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STRONG SUPMEASURABILITY OF FUNCTIONS OF TWO VARIABLES WHOSE VERTICAL SECTIONS ARE PREPONDERANTLY CONTINUOUS

Abstract

We show that functions of two variables whose vertical sections are preponderantly continuous are strongly supmeasurable.

Let $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. Let $(X, \mathcal{M})$ be a measurable space and let $\mathcal{I} \subset \mathcal{M}$ be a proper σ-ideal of subsets of X. Let Z be a Banach space.

Let $h\colon X \to Z$. Recall that h is *measurable* if $h^{-1}(U) \in \mathcal{M}$ for every open set $U \subset Z$. In [3], I introduced the following two kinds of measurability of a function. We say that h is *nearly simple* if it is measurable and its range is at most countable. We say that h is *strongly measurable* with respect to $(\mathcal{M}, \mathcal{I})$ if there exists a sequence of nearly simple functions (h_n) and a set $A \in \mathcal{I}$ such that $h_n \to h$ on $X \setminus A$.

Clearly strongly measurable functions are measurable. Easy examples show that the converse implication does not hold.

Remark. Usually in functional analysis (see, e.g., [13]), we consider another kind of strong measurability. Namely, we require that there exists a sequence of simple functions (h_n) (i.e., measurable functions with finite range) and a set $A \in \mathcal{I}$ such that $h_n \to h$ on $X \setminus A$. If Z is a separable (in particular,

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finitely dimensional) Banach space, then this notion is equivalent to the measurability, and consequently, to the strong measurability defined above. These two notions of strong measurability are equivalent to each other (but not to the notion of measurability!) also in the case that there is a σ-finite measure $\mu\colon \mathcal{M} \to [0,\infty]$ such that $\mathcal{I}$ is the σ-ideal of μ-null sets [5, Proposition 1]. However, I do not know whether the two notions of strong measurability coincide in the general case.

Recall also the following generalization of Davies' lemma [6]. (See [3, Proposition 6].)

Proposition 1. *Assume that* $(\mathcal{M},\mathcal{I})$ *satisfies condition* (ccc); *i.e., each family of pairwise disjoint elements of* $\mathcal{M}\setminus\mathcal{I}$ *is at most countable. If a function* $h\colon X\to Z$ *satisfies the following condition*:

$$\text{for each } \varepsilon>0 \text{ and each } A\in\mathcal{M}\setminus\mathcal{I} \text{ there is a set } B\in\mathcal{M}\setminus\mathcal{I} \text{ with } B\subset A \text{ such that } \operatorname{osc}_B h\le\varepsilon \tag{1}$$

(*where* $\operatorname{osc}_B h=\sup\bigl\{\|h(x)-h(y)\|\colon x,y\in B\bigr\}$), *then* h *is strongly measurable.*

Let $(Y,\mathcal{N},\nu)$ be a measure space. We assume that we have a net structure $\mathcal{J}$ in $(Y,\mathcal{N},\nu)$ (see, e.g., [1]). Recall that a *net* in Y is an at most countable cover of Y consisting of pairwise disjoint measurable sets of positive finite measure. The individual sets in the net are called *cells*. A family $\mathcal{J}=\bigcup_{n=1}^{\infty}\mathcal{J}_n$, where each $\mathcal{J}_n$ is a net, is called a *net structure*. Observe that for each $y\in Y$ and $n\in\mathbb{N}$, there is a unique cell from the net $\mathcal{J}_n$ which contains y. We will denote this cell by $J_n(y)$. Several examples of net structures can be found, e.g., in [9] or [11].

In most theorems, I consider functions from the Cartesian product $X\times Y$ into Z. We will consider the product σ-ideal $\widetilde{\mathcal{I}}$ in $X\times Y$ (the family of all subsets of the sets of the form $A\times Y$, where $A\in\mathcal{I}$) and the product σ-field $\mathcal{M}\otimes\mathcal{N}$ (the smallest σ-field containing the σ-ideal $\widetilde{\mathcal{I}}$ and the family of all sets of the form $M\times N$, where $M\in\mathcal{M}$ and $N\in\mathcal{N}$). The strong measurability of a function $h\colon X\times Y\to Z$ can be defined analogously as in the above case; i.e., a function $f\colon X\times Y\to Z$ is strongly measurable if there exists a sequence of nearly simple functions (f_n) from $X\times Y$ into Z and a set $A\in\mathcal{I}$ such that $f_n\to f$ on $(X\setminus A)\times Y$.

