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A BIRKHOFF TYPE INTEGRAL AND THE BOURGAIN PROPERTY IN A LOCALLY CONVEX SPACE

Abstract

An integral, called the Bk-integral, for functions taking values in a locally convex space is defined. Properties of Bk-integrable functions are considered and the relations with other integrals are studied. Moreover the Bk-integrability of bounded functions is compared with the Bourgain property.

1 Introduction.

In this paper we consider an integral, called the Bk-integral, which is an extension to locally convex spaces of the Birkhoff integral of [2]. Properties of the Bk-integral are considered, and it is compared with other kinds of integrals.

In §3, we present some properties of the Bk-integral. The Cauchy criterion for Bk-integrability is proved (Proposition 1).

In §4, we compare the Bk-integral with other types of integrals, and we establish that the Bk-integral lies between the Bochner integral and the Pettis integral. When the range is a Banach space, a measurable and Pettis integrable function f is also Birkhoff integrable. We prove that the same result holds for functions whose range is a Hausdorff locally convex topological vector space (Theorem 3), if we consider the measurability by seminorm instead of the measurability.

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In §5, the Bk-integrability of a function is compared with the Bourgain property of a suitable family of functions. Applying a Lemma in [4], the Bk-integrability of a bounded function is compared with the Bourgain property of a family of real valued functions.

2 Definitions and Notations.

Let X be a Hausdorff locally convex topological vector space (briefly a locally convex space) with its topology $\mathcal{T}$ and topological dual X^*. $\mathcal{P}(X)$ denotes a family of $\mathcal{T}$-continuous seminorms on X so that the topology is generated by $\mathcal{P}(X)$.

Let $(\Omega, \mathcal{F}, \mu)$ be a non-empty finite measure space. Unless specified otherwise, the terms "measure", "measurable" and "almost everywhere" (briefly "a.e.") refer to the measure μ. For a set $E \in \mathcal{F}$, we denote by χ_E the characteristic function of E. A *partition* of Ω is a countable family of disjoint measurable sets $(E_i)_{i\in\mathbb{N}}$ such that $\Omega = \bigcup_i E_i$.

A function $f : \Omega \to X$ is called *weakly-measurable* if the function $x^* f$ is measurable for every $x^* \in X^*$.

We recall that a function $f : \Omega \to X$ is called *simple* if there exist $x_1, x_2, \ldots, x_n \in X$ and $E_1, E_2, \ldots, E_n \in \mathcal{F}$ such that $f = \sum_{i=1}^n x_i \chi_{E_i}$. If $s = \sum_{i=1}^n x_i \chi_{E_i}$ and $A \in \mathcal{F}$, then $\int_A s = \sum_{i=1}^n \mu(A \cap E_i) x_i$.

We recall the following definitions (see [3], Definition 2.4).

Definition 1. A function $f : \Omega \to X$ is said to be strongly (or Bochner) integrable if there exists a sequence $(f_n)_n$ of simple functions such that:

- (i) $f_n(t) \to f(t)$ a.e.; i.e., f is strongly measurable;
- (ii) $p(f(t) - f_n(t)) \in L^1(\Omega)$ for each $n \in \mathbb{N}$ and $p \in \mathcal{P}(X)$, and $\lim_{n\to\infty} \int_\Omega p(f(t) - f_n(t))dt = 0$ for each $p \in \mathcal{P}(X)$;
- (iii) $\int_A f_n$ converges in X for each measurable subset A of Ω.

In this case we put $(B)\int_A f = \lim_{n\to\infty} \int_A f_n$.

Definition 2. A function $f : \Omega \to X$ is said to be integrable by seminorm if for any $p \in \mathcal{P}(X)$ there exist a sequence $(f_n^p)_n$ of simple functions and a subset $X_0^p \subset \Omega$, with $\mu(X_0^p) = 0$, such that:

- (i) $\lim_{n\to\infty} p(f_n^p(t) - f(t)) = 0$ for all $t \in \Omega \setminus X_0^p$; i.e., f is measurable by seminorm;
- (ii) $p(f(t) - f_n^p(t)) \in L^1(\Omega)$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} \int_\Omega p(f(t) - f_n^p(t))dt = 0$;

(*iii*) for each measurable subset A of Ω there exists an element $y_A \in X$ such that $\lim_{n\to\infty} p(\int_A f_n^p(t) - y_A) = 0$.

Then we put $\int_A f = y_A$.

Clearly a Bochner integrable function is integrable by seminorm, and the two definitions coincide in a Banach space.

Definition 3. A function $f : \Omega \to X$ is said to be Pettis integrable if $x^* f$ is Lebesgue integrable on Ω for each $x^* \in X^*$ and for every measurable set $E \subset \Omega$ there is a vector $\nu_f(E) \in X$ such that $x^*\nu_f(E) = \int_E x^* f(t)\,dt$ for all $x^* \in X^*$.

The set function $\nu_f : \mathcal{F} \to X$ is called the *indefinite Pettis integral of* f. It is known (see for example [16], p. 65) that ν_f is a countably additive vector measure, continuous with respect to the measure μ (in the sense that if $\mu(E) = 0$, then $\nu_f(E) = 0$).

3 The Bk-integral.

From now on X is a complete locally convex space. We denote by Γ a partition of Ω. The notation $\Gamma_1 \geq \Gamma_2$ for two partitions Γ_1 and Γ_2 of Ω means that Γ_1 is finer than Γ_2; that is, every set γ_1 of Γ_1 is a subset of some γ_2 of Γ_2. For a given set $\gamma \subset \Omega$, we set $f(\gamma) = \{f(t) : t \in \gamma\}$; clearly $f(\gamma) \subset \Omega$. Moreover, if $\Gamma = (\sigma_i)$ is a partition of Ω, by the symbol $\Sigma(f,\Gamma)$ we denote the formal series $\sum_i \mu(\sigma_i) f(\sigma_i)$. Let $p \in \mathcal{P}(X)$ be given. Then $p^{-1}(0)$ is a vector subspace and p defines a norm on $X/p^{-1}(0)$. If $B \subset X$ is bounded with respect to p, we use the notation $p(B) = \sup\{p(x) : x \in B\}$. Given a sequence $B_1, B_2, \dots$ of sets in X, the series $\sum_{n=1}^{\infty} B_n$ is said to be *p-convergent* provided that for every choice of $b_n \in B_n$, $n \in \mathbb{N}$, the series $\sum_{n=1}^{\infty} b_n$ is convergent in the normed space $X/p^{-1}(0)$. The series $\sum_{n=1}^{\infty} B_n$ is said to be *p-convergent to* $z \in X$ provided that for every choice of $b_n \in B_n$, $n \in \mathbb{N}$, the series $\sum_{n=1}^{\infty} b_n$ is convergent to z in the normed space $X/p^{-1}(0)$. Let $f : \Omega \to X$. We say that f is *p-unconditionally summable on* $\Gamma = (\sigma_i)$ if, for each $i \in \mathbb{N}$, $f \restriction \sigma_i$ is bounded with respect to p whenever $\mu(\sigma_i) > 0$ and the series $\Sigma(f,\Gamma)$ is p-convergent.

