_g(\nu)-\varepsilon_4)+\varepsilon_3.$$

The second point holds true due to items *1* and *2* of Proposition 2.6.

Let $p_1=\mathcal{J}(n_{j_1})$. Consider $\mathcal{G}_{p_1}$. By construction of p_1, there are $\# X_{n_{j_1}}(\alpha_{j_1})$ b-adic boxes Q of $\mathcal{G}_{p_1}$ that contain a point x_Q such that $x_Q=x_k$ for some $x_k\in X_{n_{j_1}}(\alpha_{j_1})$ (with the associated mass denoted m_Q). Each of these b-adic boxes Q satisfies $\nu(Q)\geq m_Q\geq b^{-n_{j_1}}$, which is greater than $b^{-p_1(H_g(\nu)+\varepsilon_2)}$. We can also write that $\# X_{n_{j_1}}(\alpha_{j_1})\geq b^{n_{j_1}(q_\tau(\nu)-\varepsilon_1)}\geq b^{p_1(H_g(\nu)-\varepsilon_2)(q_g(\nu)-\varepsilon_1)}=b^{p_1(H_g(\nu)q_g(\nu)-\varepsilon_5)}$, with $\varepsilon_5=|\varepsilon_1\varepsilon_2-\varepsilon_1 H_g(\nu)-\varepsilon_2 q_g(\nu)|$. By the second assumption above on j_1, we know that the cardinality of the set of cubes $Q\in\mathcal{G}_{p_1}$ such that $\nu(Q)\geq b^{-p_1(H_g(\nu)-\varepsilon_4)}$ is less than $b^{p_1(q_\tau(\nu)H_g(\nu)-\varepsilon_6)}$, with $\varepsilon_6=\varepsilon_4 q_\tau(\nu)-\varepsilon_3$.

The reader easily verifies that $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$ can be chosen in $(0,\varepsilon_0)^4$ so that $0\leq\varepsilon_5<\varepsilon_6$. In this case, since $q_g(\nu)\geq q_\tau(\nu)$, there exists $\eta_1>0$ such that there are at least

$$b^{p_1(q_g(\nu)H_g(\nu)-\varepsilon_5)}-b^{p_1(q_\tau(\nu)H_g(\nu)-\varepsilon_6)}=b^{p_1(q_g(\nu)H_g(\nu)-\eta_1)}$$

b-adic boxes of generation p_1 such that $b^{-p_1(H_g(\nu)+\varepsilon_2)} \leq \nu(Q) \leq b^{-p_1(H_g(\nu)-\varepsilon_4)}$. Let us set $\gamma_1 = \max(\varepsilon_2, \varepsilon_4) \leq \varepsilon_0$. We proved that there is a set of b-adic boxes B_{p_1}, of cardinality $b^{p_1(H_g(\nu)q_g(\nu)-\eta_1)}$ such that for every $Q \in B_{p_1}$,

- Q is also included in $S_{p_1}(H_g(\nu), \gamma_1)$,
- there is a x_Q in Q such that $x_Q = x_k$ for some $k \in \mathbb{N}$ such that k verifies $\left|\frac{\log_b m_k}{p_1} - H_g(\nu)\right| \leq \gamma_1$.

It is obvious that there exists a constant C depending only on $q_g(\nu)$ and $H_g(\nu)$ such that η_1 can be chosen so that $0 \leq \max(\gamma_1, \eta_1) \leq C\varepsilon_0$. We then construct the sequence $(p_j)_{j\geq 1}$ by induction, by iterating the same procedure at each step $i \geq 1$ (where the construction at step i is achieved using $\varepsilon_i = \dfrac{\min(\varepsilon_{i-1}, \eta_{i-1})}{2C}$ instead of ε_0).

(ii) When $q_\tau(\nu) = 1$, $q_g(\nu) = q_\tau(\nu) = 1$ and the result is proved.

(iii) Assume now that $q_\tau(\nu) \in (0,1)$.

By item 1 of Theorem 3.2, $q_g(\nu) \geq q_\tau(\nu)$. But the computations above (essentially the item 1 of Proposition 3.3 obtained with $q_g(\nu)$ instead of $q_\tau(\nu)$) yield that $f_\nu(H_g(\nu)) \geq q_g(\nu)H_g(\nu)$. Remembering that Proposition 2.6 implies that $f_\nu(h) \leq q_\tau(\nu)h$ for every $h \geq 0$, we get $q_g(\nu) \leq q_\tau(\nu)$, and thus the equality $q_g(\nu) = q_\tau(\nu)$.

(iv) We now consider the case when $q_\tau(\nu) \in (0,1)$ and $h_0 = H_\tau(\nu)$. Proposition 3.3 is a consequence of the following lemma.