Let $f\colon X\times Y\to Z$. For each $x\in X$, the function $f_x\colon Y\to Z$ defined by $f_x(y)=f(x,y)$ is called a *vertical section* of f. Similarly, for each $y\in Y$, the function $f^y\colon X\to Z$ defined by $f^y(x)=f(x,y)$ is called a *horizontal section* of f.

The notion of supmeasurability of functions from the Cartesian product $X\times Y$ into Z needs some kind of measurability of functions from X into Y.

In general, we have no topological structure in Y. Therefore we will use the net structure. One can easily see that for real functions defined on $\mathbb{R}$, the notion of supmeasurability defined below (for the net composed of half-open dyadic intervals) coincides with the classical one.

A function $\varphi\colon X \to Y$ is called $(\mathcal{M}, \mathcal{J})$-*measurable* if for each $J \in \mathcal{J}$, the inverse image $\varphi^{-1}(J) \in \mathcal{M}$. A function $f\colon X \times Y \to Z$ is *strongly supmeasurable* [5] (i.e., superpositionally measurable) if for each $(\mathcal{M}, \mathcal{J})$-measurable function $\varphi\colon X \to Y$, the superposition $x \mapsto f\big(x, \varphi(x)\big)$ is strongly measurable.

In this article, I examine the properties which guarantee the strong supmeasurability of functions of two variables. A classical theorem due to C. Carathéodory [2] states that if all vertical sections of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ are continuous, and all horizontal sections of f are Lebesgue measurable, then f is Lebesgue supmeasurable. In 1978, Z. Grande proved that each Lebesgue measurable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ all of whose vertical sections are approximately continuous is Lebesgue supmeasurable [8].

For the Baire property, we have a different case. E. Grande and Z. Grande showed in 1984 that there is a function $f\colon \mathbb{R}^2 \to [0,1]$, which is Lebesgue measurable and possesses the Baire property, all of whose vertical sections are approximately continuous (so they are derivatives), all of whose horizontal sections possess the Baire property, and which is not Baire supmeasurable [7].

To generalize the other theorem of Z. Grande, we need the following notion. We say that a function $g\colon Y \to Z$ is *preponderantly continuous* (in the sense of Denjoy) if g is measurable, and for each $y \in Y$ and each open set $U \ni g(y)$, we have

$$\liminf_{n\to\infty} \frac{\nu\big(g^{-1}(U) \cap J_n(y)\big)}{\nu\big(J_n(y)\big)} > \frac{1}{2}.$$

We say that a function $g\colon Y \to Z$ is *preponderantly continuous* (in the sense of O'Malley, cf. [10]) if g is measurable, and for each $y \in Y$ and each open set $U \ni g(y)$, we have

$$\frac{\nu\big(g^{-1}(U) \cap J_n(y)\big)}{\nu\big(J_n(y)\big)} > \frac{1}{2} \text{ for sufficiently large } n.$$

One can easily see that the the latter notion of preponderant continuity is weaker than the former one.

In 1982, W. Ślęzak proved that if all vertical sections of a Lebesgue measurable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ are preponderantly continuous in the sense of Denjoy (in the classical sense), then f is Lebesgue supmeasurable [12]. We will prove an analogous result in more general case. In the proof we will use the following lemma.

Lemma 2. *Let* $\alpha \geq 0$ *and let* E *be an* $\mathcal{M} \otimes \mathcal{N}$*-measurable set. If a function* $\varphi\colon X \to Y$ *is* $(\mathcal{M}, \mathcal{J})$*-measurable, then the set*

$$F = \left\{x \in X\colon E_x \in \mathcal{N}\ \&\ \frac{\nu\big(E_x \cap J_n(\varphi(x))\big)}{\nu\big(J_n(\varphi(x))\big)} > \alpha \text{ for sufficiently large } n\right\}$$

is $\mathcal{M}$*-measurable.*

(For each $x \in X$, we define $E_x = \{y \in Y\colon (x,y) \in E\}$; i.e., E_x is the *vertical section* of E.)