Definition 4. A function $f : \Omega \to X$ is said to be Bk-integrable with integral $z \in X$ if for every $p \in \mathcal{P}(X)$ and $\varepsilon > 0$ there exists a partition $\Gamma_p = (\sigma_i)$ of Ω for which f is p-unconditionally summable on Γ_p and

$$p\Big(\sum_{i\in\mathbb{N}} \mu(\sigma_i) f(t_i) - z\Big) < \varepsilon, \tag{1}$$

for all $t_i \in \sigma_i$. The vector z is called the value of the integral on the set Ω, and we set $z = (Bk)\int_\Omega f$.

We denote by $Bk(\Omega, X)$ the family of all Bk-integrable functions on Ω.

In order to prove the Cauchy criterion for the Bk-integral, we need the following lemma.

Lemma 1. *Let $f : \Omega \to X$ be a function. Assume that, given $p \in \mathcal{P}(X)$ and $\varepsilon > 0$ there is a partition $\Gamma_p = (\sigma_i)$ of Ω such that $f \restriction \sigma_i$ is bounded with respect to p and the series $\Sigma(f, \Gamma_p)$ is p-convergent to $z \in X$. Then, for any partition $\Gamma'_p = (\beta_k)$, $\Gamma'_p \geq \Gamma_p$; also, the series $\Sigma(f, \Gamma'_p)$ is p-convergent to $z \in X$.*

Proof. Let $p \in \mathcal{P}(X)$ and $\varepsilon > 0$ be fixed. Let $\Gamma_p = (\sigma_i)$ be a partition so that the series $\Sigma(f, \Gamma_p)$ is p-convergent to $z \in X$ and let $\Gamma'_p = (\sigma_{i,j})$ be a subpartition of Γ_p with $\cup_j \sigma_{i,j} = \sigma_i$ for each $i \in \mathbb{N}$. Set $B_i = \mu(\sigma_i) f(\sigma_i)$ and $B_{i,j} = \mu(\sigma_{i,j}) f(\sigma_{i,j})$. We show that $\sum_{i,j} B_{i,j}$ is p-convergent to z. Since $\sum_i B_i$ is p-convergent to z, for each $\varepsilon > 0$ there is $N \in \mathbb{N}$, such that for each $m > N$

$$p\Big(\sum_{n=1}^{m} B_n - z\Big) < \frac{\varepsilon}{2}, \tag{2}$$

and for each finite set $Q \subset \mathbb{N} \setminus \{1, \dots N\}$,

$$p\Big(\sum_{n \in Q} B_n\Big) < \frac{\varepsilon}{2}. \tag{3}$$

Take $M = \max\{p(f(\sigma_1)), \dots, p(f(\sigma_N)) : \mu(\sigma_i) > 0, i = 1, \dots N\}$. Since $\mu(\Omega) < \infty$ and $\mu(\Omega) = \sum_{i,j} \mu(\sigma_{i,j})$, there is K big enough such that

$$\sum_{n=1}^{N} \sum_{k > K} \mu(\sigma_{n,k}) < \frac{\varepsilon}{2M}. \tag{4}$$

We want to prove that

$$p\Big(\sum_{(n,k) \in S} B_{n,k}\Big) < \varepsilon,$$

for each finite subset S of $T = \mathbb{N} \times \mathbb{N} \setminus (\{1, \dots, N\} \times \{1, \dots, K\}$. Indeed for such a set S, let $S' = \{(n,k) \in S : 1 \leq n \leq N\}$ and $S'' = \{(n,k) \in S : n > N\}$. By

(4), we get

$$
\begin{aligned}
&p\Big(\sum_{(n,k)\in S'} B_{n,k}\Big) = p\Big(\sum_{(n,k)\in S'} \mu(\sigma_{n,k})f(\sigma_{n,k})\Big)\\
\le &\sum_{(n,k)\in S'} p(\mu(\sigma_{n,k})f(\sigma_{n,k})) \le \sum_{n=1}^{N}\sum_{k>K} M\mu(\sigma_{n,k}) < M\frac{\varepsilon}{2M} = \frac{\varepsilon}{2}.
\end{aligned}
\tag{5}
$$

Define $N' = \max\{n > N :\ \text{there is } k \text{ with } (n,k)\in S\}$. Then by (3) we obtain

$$
\begin{aligned}
&p\Big(\sum_{(n,k)\in S''} B_{n,k}\Big) = p\Big(\sum_{(n,k)\in S''} \mu(\sigma_{n,k})f(\sigma_{n,k})\Big)\\
=&p\Big(\sum_{(n,k)\in S''} \frac{\mu(\sigma_{n,k})}{\mu(\sigma_n)}\mu(\sigma_n)f(\sigma_{n,k})\Big) \le p\Big(\sum_{N<n\le N'} \operatorname{co}(B_n\cup\{0\})\Big)\\
=&p\Big(\operatorname{co}\big(\sum_{N<n\le N'} B_n\cup\{0\}\big)\Big) = p\Big(\sum_{N<n\le N'} B_n\cup\{0\}\Big)\\
=&\sup_{F\subset\{N+1,\dots,N'\}} p\Big(\sum_{k\in F} B_k\Big) < \frac{\varepsilon}{2},
\end{aligned}
\tag{6}
$$

where 0 is the null vector in X and $\operatorname{co}(B)$ is the convex hull of B. Therefore, by (5) and (6) we get

$$
p\Big(\sum_{(n,k)\in S} B_{n,k}\Big) < \varepsilon. \tag{7}
$$

By (3) there is $N\in\mathbb{N}$ such that

$$
p\Big(\sum_{n>N} B_n\Big) < \frac{\varepsilon}{2}, \tag{8}
$$

and by (7) there is $K\in\mathbb{N}$ such that if $T = \mathbb{N}\times\mathbb{N}\setminus(\{1,\dots,N\}\times\{1,\dots,K\}$,

$$
p\Big(\sum_{(n,k)\in T} B_{n,k}\Big) < \varepsilon. \tag{9}
$$

Let $r > N$ and $s > K$. By (2), (8) and (9) we have

$$\begin{aligned}
&p\Big(\sum_{n=1}^{r}\sum_{k=1}^{s} B_{n,k} - z\Big) = p\Big(\sum_{n=1}^{r}\sum_{k=1}^{s} B_{n,k} - \sum_{n} B_n\Big)\\
&+p\Big(\sum_{n} B_n - z\Big) \le p\Big(\sum_{n=1}^{N}\sum_{k=1}^{K} B_{n,k} - \sum_{n=1}^{N} B_n\Big)\\
&+p\Big(\sum_{n>N} B_n\Big) + p\Big(\sum_{n=N+1}^{r}\ \sum_{k=K+1}^{s} B_{n,k} + \sum_{n=1}^{N}\sum_{k>K} B_{n,k}\Big) + p\Big(\sum_{n} B_n - z\Big)\\
&<p\Big(\sum_{n=1}^{N}\sum_{k=1}^{K} B_{n,k} - \sum_{n=1}^{N} B_n\Big) + 2\varepsilon.
\end{aligned}$$