Lemma 3.4. *Suppose that $q_\tau(\nu) = q_g(\nu) = q_\tau \in (0,1)$ and $H_\tau(\nu) = H_\tau > 0$. Let $C_2 > \frac{2q_\tau}{1-q_\tau}$ and $\widetilde{C}_2 = C_2(1-q_\tau) - q_\tau$. If $\varepsilon > 0$ is small enough, then there exist a sequence $(p_j)_{j\geq 1}$ going to ∞, a sequence of sets of b-adic boxes $(B_j)_{j\geq 1}$ and a constant $C(\varepsilon) \in (0, \widetilde{C}_2)$ such that for all $j \geq 1$:*

1. *$B_j \subset \mathcal{S}^\nu_{p_j}(H_\tau, C(\varepsilon)\varepsilon)$ and $\# B_j \geq b^{p_j q_\tau H_\tau(1-\varepsilon)}$.*
2. *For every $Q \in B_j$, there exists $k \geq 1$ such that $x_k \in B_j$ and $b^{-p_j H_\tau(1+C_2\varepsilon)} \leq m_k \leq b^{-p_j H_\tau(1-C(\varepsilon)\varepsilon)}$.*

Proof. Recall item *5* of Theorem 2.1. Fix $\varepsilon \in (0,1)$. Then let $\eta_0 > 0$ such that for all $\eta \in (0, \eta_0)$, there exists a sequence $(p_j)_{j\geq 1}$ going to infinity such that

$$\text{for all } j \geq 1, \ \ \#\mathcal{S}^\nu_{p_j}(H_\tau, \eta H_\tau) \geq b^{p_j q_\tau(\nu) H_\tau(1-\varepsilon/2)}. \tag{9}$$

Fix now $\eta \in \big(0, \min\big(\eta_0, \widetilde{C}_2\varepsilon\big)\big)$ and write $\eta = C(\varepsilon)\varepsilon$. Let $N_0 > 0$ such that $q(n) \leq q_\tau + \varepsilon^2$ for all $n \geq N_0$, and J_1 an integer such that $p_j H_\tau(1+C_2\varepsilon) \geq N_0$ for $j \geq J_1$. It is easily seen that there is $M > 0$ independent of ε and η such that

$$\forall\ j \geq J_1, \quad \sum_{n > p_j H_\tau(\nu)(1+C_2\varepsilon)} \ \sum_{k \in K_n} m_k \leq M b^{p_j H_\tau (1+C_2\varepsilon)(1-q_\tau - \varepsilon^2)}. \tag{10}$$

Let $C_3 \in (\widetilde{C}(\varepsilon), \widetilde{C}_2)$ and let

$$R_j(C_3) = \Big\{ Q \in \mathcal{S}^\nu_{p_j}(H_\tau, \eta H_\tau) : \sum_{n > p_j H_\tau(1+C_2\varepsilon)} \ \sum_{k \in K_n :\ x_k \in Q} m_k \geq b^{-p_j H_\tau (1+C_3\varepsilon)} \Big\}.$$

Also let $B_j = \mathcal{S}^\nu_{p_j}(H_\tau, \eta H_\tau) \setminus R_j(C_3)$. By construction, each element Q of B_j must contain a point x_k such that $b^{-p_j H_\tau (1+C_2\varepsilon)} \leq m_k \leq b^{-p_j H_\tau (1-C(\varepsilon)\varepsilon)}$. Moreover, due to (10), we obtain

$$\forall\ j \geq J_1,\ \#R_j(C_3) \leq M b^{p_j H_\tau q_\tau \left[1+[C_2+(C_3-C_2)/q_\tau]\varepsilon + O(\varepsilon^2)\right]}.$$

It follows from our choice for C_2, η and C_3 that if $\varepsilon > 0$ is small enough, then

$$\forall\ j \geq J_1,\ \#R_j(C_3) \leq M b^{p_j H_\tau q_\tau (1-\varepsilon)}.$$

This yields $\#B_j \geq b^{p_j q_\tau H_\tau (1-\varepsilon)}$ for j large enough because of (9). □

3.3 Theorem 1.3(1-2): Characterization of $H_\tau(\nu)$ When $q_\tau(\nu) < 1$ and Proof of the Linear Shape of the Large Deviation Spectrum.

Proposition 3.3 yields $q_g(\nu) = q_\tau(\nu) = q_\tau$. Let h_0 be as in Proposition 3.3. A straightforward consequence of Proposition 3.3 is that $\underline{f}_\nu(h_0+\varepsilon) \geq q_\tau h_0$ for all $\varepsilon > 0$. Using item *2* of Proposition 2.6, we get $f_\nu(h_0) = q_\tau h_0$ and then $h_0 \leq H_\tau(\nu)$ by item *4* of Theorem 2.1. Let us show first that $f_\nu(h) = q_\tau h$ for every $h \in [0, H_g(\nu)]$, and then that $H_g(\nu) = H_\tau(\nu)$ when $q_\tau < 1$.