PROOF. Let $I = \{x \in X\colon E_x \notin \mathcal{N}\}$. Notice that by Fubini's theorem, we have $I \in \mathcal{I}$. One can easily see that $F = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} F_n$, where

$$F_n = \left\{x \in X \setminus I\colon \nu\big(E_x \cap J_n(\varphi(x))\big) > \alpha\nu\big(J_n(\varphi(x))\big)\right\}$$

$(n \in \mathbb{N})$. So, it suffices to show that $F_n \in \mathcal{M}$ for each n.

Fix an $n \in \mathbb{N}$. Let

$$D = \big\{(x,y) \in X \times Y\colon y \in J_n(\varphi(x))\big\}.$$

Since the function φ is $(\mathcal{M}, \mathcal{J})$-measurable, we have

$$\begin{aligned} D &= \bigcup_{J \in \mathcal{J}_n} \big\{(x,y) \in X \times Y\colon y \in J, J_n(\varphi(x)) = J\big\} \\ &= \bigcup_{J \in \mathcal{J}_n} \big\{(x,y) \in X \times Y\colon y \in J, \varphi(x) \in J\big\} = \bigcup_{J \in \mathcal{J}_n} \big(\varphi^{-1}(J) \times J\big) \in \mathcal{M} \otimes \mathcal{N}. \end{aligned}$$

So, the set $C = E \cap D \setminus (I \times Y)$ is $\mathcal{M} \otimes \mathcal{N}$-measurable, too. Observe that for each $x \in X$, the set

$$C_x = \begin{cases} E_x \cap J_n(\varphi(x)) & \text{if } x \notin I, \\ \emptyset & \text{otherwise} \end{cases}$$

is $\mathcal{N}$-measurable. Since the measure ν is σ-finite, we can use Fubini's theorem and conclude that the mapping $x \mapsto \nu\big(C_x\big)$ is $\mathcal{M}$-measurable. Hence,

$$\begin{aligned} F_n &= \bigcup_{J \in \mathcal{J}_n} \big\{x \in X \setminus I\colon \nu\big(C_x\big) > \alpha\nu\big(J_n(\varphi(x))\big)\ \&\ \varphi(x) \in J\big\} \\ &= \bigcup_{J \in \mathcal{J}_n} \big(\{x \in X\colon \nu\big(C_x\big) > \alpha\nu(J)\} \cap \varphi^{-1}(J)\big) \setminus I \in \mathcal{M}. \end{aligned}$$

□

Theorem 3. *Assume that* $(\mathcal{M},\mathcal{I})$ *satisfies condition* (ccc). *Let* $f\colon X\times Y\to Z$ *be a strongly measurable function with the property that* $\mathcal{I}$*-almost all vertical sections of* f *are preponderantly continuous in the sense of O'Malley. Then* f *is strongly supmeasurable.*

Proof. Let $\varphi\colon X\to Y$ be an $(\mathcal{M},\mathcal{J})$-measurable function. Define the function $h\colon X\to Z$ by $h(x)=f\big(x,\varphi(x)\big)$. To show that h is strongly measurable, we will use Proposition 1.

Fix $\varepsilon>0$ and $A\in\mathcal{M}\setminus\mathcal{I}$. We may assume that for each $x\in A$, the section f_x is preponderantly continuous. Since f is strongly measurable, there is an $I\in\mathcal{I}$ such that the set $f\big((X\setminus I)\times Y\big)$ is a separable subspace of Z (cf. [3, Corollary 3]). Let $\{z_k\colon k\in\mathbb{N}\}$ be a countable set dense in $f\big((X\setminus I)\times Y\big)$.

For each $k\in\mathbb{N}$ define $U_k=B(z_k,\varepsilon/4)$ (i.e., U_k is the open ball with center z_k and radius $\varepsilon/4$) and $A_k=A\cap h^{-1}(U_k)$. Since $f\big((X\setminus I)\times Y\big)\subset\bigcup_{k=1}^{\infty}U_k$, we have $A\setminus\bigcup_{k=1}^{\infty}A_k\subset I\in\mathcal{I}$. But $A\notin\mathcal{I}$, so there is a $k\in\mathbb{N}$ such that $A_k\notin\mathcal{I}$.