Consider the first term in the last inequality. Applying (5), we have

$$\begin{aligned}
&p\left(\sum_{n=1}^{N}\sum_{k=1}^{K} B_{n,k} - \sum_{n=1}^{N} B_n\right) = p\left(\sum_{n=1}^{N}\left(\sum_{k=1}^{K}\mu(\sigma_{n,k})f(\sigma_{n,k}) - \mu(\sigma_n)f(\sigma_n)\right)\right)\\
&\le p\left(\sum_{n=1}^{N}\left(\sum_{k=1}^{K}\mu(\sigma_{n,k})(f(\sigma_{n,k}) - f(\sigma_n))\right)\right) + p\left(\sum_{n=1}^{N}\sum_{k>K}\mu(\sigma_{n,k})f(\sigma_n)\right)\\
&< p\left(\sum_{n=1}^{N}\left(\sum_{k=1}^{K}\frac{\mu(\sigma_{n,k})}{\mu(\sigma_n)}\mu(\sigma_n)(f(\sigma_{n,k}) - f(\sigma_n))\right)\right) + \frac{\varepsilon}{2}\\
&\le p\left(\sum_{n=1}^{N} co((B_n - B_n)\cup\{0\})\right) + \frac{\varepsilon}{2} = p\left(co(\sum_{n=1}^{N}(B_n - B_n)\cup\{0\})\right) + \frac{\varepsilon}{2}\\
&= p\left(\sum_{n=1}^{N} B_n - \sum_{n=1}^{N} B_n \cup\{0\}\right) + \frac{\varepsilon}{2} < \varepsilon + \frac{\varepsilon}{2} = \frac{3}{2}\,\varepsilon.
\end{aligned} \tag{10}$$

Thus, we get that the series $\Sigma(f,\Gamma'_p)$ is p-convergent to $z \in X$ and the assertion holds true. □

The following proposition is a version of the Cauchy criterion.

Proposition 1. *Let $f : \Omega \to X$. Then f is Bk-integrable if and only if for every $p \in \mathcal{P}(X)$ and for every $\varepsilon > 0$ there exists a partition $\Gamma_p = (\sigma_i)$ of Ω for which f is p-unconditionally summable on Γ_p and*

$$p\Big(\sum_{i=1}^{\infty}\mu(\sigma_i)f(t_i) - \sum_{i=1}^{\infty}\mu(\sigma_i)f(v_i)\Big) < \varepsilon,$$

for all $t_i, v_i \in \sigma_i$*.*

Proof. Necessity is obvious. To prove sufficiency, let $p \in \mathcal{P}(X)$ and $\varepsilon > 0$ be fixed. Then there exists a partition $\Gamma_p = (\sigma_i)$ of Ω for which f is p-unconditionally summable on Γ_p and

$$p\left(\sum_{i=1}^{\infty}\mu(\sigma_i)f(t_i) - \sum_{i=1}^{\infty}\mu(\sigma_i)f(v_i)\right) < \varepsilon,$$

for all $t_i, v_i \in \sigma_i$. Let $\mathcal{D} = \{(\mathcal{L}, m) : \mathcal{L}$ is a finite subset of $\mathcal{P}(X)$ and $m \in \mathbb{N}\}$ and define for $(\mathcal{L}, m)$ and $(\mathcal{K}, n) \in \mathcal{D}$, $(\mathcal{L}, m) \ll (\mathcal{K}, n)$ if and only if $\mathcal{L} \subset \mathcal{K}$ and $m \leq n$. Now let $\mathcal{L} = \{p_1, \ldots, p_k\}$ be a finite subset of $\mathcal{P}(X)$. Since $\mathcal{P}(X)$ is filtering, we can find a seminorm $p_{\mathcal{L}} \in \mathcal{P}(X)$ and a constant $c_{\mathcal{L}} \geq 1$ such that for every $i \in \{1, \ldots, k\}$ and $x \in X$

$$p_i(x) \leq c_{\mathcal{L}} p_{\mathcal{L}}(x). \tag{11}$$

For each $(\mathcal{L}, m) \in \mathcal{D}$, let $\Gamma_m^{\mathcal{L}} = (\sigma_i^{\mathcal{L},m})$ be the partition corresponding by hypothesis to $p_{\mathcal{L}}$ and to $\varepsilon = \frac{1}{c_{\mathcal{L}} m}$. Then f is p-unconditionally summable on $\Gamma_m^{\mathcal{L}}$ and

$$p_{\mathcal{L}}\Big(\sum_{i=1}^{\infty}\mu(\sigma_i^{\mathcal{L},m})f(t_i^{\mathcal{L},m}) - \sum_{i=1}^{\infty}\mu(\sigma_i^{\mathcal{L},m})f(v_i^{\mathcal{L},m})\Big) < \frac{1}{c_{\mathcal{L}} m}, \tag{12}$$

for $t_i^{\mathcal{L},m}, v_i^{\mathcal{L},m} \in \sigma_i^{\mathcal{L},m}$. By Lemma 1 we may assume that $\Gamma_m^{\mathcal{L}}$ is finer than $\Gamma_m^{p_j}$, $j = 1, \ldots k$, where $\Gamma_m^{p_j}$ is the partition corresponding, by hypothesis, to the seminorm p_j and to $\varepsilon = \frac{1}{m}$. Moreover, applying again Lemma 1 we may assume that, for each fixed $\mathcal{L}$, the partition $\Gamma_m^{\mathcal{L}}$ is finer than $\Gamma_n^{\mathcal{L}}$ for $m > n$. Also, if $(\mathcal{L}, m) \ll (\mathcal{K}, m)$, we may assume that $\Gamma_m^{\mathcal{K}}$ is finer than $\Gamma_m^{\mathcal{L}}$. For any $(\mathcal{L}, m) \in \mathcal{D}$, let $\{(t_i^{\mathcal{L},m}, \sigma_i^{\mathcal{L},m})_i\}$ be a fixed family of couples, where $t_i^{\mathcal{L},m} \in \sigma_i^{\mathcal{L},m}$. Set $Q_m^{\mathcal{L}} = \sum_{i \in \mathbb{N}} \mu(\sigma_i^{\mathcal{L},m}) f(t_i^{\mathcal{L},m})$. Then $(Q_m^{\mathcal{L}})_{(\mathcal{L},m)}$ is a Cauchy net. Indeed let $p \in \mathcal{P}(X)$ and $\varepsilon > 0$ be fixed. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. Let $(\mathcal{L}, m) \gg (\{p\}, N)$ and $(\mathcal{K}, n) \gg (\{p\}, N)$. Take a couple $(\mathcal{Q}, N')$ such that $\mathcal{Q} = \mathcal{L} \cup \mathcal{K}$ and $N' = \max\{m, n\}$. Then the partition $\Gamma_{N'}^{\mathcal{Q}}$ is finer than both $\Gamma_m^{\mathcal{L}}$ and $\Gamma_n^{\mathcal{K}}$. Therefore, by Lemma 1, and applying (11) and (12) it follows that