• Let $h \in (0, h_0]$. Consider three sequences $(p_j)_{j\geq 1}$, $(\varepsilon_j)_{j\geq 1}$ and $(B_j)_{j\geq 1}$ as in Proposition 3.3. Let $m_j = [p_j h_0/h]$. For every $Q \in B_j$, there exists a unique b-adic box Q' of generation m_j containing x_Q. Let $\varepsilon \in (0,h)$. By construction, $\nu(Q') \geq m_Q \geq b^{-m_j(h+\varepsilon)}$ for j large enough, and $\underline{N}(h+\varepsilon, m_j) \geq \#B_j$. Moreover, $\lim_{j\to\infty} \log_b \#B_j/m_j = q_\tau(\nu)h$. So, for all $\eta > 0$ if j is large enough, $\underline{N}(h+\varepsilon, m_j) \geq b^{m_j(q_\tau(\nu)h - \eta)}$. This implies, by letting $j \to +\infty$, and then allowing ε and η go to zero, that $\underline{f}_\nu(h) \geq q_\tau(\nu)h$. By item *2* of Theorem 2.1, this yields $f_\nu(h) = q_\tau(\nu)h$.

• It remains to show that $H_g(\nu) = H_\tau(\nu)$ when $q_\tau(\nu) \in (0,1)$. We adopt the notation of Proposition 3.3. Let $(\varepsilon_j)_{j\geq 1}$ be a sequence of positive real numbers going to 0 and such that $\sup_{Q\in B_j}\left|\frac{\log_b m_Q}{-p_j} - H_\tau(\nu)\right| \le \varepsilon_j$ for all j. Let $(\alpha_j)_{j\ge 1}$ be another sequence of positive real numbers going to 0 such that for j large enough

$$\#B_j \ge b\cdot b^{p_j(H_\tau(\nu)+\varepsilon_j)} b^{-p_j(H_\tau(\nu)-\varepsilon_j)(1-q_\tau(\nu)-\alpha_j)}. \tag{11}$$

Such a choice is possible thanks to Proposition 3.3. Now suppose that $\forall n \in [p_j(H_\tau(\nu)-\varepsilon_j), p_j(H_\tau(\nu)+\varepsilon_j)]$, $\#K_n \cap B_j < b^{n(q_\tau(\nu)-\alpha_j)}$. This yields

$$\begin{aligned}\sum_{Q\in B_j} m_Q &= \sum_{n\in[p_j(H_\tau(\nu)-\varepsilon_j),p_j(H_\tau(\nu)+\varepsilon_j)]} \;\sum_{Q\in K_n\cap B_j} m_Q \\ &\le \sum_{n\in[p_j(H_\tau(\nu)-\varepsilon_j),p_j(H_\tau(\nu)+\varepsilon_j)]} b^{-(n-1)}\#(K_n\cap B_j) \\ &< b\cdot b^{-p_j(H_\tau(\nu)-\varepsilon_j)\left(1-q_\tau(\nu)-\alpha_j\right)}.\end{aligned} \tag{12}$$

On the other hand, $\sum_{Q\in B_j} m_Q \ge b^{-p_j(H_\tau(\nu)+\varepsilon_j)}\#B_j$. Due to (11) and the strict inequality in (12), there is a contradiction. Hence, there is an integer $n \in [p_j(H_\tau(\nu)-\varepsilon_j), p_j(H_\tau(\nu)+\varepsilon_j)]$ such that $\#K_n \cap B_j \ge b^{n(q_\tau(\nu)-\alpha_j)}$. Moreover, by construction, $\mathcal{J}(n,\alpha_n) \le p_j \le n/(H_\tau(\nu)-\varepsilon_j)$. This implies $H_g(\nu) \ge H_\tau(\nu)$. We saw that $H_g(\nu) \le H_\tau(\nu)$ (since $f_\nu(H_g(\nu) = q_\tau(\nu)H_g(\nu))$, hence the equality is true.