Choose any $x_0\in A_k$ and put

$$E=f^{-1}\big(B(h(x_0),\varepsilon/2)\big)\in\mathcal{M}\otimes\mathcal{N}.$$

By Lemma 2, the set

$$F=\Big\{x\in X\colon \frac{\nu\big(E_x\cap J_n(\varphi(x))\big)}{\nu\big(J_n(\varphi(x))\big)}>\frac{1}{2}\ \text{ for sufficiently large } n\Big\}$$

is $\mathcal{M}$-measurable. (Observe that each section E_x is $\mathcal{N}$-measurable.) We will show that $F\supset A_k$. Indeed, let $x\in A_k$. Then $h(x)=f_x\big(\varphi(x)\big)\in U_k$. Since the section f_x is preponderantly continuous at $\varphi(x)$, we have

$$\frac{\nu\big(f_x^{-1}(U_k)\cap J_n(\varphi(x))\big)}{\nu\big(J_n(\varphi(x))\big)}>\frac{1}{2}\ \text{for sufficiently large } n.$$

If $y\in f_x^{-1}(U_k)$, then

$$\big\|f(x,y)-f\big(x_0,\varphi(x_0)\big)\big\|\le\big\|f_x(y)-z_k\big\|+\big\|z_k-f\big(x_0,\varphi(x_0)\big)\big\|<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2},$$

whence $(x,y)\in E$. Consequently, $f_x^{-1}(U_k)\subset E_x$ and $x\in F$.

Let $B=A\cap F$. Then $B\in\mathcal{M}$ and $B\supset A_k\notin\mathcal{I}$, whence $B\notin\mathcal{I}$. To complete the proof suppose that $\operatorname{osc}_B h>\varepsilon$. Then

$$\big\|h(x_1)-h(x_0)\big\|=\big\|f\big(x_1,\varphi(x_1)\big)-h(x_0)\big\|>\frac{\varepsilon}{2}$$

for some $x_1 \in B$, whence

$$\varphi(x_1) \in f_{x_1}^{-1}\bigl(Z \setminus \operatorname{cl} B\bigl(h(x_0), \varepsilon/2\bigr)\bigr),$$

where cl stands for the closure operator. Since $x_1 \in F$, we have

$$\nu\bigl(E_{x_1} \cap J_n(\varphi(x_1))\bigr) > \nu\bigl(J_n(\varphi(x_1))\bigr)/2 \text{ for sufficiently large } n.$$

On the other hand, since the section f_{x_1} is preponderantly continuous, we obtain

$$\nu\bigl(f_{x_1}^{-1}(Z \setminus \operatorname{cl} B(h(x_0), \varepsilon/2)) \cap J_n(\varphi(x_1))\bigr) > \nu\bigl(J_n(\varphi(x_1))\bigr)/2$$

for sufficiently large n. Using the two preceding inequalities, we can choose a point

$$y \in E_{x_1} \cap f_{x_1}^{-1}\bigl(Z \setminus \operatorname{cl} B\bigl(h(x_0), \varepsilon/2\bigr)\bigr).$$

Then $(x_1, y) \in E$, so

$$f(x_1, y) \in B\bigl(h(x_0), \varepsilon/2\bigr) \cap \bigl(Z \setminus \operatorname{cl} B\bigl(h(x_0), \varepsilon/2\bigr)\bigr) = \emptyset,$$

an impossibility. This contradiction completes the proof. □

From the following theorem due to E. Grande and Z. Grande [7] it follows that the assumptions in the above theorem are hard to weaken.

Theorem 4. *There is a Lebesgue measurable function $f\colon \mathbb{R}^2 \to \mathbb{R}$, which is not Lebesgue supmeasurable, such that for all $x, y \in \mathbb{R}$ and each open set $U \ni f(x,y)$, we have*

$$\liminf_{\delta \to 0^+} \frac{\lambda\bigl(\operatorname{int}\bigl((y-\delta, y+\delta) \cap f_x^{-1}(U)\bigr)\bigr)}{2\delta} \geq \frac{1}{2}.$$

(The letter λ denotes the Lebesgue measure in $\mathbb{R}$, and the symbol int stands for the interior operator.)