$$\begin{aligned}
&p\Big(\sum_i \mu(\sigma_i^{\mathcal{L},m})f(t_i^{\mathcal{L},m}) - \sum_j \mu(\sigma_j^{\mathcal{K},n})f(t_j^{\mathcal{K},n})\Big)\\
\leq & c_{\mathcal{L}} p_{\mathcal{L}}\Big(\sum_i \mu(\sigma_i^{\mathcal{L},m})f(t_i^{\mathcal{L},m}) - \sum_l \mu(\sigma_l^{Q,N'})f(t_l^{Q,N'})\Big)\\
+ & c_{\mathcal{K}} p_{\mathcal{K}}\Big(\sum_l \mu(\sigma_l^{Q,N'})f(t_l^{Q,N'}) - \sum_j \mu(\sigma_j^{\mathcal{K},n})f(t_j^{\mathcal{K},n})\Big) < \frac{1}{m} + \frac{1}{n} < \varepsilon.
\end{aligned} \tag{13}$$

As the space is complete, there is a vector L such that the net $(Q_m^{\mathcal{L}})_{(\mathcal{L},m)}$ converges to L. We want to prove that L is the Bk-integral of f. To do this, fix $p \in \mathcal{P}(X)$ and $\varepsilon > 0$. Since $(Q_m^{\mathcal{L}})_{(\mathcal{L},m)}$ converges to L, there is a natural number N, with $\frac{1}{N} < \frac{\varepsilon}{2}$ such that

$$p\Big(\sum_{i\in\mathbb{N}} \mu(\sigma_i^{m,\mathcal{L}}) f(t_i^{m,\mathcal{L}}) - L\Big) < \frac{\varepsilon}{2} \tag{14}$$

whenever $(\mathcal{L}, m) \gg (\{p\}, N)$. Let $v_k \in \sigma_k^{\{p\},N}$. Since $(\Gamma_m^{\mathcal{L}})$ is finer than $\Gamma_N^{\{p\}}$, applying once again Lemma 1, we get

$$\begin{aligned}
&p\Big(\sum_k \mu(\sigma_k^{N,\{p\}}) f(v_k) - L\Big)\\
\leq &p\Big(\sum_k \mu(\sigma_k^{N,\{p\}}) f(v_k) - \sum_i \mu(\sigma_i^{m,\mathcal{L}}) f(t_i^{m,\mathcal{L}})\Big)\\
+&p\Big(\sum_i \mu(\sigma_i^{m,\mathcal{L}}) f(t_i^{m,\mathcal{L}}) - L\Big) < \frac{1}{N} + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

□

Proposition 2. *If $f : \Omega \to X$ is Bk-integrable, then for every $\gamma \in \mathcal{F}$ the function $f \restriction \gamma$ is Bk-integrable.*

Proof. Let $p \in \mathcal{P}(X)$ and $\varepsilon > 0$ be fixed. According to Proposition 1, let $\Gamma_p = (\sigma_i)$ be a partition of Ω for which f is p-unconditionally summable on Γ_p and

$$p\Big(\sum_{i=1}^{\infty} \mu(\sigma_i) f(t_i) - \sum_{i=1}^{\infty} \mu(\sigma_i) f(v_i)\Big) < \varepsilon,$$

for all $t_i, v_i \in \sigma_i$. Without loss of generality, by Lemma 1, we can assume that, for each i, $\sigma_i \subset \gamma$ or $\sigma_i \cap \gamma = \emptyset$. Let $I = \{i : \sigma_i \subset \gamma\}$. For all $t_i, v_i \in \sigma_i$, we get

$$\begin{aligned}
&p\Big(\sum_{i\in I} \mu(\sigma_i) f(t_i) - \sum_{i\in I} \mu(\sigma_i) f(v_i)\Big)\\
=&p\Big(\sum_{i\in I} \mu(\sigma_i) f(t_i) + \sum_{i\notin I} \mu(\sigma_i) f(t_i) - \sum_{i\in I} \mu(\sigma_i) f(v_i) - \sum_{i\notin I} \mu(\sigma_i) f(t_i)\Big)\\
=&p\Big(\sum_{i=1}^{\infty} \mu(\sigma_i) f(x_i) - \sum_{i=1}^{\infty} \mu(\sigma_i) f(y_i)\Big) < \varepsilon,
\end{aligned}$$

where $x_i = t_i$ and $y_i = v_i$ if $i \in I$, while $x_i = y_i = t_i$ if $i \notin I$. Then, by Proposition 1, $f \restriction \gamma$ is Bk-integrable. □

The property of linearity of the Bk-integral holds and can be proved in a standard way.

Proposition 3. *Let $f : \Omega \to X$ and $g : \Omega \to X$ be two Bk-integrable functions, then:*
(i) *the function $f + g$ is Bk-integrable;*
(ii) *for each $\alpha \in \mathbb{R}$ the function αf is Bk-integrable;*
(iii) *if $x^* \in X^*$, the real valued function $x^* f$ is Lebesgue integrable.*

We recall that if X is a Banach space with norm $\|\cdot\|$, then a function $f : \Omega \to X$ is Birkhoff integrable, with Birkhoff integral $z \in X$, if for every $\varepsilon > 0$ there is a partition $\Gamma = (\sigma_i)$ of Ω into measurable sets such that for all $t_i \in \sigma_i$ $\sum_{i\in\mathbb{N}} \mu(\sigma_i) f(t_i)$ is unconditionally convergent, and

$$\Big\| \sum_{i\in\mathbb{N}} \mu(\sigma_i) f(t_i) - z \Big\| < \varepsilon \tag{15}$$