4 Theorem 1.3, Item *3*: Sharpness of $H_g(\nu)$ When $q_\tau(\nu) = 1$.

If $q_\tau(\nu) = q_g(\nu) = 1$ and **(H)** holds, then by Theorem 2.1, $f_\nu(H_\tau(\nu)) = H_\tau(\nu)$. By the work achieved in the previous section, we also have $f_\nu(h) = h$ for every $h \in [0, H_g(\nu)]$. Hence, it is natural to ask whether the large deviation spectrum is still linear for $H_g(\nu) < h < H_\tau(\nu)$. The answer is negative (as stated by item *3* of Theorem 1.3). The optimality of $H_g(\nu)$ in item *2* of Theorem 1.3 is a consequence of the examples below. These examples depend on Propositions 4.1 and 4.2 whose long and technical proofs are available in [7]. However, to give the reader a flavor of the proof, we give in Section 5 a one-dimensional measure for which $1/3 = H_g(\nu) < H_\tau(\nu) = 1/2$ and $f_\nu(h) = h$ for all $h \in [0, H_\tau(\nu)]$.

4.1 Scheme of the General Construction for Theorem 1.3, Item *3(a)*.

Let $0 < h_0 < h_1 \le d$. In the sequel, when $\rho \in (0, 1/2]$, μ_ρ stands for the measure on $[0,1]^d$ obtained as the tensor product of d binomial measures of parameter ρ on $[0,1]$. Recall that

$$\tau_{\mu_\rho}(q) = -d\log_2(\rho^q + (1-\rho)^q).$$

Let us consider two parameters $\rho_0 \leq \rho_1$ in $(0,1/2]$, as well as μ_{ρ_0} and μ_{ρ_1} the tensor products of d binomial measures on $[0,1]$ of parameters ρ_0 and ρ_1 respectively. Recalling that $\tau'_{\mu_\rho}(1) = -d(\rho\log_2\rho + (1-\rho)\log_2(1-\rho))$, the parameters ρ_0 and ρ_1 can be chosen so that $\tau'_{\mu_{\rho_0}}(1) = h_0$ and $\tau'_{\mu_{\rho_1}}(1) = h_1$. Now, let $(\varepsilon_j)_{j\geq 1}$ be a positive sequence going to 0 at ∞. For $i \in \{0,1\}$ and any $j \geq 1$, let

$$E_j^i = \bigcap_{j'\geq j} \bigcup_{Q\in\mathcal{G}_{j'}:\ 2^{-j'h_i(1+\varepsilon_{j'})}\leq\mu_{\rho_i}(Q)\leq 2^{-j'h_i(1-\varepsilon_{j'})}} Q.$$

It is well-known that μ_{ρ_i} is carried by the set $\widehat{E}_{h_i}^{\rho_i}$. (See Remark 2.3 for the definition of this set.) Thus, the sequence $(\varepsilon_j)_{j\geq 1}$ can be chosen so that $\mu_{\rho_i}\big(\bigcup_{j\geq 1} E_j^i\big) = 1$ for $i \in \{0,1\}$. Moreover, the sets E_j^i form a non-decreasing sequence, so we can fix l_i, $i \in \{0,1\}$ such that $\mu_{\rho_i}(E_{l_i}^i) \geq 1/2$. Let us consider, for $j \geq 1$, the subset $\mathcal{G}_j^{(i)}$ of intervals of $\mathcal{G}_j$ defined by

$$\mathcal{G}_j^{(i)} = \{Q \in \mathcal{G}_j : Q \cap E_{l_i}^i \neq \emptyset\}.$$

Notice that by construction $\lim_{j\to\infty} \frac{\log_2 \#\mathcal{G}_j^{(i)}}{j} = h_i$, for any $i \in \{0,1\}$.

For $n \geq 1$, we build the sequence of purely discontinuous measures

$$\nu_n^0 = \sum_{Q\in\mathcal{G}_n^{(0)}} \mu_{\rho_0}(Q)\delta_{x_Q}.$$

Set $j_1 = 2$, $n_1 = 4$, and for every $k \geq 2$, $j_k = 2^{2^{n_{k-1}}}$ and then $n_k = 2^{j_k}$. When k is large, $n_{k-1} = o(j_k)$ and $j_k = o(n_k)$. Then define

$$\nu = \sum_{k\geq 1} 2^{-k} \sum_{Q\in\mathcal{G}_{j_k}^{(1)}} \mu_{\rho_1}(Q)\, \nu_{n_k}^0 \circ f_Q^{-1},$$

where f_Q stands for a similitude mapping $[0,1]^d$ onto Q. In particular, notice that the Dirac masses used in this construction take values $2^{-k}\mu_{\rho_1}(Q_k)\mu_{\rho_0}(Q'_k)$ at $x_{f_{Q_k}(Q'_k)}$, with $(Q_k, Q'_k) \in \mathcal{G}_{j_k}^{(1)} \times \mathcal{G}_{n_k}^{(0)}$. Then the next assertion follows.

Proposition 4.1. *We have* $q_\tau(\nu) = q_g(\nu) = 1$, $H_\tau(\nu) = h_1$, $H_g(\nu) = h_0$, *and* $f_\nu(h) = h$ *for every* $h \in [0,h_1]$.