Now we turn our attention to real functions. Another interesting generalization of Z. Grande's theorem was given by W. Ślęzak [12]. His condition can be described as $\frac{3}{4}$-upper and lower preponderant semicontinuity in the sense of Denjoy of the vertical sections of f.

Theorem 5. *Assume that a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is Lebesgue measurable, and for all $x, y \in \mathbb{R}$ and each $a \in \mathbb{R}$,*

- *if $f(x,y) < a$, then* $\displaystyle\liminf_{\delta \to 0^+} \frac{\lambda\bigl(f_x^{-1}((-\infty, a)) \cap (y-\delta, y+\delta)\bigr)}{2\delta} > \frac{3}{4}$;

- *if* $f(x,y) > a$, *then* $\liminf_{\delta\to 0^+} \dfrac{\lambda\big(f_x^{-1}((a,\infty)) \cap (y-\delta, y+\delta)\big)}{2\delta} > \dfrac{3}{4}$.

Then f *is Lebesgue supmeasurable.*

My next goal is to prove that the fractions $\frac{3}{4}$ can be diminished to $\frac{2}{3}$. This result is new even for functions defined on $\mathbb{R}^2$.

Theorem 6. *Assume that* $(\mathcal{M},\mathcal{I})$ *satisfies condition* (ccc). *Let* $f\colon X\times Y \to \mathbb{R}$ *be a measurable function with the property that there exists an* $I \in \mathcal{I}$ *such that for each* $x \in X\setminus I$, *the following conditions are satisfied*:

- *the section* f_x *is* $\mathcal{N}$*-measurable*;
- *if* $y \in Y$, $a \in \mathbb{R}$, *and* $f(x,y) < a$, *then* $\dfrac{\nu\big(f_x^{-1}((-\infty,a)) \cap J_n(y)\big)}{\nu\big(J_n(y)\big)} > \dfrac{2}{3}$ *for sufficiently large* n;
- *if* $y \in Y$, $a \in \mathbb{R}$, *and* $f(x,y) > a$, *then* $\dfrac{\nu\big(f_x^{-1}((a,\infty)) \cap J_n(y)\big)}{\nu\big(J_n(y)\big)} > \dfrac{2}{3}$ *for sufficiently large* n.

Then f *is supmeasurable.*

Proof. Let $\varphi\colon X \to Y$ be an $(\mathcal{M},\mathcal{J})$-measurable function. Define the function $h\colon X \to \mathbb{R}$ by $h(x) = f\big(x,\varphi(x)\big)$. To show that h is strongly measurable, we will use Proposition 1.

Fix $\varepsilon > 0$ and $A \in \mathcal{M}\setminus\mathcal{I}$. Clearly we may assume that $A \cap I = \emptyset$. Arrange all open intervals with rational endpoints, whose length is less than $\varepsilon/2$, in a sequence, say (U_k). For each $k \in \mathbb{N}$ define $A_k = A \cap h^{-1}(U_k)$. Since $A = \bigcup_{k=1}^{\infty} A_k \notin \mathcal{I}$, there is a $k \in \mathbb{N}$ such that $A_k \notin \mathcal{I}$. Let $U_k = (a_1, a_2)$. Choose any $x_0 \in A_k$ and put

$$E = f^{-1}\big(\big(h(x_0) - \varepsilon/2, h(x_0) + \varepsilon/2\big)\big) \in \mathcal{M}\otimes\mathcal{N}.$$

By Lemma 2, the set

$$F = \Big\{x \in X\colon \frac{\nu\big(E_x \cap J_n(\varphi(x))\big)}{\nu\big(J_n(\varphi(x))\big)} > \frac{1}{3} \text{ for sufficiently large } n\Big\}$$

is $\mathcal{M}$-measurable. (Notice that each section E_x is $\mathcal{N}$-measurable.) We will show that $F \supset A_k$.