([8] Definition 3). Note that if X is a Banach space, Definition 4 is equivalent to the above definition of the Birkhoff integral. Indeed, at first we observe that $\mathcal{T}$ is the norm topology and $\mathcal{P}(X) = \{\|\cdot\|\}$. Moreover, let $f : \Omega \to X$ be a Birkhoff integrable function. Then there exists a vector z satisfying that, given $\varepsilon > 0$ there is a partition $\Gamma = (\sigma_i)$ of Ω such that $\sum_i \mu(\sigma_i) f(\sigma_i)$ is unconditionally convergent and (15) holds. Therefore, f is unconditionally summable with respect to the norm $\|\cdot\|$ and hence it is Bk-integrable. Conversely, assume that $f : \Omega \to X$ is Bk-integrable. Given $\varepsilon > 0$, for each $n = 0, 1, 2, \dots$ there exists $\Gamma_n = (\sigma_i^n)$, where $\Gamma_{n+1} \geq \Gamma_n$, and a finite set of natural numbers π_n such that if $\pi \geq \pi_n$, then $\|\sum_\pi \mu(\sigma_i^n) f(\sigma_i^n) - z\| < \frac{\varepsilon}{2^n}$. Let N_n be the greatest of the integers in π_n. If $m, i_k \geq N_n$, $k = 1, \dots, l$, we get for $n = 0, 1, 2, \dots,$

$$\Big\| \sum_{i=1}^{m} \mu(\sigma_i^n) f(\sigma_i^n) - z \Big\| < \frac{\varepsilon}{2^n} \text{ and } \Big\| \sum_{k=1}^{l} \mu(\sigma_{i_k}^n) f(\sigma_{i_k}^n) \Big\| < 2\frac{\varepsilon}{2^n}.$$

Let M_n be such that $\sum_{i=M_n}^{\infty} \mu(\sigma_i^n) < \frac{1}{2^n}$ and set $P_n = \max\{N_n, M_n\}$. Define a sequence of sets $\sigma_1, \sigma_2, \dots$ inductively as follows. $\sigma_1 = \sigma_1^0$; if σ_k is in Γ_n then σ_{k+1} is the set of lowest subscript of Γ_n which is disjoint from any of the previously chosen sets unless all of such sets have subscript greater than P_n. In this case, let $R_n = k$ and σ_{k+1} is the set of lowest subscript of Γ_{n+1} which is disjoint from any of the previously chosen sets. Since $\sum_{i=1}^{R_n} \mu(\sigma_i) \geq \mu(\Omega) - \frac{1}{2^n}$, $\Gamma = (\sigma_i)$ is a partition of Ω unless a set of measure 0. As $\Gamma \geq \Gamma_0$, there exists, by Lemma 1, N_Γ such that for $n \geq N_\Gamma$,

$$\Big\| \sum_{i=1}^{n} \mu(\sigma_i) f(\sigma_i) - z \Big\| < 4\varepsilon.$$

Moreover, by the choice of (Γ_n), if $i_k > R_n$, then σ_{i_k} is a subset of some σ_i^n, $i \geq N_n$. Thus, there exists a finite set of integers $\pi \subset \mathbb{N} \setminus \{1, \dots N_n\}$ so that

$$\Big\| \sum_{k=1}^{l} \mu(\sigma_{i_k}) f(\sigma_{i_k}) \Big\| \leq \Big\| \sum_{\pi} co(\mu(\sigma_i^n) f(\sigma_i^n) + 0) \Big\|$$
$$\leq \sup \left\{ \Big\| \sum_{\pi'} \mu(\sigma_i^n) f(\sigma_i^n)) \Big\| : \pi' \leq \pi \right\} \leq \frac{\varepsilon}{2^{n-1}}.$$

Therefore, $\sum_i \mu(\sigma_i) f(\sigma_i)$ is unconditionally convergent and it follows that f is Birkhoff integrable.

Remark 1. Observe that Definition 4 is equivalent to the extension to locally convex spaces of the Birkhoff integral, given by Phillips and called $\mathcal{V}$-integral (see [13] Definition 2.1). In Definition 4, following the idea of [2], (see also [4] and [8]), it is required that for each $p \in \mathcal{P}(X)$ the series $\sum_i \mu(\sigma_i) f(\sigma_i)$ is p-convergent. Phillips' definition, instead, requires that given a seminorm p there exist a subdivision $\Gamma_p = (\sigma_i)$ and a finite set of natural numbers π_p, such that $\sum_{\pi_p} \mu(\sigma_i) f(\sigma_i)$ is contained in a neighborhood of the vector z.

The following proposition, for Banach valued functions, has been proved in [8], Lemma 9 (see also [4], Lemma 3.2).

Proposition 4. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space, let $f : \Omega \to X$ be a function and $(S_i)_{i \in \mathbb{N}}$ be a cover of Ω by measurable sets. Then f is Bk-integrable if and only if f is Pettis integrable and $f \upharpoonright S_i : S_i \to X$ is Bk-integrable for every $i \in \mathbb{N}$.*

Proof. Assume first that f is Bk-integrable. By Proposition 2, $f \upharpoonright S$ is Bk-integrable for each $S \in \mathcal{F}$. Moreover, if $x^* \in X^*$, the real valued function $x^* f$ is Lebesgue integrable, then f is Pettis integrable. Now assume that f is Pettis integrable and $f \upharpoonright S_i : S_i \to X$ is Bk-integrable for every $i \in \mathbb{N}$. Without loss of generality, we can suppose that the sets S_i are disjoint. Let $p \in \mathcal{P}(X)$ and $\varepsilon > 0$. Recall that $\nu_f : \mathcal{F} \to X$ is countably additive. For each $i \in \mathbb{N}$, let $(\sigma_{ij})_{j \in \mathbb{N}}$ be a partition of S_i such that $\sum_{j \in \mathbb{N}} \mu(\sigma_{ij}) f(t_{ij})$ is p-unconditionally convergent and

$$p\Big(\sum_{j \in \mathbb{N}} \mu(\sigma_{ij}) f(t_{ij}) - (Bk)\int_{S_i} f \Big) < \frac{\varepsilon}{2^i}. \tag{16}$$

Let $\delta > 0$ such that whenever $\mu(E) < \delta$ then $p(\nu_f(E)) \leq \varepsilon$. Since $\mu(\Omega) = \sum_{i=1}^{\infty} \mu(S_i) < \infty$, there is a $N \in \mathbb{N}$ such that $\sum_{i > N} \mu(S_i) = \mu(\bigcup_{i > N} S_i) < \delta$.

Moreover, let L be a finite subset of $\mathbb{N}$ and fix $n \in \mathbb{N}$. If $B_{n,L} = \bigcup_{l \in L} \sigma_{n,l}$, we get by Proposition 2 that $f \restriction B_{n,L}$ is Bk-integrable, and

$$p\Big(\sum_{l \in L} \mu(\sigma_{nl}) f(t_{nl}) - (Bk)\int_{B_{n,L}} f\Big) < \frac{\varepsilon}{2^n}.$$

Thus, if $\Gamma = (\sigma_{i,j})$ we get that $\Sigma(f, \Gamma)$ is p-convergent. For $n > N$,

$$p\Big(\sum_{i=1}^{n} \sum_{j \in \mathbb{N}} \mu(\sigma_{ij}) f(t_{ij}) - \nu_f(\Omega)\Big) \leq \sum_{i=1}^{n} p\Big(\sum_{j \in \mathbb{N}} \mu(\sigma_{ij}) f(t_{ij}) - \nu_f(S_i)\Big)$$
$$+p(\nu_f(\bigcup_{i>n} S_i)) \leq 2\varepsilon + \varepsilon = 3\varepsilon. \qquad \square$$

4 Relation Between the McShane Integral and the Bk-integral and Other Types of Integral.

In this section we establish some relations between the Bk-integral and the Bochner and the McShane integrals.