4.2 Scheme of the General Construction for Theorem 1.3, Item *3(b)*.

We adopt the notation of the previous section and suppose that $h_0 < h_1$. Let $(\theta_k)_{k\geq 1}$ be an increasing sequence of integers such that $\theta_k j_k = o(n_k)$ as $k \to \infty$. Then let

$$\text{for } k \geq 1, \quad \mu^{\theta_k} = \sum_{Q \in \mathcal{G}_{\theta_k j_k}} \mu_{\rho_1}(Q)\, \mu_{p_0} \circ f_Q^{-1}.$$

Now for $k, n \geq 1$ consider the measure $\nu_n^{\theta_k} = \sum_{Q \in \mathcal{G}_n^{(0)}} \mu^{\theta_k}(Q)\delta_{x_Q}$. Finally let

$$\nu = \sum_{k \geq 1} 2^{-k} \sum_{Q \in \mathcal{G}_{j_k}^{(1)}} \mu_{\rho_1}(Q)\, \nu_{n_k}^{\theta_k} \circ f_Q^{-1}.$$

Proposition 4.2. *We have $q_\tau(\nu) = q_g(\nu) = 1$, $H_\tau(\nu) = h_1$, $H_g(\nu) = h_0$, and $f_\nu(h) < h$ for every $h \in (h_0, h_1)$.*

4.3 Theorem 1.3, Item *4*: A Condition to Have $H_g(\nu) = H_\tau(\nu)$.

Let $\varepsilon \in (0, H_\tau(\nu)^{-1})$. Let (n_j) be as in the item *4* of Theorem 1.3. Let $\eta > 0$ and suppose that there exists an integer j_0 such that for $j \geq j_0$ the set X_{n_j} is included in the union of less than $b^{n_j(1-\eta)}$ b-adic boxes of generation $[n_j(H_\tau(\nu)^{-1} - \varepsilon)]$. This implies that $\bigcup_{k \in K_{n_j}} B(x_k, b^{-n_j(H_\tau(\nu)^{-1}-\varepsilon)})$ is covered by at most $3^d b^{n_j(1-\eta)}$ b-adic boxes of generation $[n_j(H_\tau(\nu)^{-1} - \varepsilon)]$. Elementary computations yield

$$\dim\Big(\limsup_{j\to\infty} \bigcup_{k \in K_{n_j}} B(x_k, b^{-n_j(H_\tau(\nu)^{-1}-\varepsilon)})\Big) \leq \frac{1-\eta}{H_\tau(\nu)^{-1} - \varepsilon}.$$

If $\eta \in (H_\tau(\nu)\varepsilon, 1)$, then $\frac{1-\eta}{H_\tau(\nu)^{-1}-\varepsilon} < H_\tau(\nu)$. This yields a contradiction with our assumption. Consequently, there is an increasing sequence $(j_p)_{p\geq 1}$ such that for every $p \geq 1$, the set $X_{n_{j_p}}$ is included in the union of at least $b^{n_{j_p}(1-\eta)}$ b-adic boxes of generation $[n_{j_p}(H_\tau(\nu)^{-1} - \varepsilon)]$. We let the reader check that this implies $H_g(\nu) \geq H_\tau(\nu)$.

5 A Simple Example Illustrating Theorem 1.3, Item *3(a)*.

The idea is to replace the binomial measures in Section 4 by uniform measures on Cantor sets, which are known to be monofractal and are easier to deal with.

(i) Preliminary step: Let us consider the Cantor set K_0 defined by this recursive scheme. Divide the interval $I = [0,1]$ into 8 subintervals of length $1/8$, and keep only the first and last ones, denoted respectively by I_0 and I_1. The same dividing scheme applied to I_0 (*resp.* I_1) yields two intervals $I_{0,0}$ and $I_{0,1}$ (*resp.* $I_{1,0}$ and $I_{1,1}$) of length $(1/8)^2$. Iterating this procedure yields, at every generation $m \geq 1$, 2^m intervals of same length $(1/8)^n = (1/2)^{3n}$. Denote by $E_m^0 \subset \mathcal{G}_{3m}$ this set of intervals, and $K_0 = \bigcap_{m\geq 1} E_m^0$. For every $m \geq 1$, consider the probability measure $\mu^0(m)$ which is uniformly distributed on E_m^0; i.e., $\mu^0(m)$ has a density $f_{\mu^0(m)}$ equal to

$$f_{\mu^0(m)} = \sum_{I\in E_m^0} 2^{-m}\mathbf{1}_I(x), \text{ where } \mathbf{1}_I \text{ is the indicator function of the interval } I,$$

and the discontinuous measure $\nu^0(m)$ defined by

$$\nu^0(m) = \sum_{I\in E_m^0} 2^{-m}\delta_{x_I}, \text{ where } x_I \text{ is the left end-point of the interval } I.$$

Similarly, consider the Cantor set K_1 where each interval is split into only four equal parts and where the two extremal intervals are kept at each generation. Again, the m-th generation of the construction is denoted by $E_m^1 \subset \mathcal{G}_{2m}$ and $K_1 = \bigcap_{m\geq 1} E_m^1$. Finally, for every $m \geq 1$ two measures $\mu^1(m)$ and $\nu^1(m)$ are built using the same scheme as the one used for $\mu^0(m)$ and $\nu^0(m)$.