Indeed, let $x \in A_k$. Then

$$h(x) = f_x\big(\varphi(x)\big) \in U_k = (a_1,\infty) \cap (-\infty, a_2).$$

By assumptions,

$$\frac{\nu\big(f_x^{-1}((a_1,\infty)) \cap J_n(\varphi(x))\big)}{\nu\big(J_n(\varphi(x))\big)} > \frac{2}{3} \text{ and } \frac{\nu\big(f_x^{-1}((-\infty,a_2)) \cap J_n(\varphi(x))\big)}{\nu\big(J_n(\varphi(x))\big)} > \frac{2}{3},$$

whence

$$\frac{\nu\big(f_x^{-1}(U_k)\cap J_n(\varphi(x))\big)}{\nu\big(J_n(\varphi(x))\big)} > \frac{1}{3}.$$

If $y \in f_x^{-1}(U_k)$, then $\big|f(x,y)-h(x_0)\big| < \varepsilon/2$, whence $(x,y)\in E$. Consequently, $f_x^{-1}(U_k)\subset E_x$ and $x\in F$.

Let $B = A\cap F$. Then $B\in\mathcal{M}$ and $B\supset A_k\notin\mathcal{I}$, whence $B\notin\mathcal{I}$. To complete the proof suppose that $\operatorname{osc}_B h > \varepsilon$. Then

$$\big|h(x_1)-h(x_0)\big| = \big|f\big(x_1,\varphi(x_1)\big)-h(x_0)\big| > \frac{\varepsilon}{2}$$

for some $x_1\in B$. Assume that $h(x_1) > h(x_0)+\varepsilon/2$. (The other case is analogous.) Then

$$\varphi(x_1)\in f_{x_1}^{-1}\big((h(x_0)+\varepsilon/2,\infty)\big).$$

Since $x_1\in F$, we have

$$\frac{\nu\big(E_{x_1}\cap J_n(\varphi(x_1))\big)}{\nu\big(J_n(\varphi(x_1))\big)} > \frac{1}{3} \text{ for sufficiently large } n.$$

On the other hand, by assumption, we have

$$\frac{\nu\big(f_{x_1}^{-1}\big((h(x_0)+\varepsilon/2,\infty)\big)\cap J_n(\varphi(x_1))\big)}{\nu\big(J_n(\varphi(x_1))\big)} > \frac{2}{3}$$

for sufficiently large n. Using the two above inequalities, we can choose a point

$$y\in E_{x_1}\cap f_{x_1}^{-1}\big((h(x_0)+\varepsilon/2,\infty)\big).$$

Then $(x_1,y)\in E$, so $\big|f(x_1,y)-h(x_0)\big| < \varepsilon/2$. But $f_{x_1}(y) > h(x_0)+\varepsilon/2$, an impossibility. This contradiction completes the proof. □

Problem. Does there exist a Lebesgue measurable function $f\colon\mathbb{R}^2\to\mathbb{R}$, which is not Lebesgue supmeasurable, and such that:

- all vertical sections f_x are Lebesgue measurable;
- if $y\in Y$, $a\in\mathbb{R}$, and $f(x,y)<a$, then

$$\liminf_{n\to\infty}\frac{\lambda\big(f_x^{-1}((-\infty,a))\cap J_n(y)\big)}{\nu\big(J_n(y)\big)} \geq \frac{2}{3};$$

- if $y\in Y$, $a\in\mathbb{R}$, and $f(x,y)>a$, then

$$\liminf_{n\to\infty}\frac{\lambda\big(f_x^{-1}((a,\infty))\cap J_n(y)\big)}{\nu\big(J_n(y)\big)} \geq \frac{2}{3}?$$

The following theorem due to E. Grande and Z. Grande [7] shows that even the approximate upper semicontinuity of all sections of a measurable function does not imply its supmeasurability.

Theorem 7. *There is a Lebesgue measurable function* $f\colon \mathbb{R}^2 \to \mathbb{R}$*, which is not Lebesgue supmeasurable, though all its horizontal sections are Borel measurable, and it satisfies the following condition*:

> *for each* $y \in \mathbb{R}$ *and each* $\varepsilon > 0$*, there is an open set* $U \subset \mathbb{R}$*, whose density at* y *is equal to* 1*, such that* $f(x,v) - f(x,y) < \varepsilon$ *for all* $x \in \mathbb{R}$ *and* $v \in U$*.*

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