We prove that each integrable by seminorm function is Bk-integrable. First, we need the following lemma.

Lemma 2. *If $f : \Omega \to X$ is a simple function, then $f \in Bk(\Omega, X)$.*

Proof. Since the Bk-integral is linear, it is sufficient to consider the case $f(x) = \chi_E(x) \cdot w$ where E is a measurable set in Ω and w is a non null vector in X. Let $p \in \mathcal{P}(X)$ and $\varepsilon > 0$ be fixed. If $\Gamma^p = (E, E^c)$, then Γ^p is a partition of Ω and f is p-unconditionally summable on Γ^p. Moreover, inequality (1) is satisfied with $z = \mu(E) \cdot w$. Therefore, $f \in Bk(\Omega, X)$ and for each $\gamma \in \Omega$, $(Bk)\int_\gamma f = \mu(E \cap \gamma) \cdot w$. $\square$

Lemma 3. *Let $f : \Omega \to X$ be a function. Given $p \in \mathcal{P}(X)$ and $\varepsilon > 0$, there is a partition $\Gamma_p = (\sigma_i)$ of Ω into disjoint measurable sets such that*

$$\sum_{i=1}^{\infty} p(f(x_i))\mu(\sigma_i) \leq \overline{\int_\Omega} p(f(t))dt + \epsilon$$

for all $x_i \in \sigma_i$, where the integral in the last inequality is the upper Lebesgue integral.

Proof. Let $p \in \mathcal{P}(X)$. We can consider only the case $\overline{\int_\Omega} p(f(t))dt < \infty$, otherwise the inequality is obvious. Choose a real-valued function g on Ω such that $g(t) \geq p(f(t))$ for all t and $\int_\Omega g(t)dt = \overline{\int_\Omega} p(f(t))dt$. Since g is Lebesgue integrable for each fixed $\varepsilon > 0$, there is a decomposition $\Gamma_p = (\sigma_i)$ of Ω into disjoint measurable sets such that g is p-unconditionally summable on Γ_p. Moreover,

$$\Big|\sum_{i=1}^{\infty} g(t_i)\mu(\sigma_i) - \overline{\int_\Omega} p(f(t))dt\Big| < \varepsilon$$

whenever $t_i \in \sigma_i$. Therefore, we have

$$\sum_{i=1}^{\infty} p(f(t_i))\mu(\sigma_i) \leq \sum_{i=1}^{\infty} g(t_i)\mu(\sigma_i) \leq \overline{\int_\Omega} p(f(t))dt + \epsilon,$$

as required. □

Proposition 5. *Let $f : \Omega \to X$ be a function which is integrable by seminorm. Then it is Bk-integrable and the two integrals coincide.*

Proof. Choose $p \in \mathcal{P}(X)$ and fix $\epsilon > 0$. Let $\phi_p : \Omega \to X$ be a simple function such that

$$\int_\Omega p(f(t) - \phi_p(t))dt < \frac{\epsilon}{4}. \tag{17}$$

According to Lemma 2, the function ϕ_p is Bk-integrable. Thus, there is a partition $\Gamma_p^1 = (\sigma_i)$ of Ω such that

$$p\Big(\sum_{i=1}^{\infty} \mu(\sigma_i)\phi_p(t_i) - \int_\Omega \phi_p\Big) < \frac{\epsilon}{4} \tag{18}$$

for all $t_i \in \sigma_i$. Moreover, by Lemma 3 there is a partition $\Gamma_p^2 = (\sigma_i')$ such that

$$\sum_{i=1}^{\infty} p(f(t_i) - \phi_p(t_i))\mu(\sigma_i') \leq \int_\Omega p(f(t) - \phi_p(t))dt + \frac{\epsilon}{4} \tag{19}$$

for all $t_i \in \sigma_i'$. If $\Gamma_p = \Gamma_p^1 \cap \Gamma_p^2$, we get by (17), (18) and (19),

$$\begin{aligned}
&p\Big(\sum_{i=1}^{\infty} \mu(\sigma_i)f(t_i) - \int_\Omega f\Big) \leq p\Big(\sum_{i=1}^{\infty}(\mu(\sigma_i)f(t_i) - \mu(\sigma_i)\phi_p(t_i))\Big) \\
&+p\Big(\sum_{i=1}^{\infty} \mu(\sigma_i)\phi_p(t_i) - \int_\Omega \phi_p\Big) + p\Big(\int_\Omega \phi_p - \int_\Omega f\Big) \\
&\leq \int_\Omega p(f(t) - \phi_p(t))dt + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \int_\Omega p(f(t) - \phi_p(t)) < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon
\end{aligned}$$

for all $t_i \in \sigma_i$. Therefore, the Bk-integrability of f follows. □

Since a Bochner integrable function is integrable by seminorm, we get, as a consequence of the previous proposition, that each Bochner integrable function is Bk-integrable.

We see when a Bk-integrable function is integrable by seminorm.

Theorem 1. *Let $f : \Omega \to X$ be a function which is measurable by seminorm. Then f is integrable by seminorm if and only if f is Bk-integrable and for each $p \in \mathcal{P}(X)$, the real valued function $p(f(x))$ is Bk-integrable.*

Proof. Necessity has been proved in Proposition 5. To prove the converse, let $f : \Omega \to X$ be a measurable by seminorm function such that for each $p \in \mathcal{P}(X)$, the real valued function $p(f(x))$ is integrable. Then, the assertion follows by ([3], Theorem 2.10). □

We investigate now the relationship between the Bk-integral and the McShane integral. Since the McShane integral involves a topology, from now on $(\Omega, \mathcal{T}, \mathcal{F}, \mu)$ is a non-empty finite quasi-Radon measure space (see [7]).

A *generalized McShane partition* (or simply a *McS-partition*) in Ω is a countable (eventually finite) set of pairs $P = \{(E_i, \omega_i) : i = 1, 2, \dots\}$ where $(E_i)_i$ is a disjoint family of measurable sets of finite measure and $\omega_i \in \Omega$ for each $i = 1, 2 \dots$. If $\mu(\Omega \setminus \cup_i E_i) = 0$, we say that P is a *McS-partition of* Ω. A *gauge* on Ω is a function $\Delta : \Omega \to \mathcal{T}$ such that $\omega \in \Delta(\omega)$ for each $\omega \in \Omega$. We say that a McS-partition $\{(E_i, \omega_i) : i = 1, 2, \dots\}$ is *subordinate to a gauge* Δ if $E_i \subset \Delta(\omega_i)$ for $i = 1, 2, \dots$.