The reader can verify the next lemma, which follows from classical self-similarity properties of the construction and of the monofractal measure on the uniform Cantor sets we deal with.

Lemma 5.1. *Set $h_0 = 1/3$ and $h_1 = 1/2$. For every $m \geq 2$,*

1. *If I is a dyadic interval of generation $1 \leq j \leq 3m$, then either $\nu^0(m)(I) = \mu^0(m)(I) = 0$ or $|I|^{h_0}/8 \leq \nu^0(m)(I) = \mu^0(m)(I) \leq 8 \cdot |I|^{h_0}$.*
   The cardinality N_j^0 of the set $\{I \in \mathcal{G}_j : \nu^0(m)(I) > 0\}$ satisfies $2^{jh_0}/8 \leq N_j^0 \leq 8 \cdot 2^{jh_0}$.
2. *If I is a dyadic interval of generation $1 \leq j \leq 2m$, then either $\nu^1(m)(I) = \mu^1(m)(I) = 0$ or $|I|^{h_1}/4 \leq \nu^1(m)(I) = \mu^1(m)(I) \leq 4 \cdot |I|^{h_1}$.*
   The cardinality N_j^1 of the set $\{I \in \mathcal{G}_j : \nu^1(m)(I) > 0\}$ satisfies $2^{jh_1}/4 \leq N_j^1 \leq 4 \cdot 2^{jh_1}$.
3. *For any $\varepsilon > 0$, there exists an integer m_ε such that for every $m \geq m_\varepsilon$, any subset $E \subset \{x_I : I \in E_m^0\}$ of cardinality greater than $2^{m(1-\varepsilon/2)} = (\#E_m^0)^{1-\varepsilon}$ contains two points x and y such that $|x-y| \leq 2^{-3m(1-\varepsilon)}$.*

(ii) Construction of the measure ν: Two sequences of integers $(j_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ are needed. They are built recursively, using the same scheme as in Section 4. Set $j_1 = 2$, $n_1 = 4$, and $\forall k \geq 2$ $j_k = 2^{2^{n_{k-1}}}$ and then $n_k = 2^{j_k}$.

For every dyadic interval I, we denote by f_I the increasing affine mapping which maps I onto $[0,1]$. Finally we set

$$\nu = \sum_{k\geq 1} \frac{1}{j_k} \sum_{I\in E^1_{j_k}} 2^{-j_k}\nu^0(n_k)\circ f_I. \tag{13}$$

Note that:

• The Dirac masses which appear in (13) all have an intensity of order $\dfrac{2^{-(j_k+n_k)}}{j_k}$ and when k is fixed, there are $2^{j_k+n_k}$ such Dirac masses. Moreover, still for a given $k\geq 1$, these masses are located at dyadic points of scale $2^{-(2j_k+3n_k)}$.

• For every $k\geq 1$, the Dirac masses of generation $k'\geq k$ which appear in the sum (13) all belong to one of the intervals $I\in E^1_{j_k}$.

• For every $k\geq 1$, (recall that $j_{k+1}=2^{2^{n_k}}$, $n_{k-1}=o(j_k)$ and $j_k=o(n_k)$)

$$\sum_{k'\geq k+1} \frac{1}{j_k} \sum_{I\in E^1_{j_k}} 2^{-j_k} = o(2^{-(j_k+n_k)}). \tag{14}$$

• The structure of ν is comparable to the one of Cantor sets with different upper and lower box dimensions (see [12]).

(iii) Properties of ν: They follow from Propositions 5.2 and 5.3.

Proposition 5.2. *We have $q_g(\nu)=q_\tau(\nu)=1$, $H_\tau(\nu)=h_1$ and $H_g(\nu)=h_0$.*

PROOF. **(i)** At every scale $j=2j_k$, there are by construction 2^{j_k} intervals whose ν-mass is larger than $2^{j_k}/j_k$. This holds for every $k\geq 1$, and thus $\underline{f}_\nu(h_1)\geq h_1$. By Proposition 2.6, $q_\tau(\nu)=1$ and $f_\nu(h_1)=h_1$, and finally $H_\tau(\nu)\geq h_1$.

In order to get $H_\tau(\nu)=h_1$, it is enough to prove that $\tau_\nu(q)\geq (q-1)h_1$ near 1^- (indeed, this clearly implies that $H_\tau(\nu)=\tau'_\nu(1^-)\leq h_1$). Let $q\in(0,1)$ and $j\geq 1$. Let k be the unique integer such that $2j_k\leq j<2j_{k+1}$.

- Assume $2j_k\leq j<2j_k+3n_k$**.** Let us evaluate $s_j(q)$ (defined in (2)) for the measure ν. There are only two types of dyadic intervals of non-zero ν-mass at scale j.

- Those which contain a Dirac mass of the form $2^{-(j_{k'}+n_{k'})}/j_{k'}$ for $k'\in\{1,\dots,k-1\}$. For each such k', there are at most $2^{(j_{k'}+n_{k'})}$ of them.
- By item *1* of Lemma 5.1, if an interval at scale j does not contain a mass of generation $<k$, then either its ν-mass is 0, or it is equivalent to $2^{-(j-2j_k)h_0}2^{-j_k}/j_k$. (We implicitly use (14) which ensures that the masses of next generations do not interfere.) Still by Lemma 5.1, the number of such intervals is $N^0_{j-2j_k}$, which is approximately $2^{(j-2j_k)h_0}$.

Due to the sub-additivity of the function $x \mapsto x^q$ on $\mathbb{R}_+$ ($q < 1$), we get

$$\begin{aligned} s_j(q) &= \sum_{I \in \mathcal{G}_j} \nu(I)^q \\ &\leq \sum_{k'=1}^{k-1} 2^{j_{k'}+n_{k'}} \Big(\frac{2^{-(j_{k'}+n_{k'})}}{j_{k'}} \Big)^q + N^0_{j-2j_k} \Big(\frac{2^{-(j-2j_k)(h_0)} \cdot 2^{-j_k}}{j_k} \Big)^q \\ &\leq C \Big(2^{(j_{k-1}+n_{k-1})(1-q)} + 2^{(j-2j_k)(1-q)-qj_k} \Big). \end{aligned}$$

Given $\varepsilon > 0$, using that $j_{k-1} + n_{k-1} = o(j_k)$, taking the log and then dividing by $-j$, we find that as soon as j (and thus k) is large enough,

$$\frac{\log s_j(q)}{-j} \geq h_0(q-1) + q\frac{2j_k}{j} - \varepsilon.$$

which is always greater than $h_0(q-1) - \varepsilon$, for every ε when k is large enough.

-Assume $2j_k + 3n_k \leq j < 2j_{k+1}$**.** Again there are two types of intervals.

- Those which contain a Dirac mass of the form $2^{-(j_{k'}+n_{k'})}/j_{k'}$ for $k' \in \{1, \dots, k\}$. For each $k' \in \{1, \dots, k\}$, there are at most $2^{(j_{k'}+n_{k'})}$ of them.
- By item *2* of Lemma 5.1, if an interval of scale j does not contain a mass of generation $\leq k$, then its ν-mass is either 0 or is equivalent to $|I|^{h_1}/j_{k+1} = 2^{-jh_1}/j_{k+1}$. (Again, (14) is used.) Still by Lemma 5.1, the number of such intervals is N^1_j.

Hence, the same estimates as above yield

$$\begin{aligned} s_j(q) &\leq \sum_{k'=1}^{k} 2^{j_{k'}+n_{k'}} \Big(\frac{2^{-(j_{k'}+n_{k'})}}{j_{k'}} \Big)^q + N^1_j \Big(\frac{2^{-jh_1}}{j_{k+1}} \Big)^q \\ &\leq C \Big(2^{(j_k+n_k)(1-q)} + \frac{2^{jh_1(1-q)}}{j^q_{k+1}} \Big). \end{aligned}$$

Given $\varepsilon > 0$, using that $j_{k-1} + n_{k-1} = o(j_k)$, taking the log and then dividing by $-j$, we find that when j (and thus k) is large enough,

$$\begin{aligned} \frac{\log s_j(q)}{-j} &\geq (q-1)\Big(h_1 + \frac{j_k + n_k}{j} \Big) + q\frac{\log j_{k+1}}{j} \\ &\geq (q-1)\Big(h_1 + \frac{j_k + 2^{j_k}}{j} \Big) + q\frac{2^{n_k}}{j}. \end{aligned}$$

Let $\varepsilon > 0$. Since $j_k = o(n_k)$, when k is large enough, we finally obtain

$$\frac{\log s_j(q)}{-j} \geq h_1(q-1) + q\frac{2^{n_k(1-\varepsilon)}}{j},$$

which is always greater than $h_1(q-1)$ (and actually which is equivalent to $h_1(q-1)$ when j is close to $2j_{k+1}$).