Definition 5. A function $f : \Omega \to X$ is said to be McShane integrable (briefly McS-integrable) on Ω if there exists a vector $w \in X$ satisfying the following property. Given $\varepsilon > 0$ and $p \in \mathcal{P}(X)$ there exists a gauge Δ_p on Ω such that for each McS-partition $P = \{(E_i, t_i) : i = 1, 2, \dots\}$ of Ω subordinate to Δ_p, we have

$$\limsup_{n\to\infty} p\Big(\sum_{i=1}^{n} \mu(E_i) f(t_i) - w\Big) < \varepsilon\ .$$

We denote by $McS(\Omega, X)$ the family of all McS-integrable functions on Ω, and we set $w = (McS)\int_\Omega f$. Definition 5 extends to locally convex spaces Definition 1A of [6]. Properties of the McS-integral for functions defined on a bounded subinterval of the real line and taking values in a locally convex space have been studied in [10].

For each $p \in \mathcal{P}(X)$, let X_p be the completion of the normed linear space $X/p^{-1}(0)$ and let i_p be the canonical mapping of X into X_p (see [14], 0.11.1).

Given a function $f : \Omega \to X$ and a seminorm $p \in \mathcal{P}(X)$, define the function $f_p : \Omega \to X_p$ by

$$f_p(t) = (i_p \circ f)(t) = i_p(f(t)).$$

Remark 2. We note that if $f : \Omega \to X$ is Bk-integrable then also $f_p : \Omega \to X_p$ is Bk-integrable. Indeed let z denote the Bk-integral of f. Fix $p \in \mathcal{P}(X)$ and $\varepsilon > 0$. Let $\Gamma_p = (\sigma_i)$ be a partition of Ω for which f is p-unconditionally summable on Γ_p, and

$$p\Big(\sum_{i=1}^{\infty} \mu(\sigma_i) f(t_i) - z\Big) < \varepsilon \tag{20}$$

for all $t_i \in \sigma_i$. Since

$$p\Big(\sum_{i=1}^{\infty} \mu(\sigma_i) i_p \circ f(t_i) - i_p(z)\Big) = p\Big(\sum_{i=1}^{\infty} \mu(\sigma_i) f(t_i) - z\Big),$$

from (20), we get for $t_i \in \sigma_i$,

$$p\Big(\sum_{i=1}^{\infty} \mu(\sigma_i) f_p(t_i) - i_p(z)\Big) < \varepsilon.$$

Theorem 2. *Let $f : \Omega \to X$ be a Bk-integrable function. Then the function f is McS-integrable and then Pettis integrable.*

Proof. By the previous remark, we get that for each $p \in \mathcal{P}(X)$ the function $f_p : \Omega \to X_p$ is Bk-integrable. Then, by [8] Proposition 4 f_p is McS-integrable with integral $(McS)\int_\Omega f_p = i_p((Bk)\int_\Omega f)$. Let $\varepsilon > 0$ be fixed. Then there is a gauge Δ_p such that if $P = \{(E_i, t_i) : i = 1, 2, \ldots\}$ is a McS-partition of Ω subordinate to Δ_p, we have for n big enough

$$p\Big(\sum_{i=1}^{n} \mu(E_i) f_p(t_i) - (McS)\int_\Omega f_p\Big) < \varepsilon. \tag{21}$$

Since

$$p\Big(\sum_{i=1}^{n} \mu(E_i) f_p(t_i) - (McS)\int_\Omega f_p\Big) = p\Big(i_p\Big(\sum_{i=1}^{n} \mu(E_i) f(t_i) - (Bk)\int_\Omega f\Big)\Big),$$

we obtain by (21),

$$p\Big(i_p\Big(\sum_{i=1}^{n} \mu(E_i) f(t_i) - (Bk)\int_\Omega f\Big)\Big) = p\Big(\sum_{i=1}^{n} \mu(E_i) f(t_i) - (Bk)\int_\Omega f\Big) < \varepsilon.$$

Therefore, f is McS-integrable. The proof that each McS-integrable function is Pettis integrable follows as [10] Theorem 2. □

In the case when the range X is a Banach space, each measurable Pettis integrable function is Birkhoff integrable (see [12] Corollary 5.11). For functions taking values in a locally convex space, an analogous result is true if we consider measurability by seminorm instead of measurability.

Theorem 3. *Let $f : \Omega \to X$ be a function which is Pettis integrable and measurable by seminorm. Then it is Bk-integrable and the two integrals coincide.*

Proof. Choose $p \in \mathcal{P}(X)$ an fix $\epsilon > 0$. According to ([1] Theorem 6), let $\phi_p : \Omega \to X$ be a simple function and X_0^p a null set such that

$$p(f(t) - \phi_p(t)) < \frac{\epsilon}{2\mu(\Omega)} \text{ for all } t \in \Omega \setminus X_0^p \tag{22}$$

and

$$p\Big(\int_\Omega (f(t) - \phi_p(t))dt\Big) < \frac{\epsilon}{2}. \tag{23}$$

Let (E_i) be disjoint measurable sets with $\cup E_i = \Omega \setminus X_0^p$ and $\phi_p(t) = y_i$ on E_i. Then, $\int_{\Omega \setminus X_0^p} \phi_p = \sum_i \mu(E_i) y_i$ and the series is p-convergent. Hence, since by (22) we have that for $s_i \in E_i$,

$$\sum_{i=1}^{\infty} p(\mu(E_i)(f(s_i) - y_i)) < \frac{\epsilon}{2}, \tag{24}$$

the series

$$\sum_{i=0}^{\infty} \mu(E_i) f(s_i) = \sum_{i=1}^{\infty} \mu(E_i) f(s_i) = \sum_{i=1}^{\infty} \mu(E_i)(f(s_i) - y_i) + \sum_{i=1}^{\infty} \mu(E_i) y_i$$

is p-convergent, being the sum of two such series. Therefore, applying (23) and (24), we get

$$\begin{aligned} &p\Big(\sum_{i=0}^{\infty} \mu(E_i) f(s_i) - \int_\Omega f\Big) = p\Big(\sum_{i=0}^{\infty} \mu(E_i) f(s_i) - \sum_{i=0}^{\infty} \mu(E_i) \phi_p(s_i)\Big) \\ &+p\Big(\sum_{i=1}^{\infty} \mu(E_i) \phi_p(s_i) - \int_\Omega f\Big) = p\Big(\sum_{i=0}^{\infty} \mu(E_i) f(s_i) - \sum_{i=0}^{\infty} \mu(E_i) y_i\Big) \\ &+p\Big(\sum_{i=1}^{\infty} \mu(E_i) y_i - \int_\Omega f\Big) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

Proposition 6. *Let $f : \Omega \to X$ be a measurable by seminorm function. Then f is McS-integrable if and only if it is Pettis integrable.*

Proof. If f is McS-integrable, by Theorem 2, it is Pettis integrable. The proof of the sufficiency part follows as in [11] Theorem 3, applying [6] Corollary 4C instead of [9] Theorem 17. □

Remark 3. In separable by seminorm spaces the Pettis integral, the McShane integral and the Bk-integral all are equivalent. Indeed, by the Pettis measurability Theorem ([3] Theorem 2.2) the Pettis integral and the Bk-integral coincide. The McShane integrability follows from Proposition 6.