(ii) To conclude the proof, we need to establish that $H_g(\nu) = h_0$.

For every $k \geq 1$, let us denote by l_k the unique integer such that $2^{-l_k} \leq 2^{-(j_k+n_k)}/j_k \leq 2^{-l_k+1}$. By construction, for any integer $n \geq 1$, either $q(n) = 0$, or there is k such that $n = l_k$ and thus $q(n) = q(l_k) = \frac{\log_2 2^{j_k+n_k}}{l_k}$, which clearly tends to 1 when k goes to infinity. Hence $q_g(\nu) = q_\tau(\nu) = 1$.

Let $\widetilde{\alpha} = (\alpha_k)_{k\in\mathbb{N}}$ be a positive sequence converging to 0. Let K be such that for every $k \geq K$, $\alpha_k \leq \varepsilon/4$. Consider such an integer $k \geq K$. Let E be any subset of $X(l_k)$ of cardinality greater than $(\#X(l_k))^{(1-\alpha_k)} = 2^{(j_k+n_k)(1-\alpha_k)}$. By construction (self-similarity of the Cantor set), it is obvious that $\mathcal{J}(l_k, \alpha_k) \geq 2j_k + 3n_k$

The points of $X(l_k)$ can be separated into 2^{j_k} packets of 2^{n_k} Dirac masses, where each packet corresponds to one term $\nu^0(n_k) \circ f_I$ ($I \in E^1_{j_k}$) in the definition (13) of ν. As a consequence, there is one packet such that the set E contains (at least) $2^{(j_k+n_k)(1-\alpha_k)}/2^{j_k}$ of the initial Dirac masses of this packet. Since $j_k = o(n_k)$, for k large enough, E contains at least $2^{n_k(1-\varepsilon/2)}$ Dirac masses.

By item *3* of Lemma 5.1, any such subset $E \subset X(l_k)$ contains two points x and y such that $|x-y| \leq 2^{-j_k}2^{-3n_k(1-\varepsilon)}$. Hence, $\mathcal{J}(l_k, \alpha_k) \leq j_k + 3n_k(1-\varepsilon)$. Using the two bounds for $\mathcal{J}(l_k, \alpha_k)$, we get that $\lim_{k\to+\infty} \frac{l_k}{\mathcal{J}(l_k,\alpha_k)} = 1/3 = h_0$. Hence $H_g(\nu) = h_0$. This ends the proof. $\square$

Proposition 5.3. *For every $h \in (h_0, h_1)$, $f_\nu(h) = h$.*

Proof. Let $k \geq 1$ be large enough and $j \geq 1$ be such that $2j_k \leq j < 2j_k + 3n_k$. Let $\varepsilon > 0$. As explained above in Proposition 5.2, there are, at scale j, at least $2^{(j-2j_k)h_0(1-\varepsilon)}2^{-j_k}$ intervals I of length 2^{-j} such that $\nu(I) \geq 2^{-(j-2j_k)h_0}2^{-j_k}/j_k$. But $2^{-(j-2j_k)h_0}2^{-j_k}/j_k = 2^{-jh_j^{(k)}}$, where

$$h_j^{(k)} = h_0 + j_k/j(1-2h_0) + (\log_2 j_k)/j = h_0 + j_k/j(1-2h_0) + n_{k-1}/j.$$

Note that the exponents $h_j^{(k)}$ range in $[h_0, h_0 + 1/2 - h_0] = [h_0, h_1]$ when j describes $\{2j_k, \ldots, 2j_k + 3n_k\}$ and that any $h \in [h_0, h_1]$ is the limit of a sequence of such points $h_j^{(k)}$ fully when $k \to +\infty$.

Let $h > 0$ and assume that $\varepsilon' > 0$ is so small that $[h-\varepsilon', h+\varepsilon'] \subset (h_0, h_1)$. Assume also that k is large enough so that there is $j \in [2j_k + 3n_k, 2j_{k+1}]$ such that $h_j^{(k)} \in [h-\varepsilon', h+\varepsilon']$. As proved just above, the number of intervals I of scale j such that $\nu(I) \geq 2^{-jh_j^{(k)}}$ is greater than $2^{(j-2j_k)h_0(1-\varepsilon)}2^{j_k} \geq 2^{jh_j^{(k)}(1-\varepsilon)}$. This occurs for an infinite number of scales j and for every $\varepsilon' > 0$ and $\varepsilon > 0$, hence $\underline{f}_\nu(h) \geq h$. By item *2* of Proposition 2.6, $f_\nu(h) = h$. $\square$

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