5 The Bk-integral and the Property of Bourgain.

In [4], the Birkhoff integral for functions taking values in a Banach space is considered. In particular, it is compared with the Bourgain property of the set of compositions of the function with elements of the unit ball of the dual. We extend this result to a locally convex space.

Definition 6. ([4], Definition 2) A family $\mathcal{A}$ of real valued functions defined on Ω is said to satisfy the Birkhoff property if for every $\varepsilon > 0$ there is a partition $\Gamma = (\sigma_n)$ of Ω such that for each $x_k, y_k \in \sigma_k$, $k \in \mathbb{N}$, we have

$$\Big|\sum_{k=1}^{m} f(x_k)\mu(\sigma_k) - \sum_{k=1}^{m} f(y_k)\mu(\sigma_k)\Big| < \varepsilon,$$

for every $m \in \mathbb{N}$ and $f \in \mathcal{A}$.

In other words, a family $\mathcal{A}$ of real valued functions satisfies the Birkhoff property if Cauchy criterion is satisfied uniformly for every function of the set $\mathcal{A}$.

To simplify the notation, in the following we write $|x^*| \leq p$ instead of $|x^*(x)| \leq p(x)$ for each $x \in X$, and we let $X_p^* = \{x^* \in X^* : |x^*| \leq p\}$. We recall that a seminorm $p \in \mathcal{P}(X)$ is called *representable* if

$$p(x) = \sup_{X_p^*} |x^*(x)| \tag{25}$$

for all $x \in X$. If equality (25) holds for all $p \in \mathcal{P}(X)$, the space X is said to be *representable by seminorm* (see [5], p. 185). If a space X is separable by seminorm, then it is representable by seminorm. We characterize the Bk-integrability of $f : \Omega \to X$ in terms of the Birkhoff property of the set of composition of f with elements of X_p^*;

$$Z_f^p = \{x^* f : x^* \in X_p^*\}.$$

Proposition 7. *Let X be a representable by seminorm locally convex space and $f : \Omega \to X$ be a function. The following conditions are equivalent:*

(i) *f is Bk-integrable;*

(ii) *for each seminorm p, f is p-unconditionally summable with respect to some countable partition Γ_p^0 of Ω and Z_f^p has the Birkhoff property.*

Proof. Assume that f is Bk-integrable. Let $p \in \mathcal{P}(X)$ and $\varepsilon > 0$ be fixed. Then, the function $f_p : \Omega \to X_p$ is Bk-integrable. Let $x^* \in X_p^*$. By ([4], Proposition 2.2), we have

$$\begin{aligned}
&\Big|\sum_{k=1}^{m} x^* f(x_k)\mu(\sigma_k) - \sum_{k=1}^{m} x^* f(y_k)\mu(\sigma_k)\Big| \\
=&\Big|x^*\Big(\sum_{k=1}^{m} f(x_k)\mu(\sigma_k) - \sum_{k=1}^{m} f(y_k)\mu(\sigma_k)\Big)\Big| \\
\leq& p\Big(\sum_{k=1}^{m} \mu(\sigma_k)f(x_k) - \sum_{k=1}^{m} \mu(\sigma_k)f(y_k)\Big) < \varepsilon.
\end{aligned}$$

Conversely, fix $p \in \mathcal{P}(X)$ and $\varepsilon > 0$. Since Z_f^p has the Birkhoff property, there is a partition $\Gamma_p^1 = (\sigma_i)$ such that for all $x_k, y_k \in \sigma_k$ and all $m \in \mathbb{N}$,

$$\Big|\sum_{k=1}^{m} x^* f(x_k)\mu(\sigma_k) - \sum_{k=1}^{m} x^* f(y_k)\mu(\sigma_k)\Big| < \frac{\varepsilon}{2}$$

for every $m \in \mathbb{N}$ and $x^* \in X_p^*$. If $\Gamma_p = (\alpha_k)$ is finer than Γ_p^0 and Γ_p^1, we get that f is p-unconditionally summable with respect to Γ_p, and for all $x^* \in X_p^*$,

$$\begin{aligned}
\Big|\sum_{k=1}^{s}(x^* f(x_k) - x^* f(y_k))\mu(\alpha_k)\Big| &\leq \Big|\sum_{i=1}^{m} \sum_{\alpha_k \subset \sigma_i} (x^* f(x_k) - x^* f(y_k))\mu(\alpha_k)\Big| \\
&\leq \sum_{i=1}^{m} \Big|x^* f(x_i) - x^* f(y_i)\Big|\mu(\sigma_i) < \varepsilon.
\end{aligned}$$

Since the space X is representable by seminorm, we get

$$p\Big(\sum_{k=1}^{s} \mu(\alpha_k)f(x_k) - \sum_{k=1}^{s} \mu(\alpha_k)f(y_k)\Big) < \varepsilon.$$

Therefore, by Proposition 1, it follows that f is Bk-integrable. $\square$

Definition 7. (see [15], Definition 10) A family $\mathcal{A}$ of real valued functions defined on Ω is said to satisfy the Bourgain property if for each set K of positive measure and for every $\varepsilon > 0$ there is a finite collection F of subsets of positive measure of K such that for every $f \in \mathcal{A}$, the inequality

$$\sup f(B) - \inf f(B) < \varepsilon$$

holds for some member $B \in F$.

In order to link the Bk-integrability with the Bourgain property of the set Z_f^p, we need the following lemma.

Lemma 4 ([4] Lemma 2.3)**.** *Let $\mathcal{A}$ be a family of real valued functions defined on Ω. The following statements hold:*

(i) *if $\mathcal{A}$ has the Birkhoff property, then $\mathcal{A}$ has the Bourgain property;*

(ii) *if $\mathcal{A}$ is uniformly bounded and has the Bourgain property, then $\mathcal{A}$ has the Birkhoff property.*

Theorem 4. *Let X be a representable by seminorm locally convex space and $f : \Omega \to X$ be a bounded function. Then the following conditions are equivalent:*

(i) *f is Bk-integrable;*

(ii) *for each seminorm p, f is p-unconditionally summable with respect to some countable partition Γ_p^0 of Ω and Z_f^p has the Bourgain property.*

Proof. Let $p \in \mathcal{P}(X)$. If $f : \Omega \to X$ is a bounded function, then the set $Z_f^p = \{x^* f : x^* \in X_p^*\}$ is a family of uniformly bounded functions. If f is Bk-integrable, by Proposition 7 the set $Z_f^p = \{x^* f : x^* \in X_p^*\}$ has the Birkhoff property, therefore by Lemma 4, it has the Bourgain property. Conversely if the set $Z_f^p = \{x^* f : x^* \in X_p^*\}$ has the Bourgain property, applying again Lemma 4 we get that Z_f^p has the Birkhoff property, and the assertion follows by Proposition 7. $\square$

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