Real Analysis Exchange
Vol. 28(2), 2002/2003, pp. 269–279

Józef Myjak, Dipartimento di Matematica Pura ed Applicata, Università di L'Aquila, Via Vetoio, 67100 L'Aquila, Italy. e-mail: `myjak@univaq.it`

Ryszard Rudnicki*, Institute of Mathematics, Polish Academy of Sciences and Institute of Mathematics, Silesian University, Bankowa 14, 40-007 Katowice, Poland. e-mail: `rudnicki@us.edu.pl`

TYPICAL PROPERTIES OF CORRELATION DIMENSION

Abstract

Let (X, ρ) be a complete separable metric space and $\mathcal{M}$ be the set of all probability Borel measures on X. We show that if the space $\mathcal{M}$ is equipped with the weak topology, the set of measures having the upper (resp. lower) correlation dimension zero is residual. Moreover, the upper correlation dimension of a typical (in the sense of Baire category) measure is estimated by means of the local lower entropy and local upper entropy dimensions of X.

1 Introduction

The correlation dimension introduced by Procaccia, Grassberger and Hentschel [9] is frequently used in the theory of dynamical systems. A rigorous mathematical treatment of this dimension was given by Pesin [5]. For further results see [1, 3, 4, 6, 7, 8, 10, 11].

In this note we investigate some typical properties of the correlation dimension. Recall that a set in a metric space is called nowhere dense if its closure has empty interior. A countable union of nowhere dense sets is said to be of the first Baire category. A subset A of a complete metric space X is said to be residual in X if its complement is of the first Baire category. If the set of all elements of X satisfying some property P is residual in X, then the property P is called typical or generic. We also say that a typical element of X has property P.

Key Words: measure, dimension, residual subset
Mathematical Reviews subject classification: Primary: 28A80; Secondary: 54E52
Received by the editors March 15, 2001
*This research was supported by the State Committee for Scientific Research (Poland) Grant No. 2 P03A 010 16 (RR).

Let $\mathcal{M}$ be the space of all probability Borel measures on a complete separable metric space X. We show that a typical measure in the space $\mathcal{M}$ endowed with the strong topology has upper correlation dimension zero. If the space $\mathcal{M}$ is endowed with the weak topology, then a typical measure has lower correlation dimension zero and upper correlation dimension no smaller than the smallest local lower entropy dimension of X and no greater than the smallest local upper entropy dimension of X.

These results are in the spirit of that by Gruber [2], who studied typical properties of entropy dimension of compact subsets of X. Namely, he considered the space $\mathcal{C}$ of all compact subsets of X equipped with the Hausdorff metric. He proved that a typical compact subset of X has lower entropy dimension zero. He also proved that if the compact subsets of X having lower entropy dimension at least δ are dense in $\mathcal{C}$, then a typical compact subset of X has upper dimension at least δ.

The paper is divided into three sections. In Section 2 we formulate the main results. Section 3 contains the proofs. In Section 4 we present two examples which show that the estimation of the upper correlation dimension given in Section 2 cannot be improved.

2 Main Result

Let (X,ρ) be a complete separable metric space and let $B(x,r)$ denote the open ball in X with center at x and radius $r>0$. By $\mathcal{B}$ we denote the σ-algebra of Borel subsets of X and by $\mathcal{M}$ we denote the set of all probability Borel measures on X.

For $\mu_1,\mu_2\in\mathcal{M}$ we consider the distance d_1 given by the *supremum norm*; i.e.,

$$d_1(\mu_1,\mu_2)=\sup_{A\in\mathcal{B}}|\mu_1(A)-\mu_2(A)|$$

and the *Fortet-Mourier distance* d_2 given by the formula

$$d_2(\mu_1,\mu_2)=\sup\Big\{\Big|\int_X f(x)\,d\mu_1(x)-\int_X f(x)\,d\mu_2(x)\Big| : f\in\mathcal{L}\Big\},$$

where $\mathcal{L}$ is the subset of $C(X)$ which contains all the functions f such that $|f(x)|\le 1$ and $|f(x)-f(y)|\le\rho(x,y)$ for $x,y\in X$. It can be proved that the sequence (μ_n), $\mu_n\in\mathcal{M}$, is weakly convergent to a measure $\mu\in\mathcal{M}$ if and only if $\lim_{n\to\infty}d_2(\mu_n,\mu)=0$. It is well known that the spaces $(\mathcal{M},d_1)$ and $(\mathcal{M},d_2)$ are complete.

Let $\mu\in\mathcal{M}$. The quantities

$$\overline{\dim}_c\,\mu=\overline{\lim}_{r\to 0}\frac{1}{\log r}\log\int_X \mu(B(x,r))\,d\mu(x)$$

and

$$\underline{\dim}_c \mu = \underline{\lim}_{r\to 0} \frac{1}{\log r} \log \int_X \mu(B(x,r))\, d\mu(x)$$

are called the *upper* and *lower correlation dimension* of μ, respectively. From the definition of the upper correlation dimension it follows immediately that if $\mu(\{x\}) > 0$ for some $x \in X$, then $\overline{\dim}_c \mu = 0$.

Finally we recall that the *upper* and *lower entropy dimensions* of a set $K \subset X$ are defined, respectively, by the formulae

$$\overline{\dim}\, K = \limsup_{r\to 0^+} \frac{\log N(K,r)}{\log(1/r)} \text{ and } \underline{\dim}\, K = \liminf_{r\to 0^+} \frac{\log N(K,r)}{\log(1/r)},$$

where $N(K,r)$ is the least number of balls of radius r which cover the set K. Note that if the set K is closed and non-compact, then $\overline{\dim}\, K = \underline{\dim}\, K = \infty$.

Remark 1. In the definitions of entropy dimensions we can replace the number $N(K,r)$ by

$$M(K,r) = \sup\{\operatorname{card} F : F \subset K \text{ and } \rho(x,y) \geq r \text{ for every } x,\, y \in F,\, x \neq y\}.$$

Now we are ready to formulate our main result.

Theorem 1. *Let* $\alpha = \inf\{\underline{\dim}\, B(x,a) : x \in X,\, a > 0\}$ *and* $\beta = \inf\{\overline{\dim}\, B(x,a) : x \in X,\, a > 0\}$. *Then*

(a) the set $\mathcal{M}^0 = \{\mu \in \mathcal{M} : \overline{\dim}_c \mu = 0\}$ *is residual in the space* $(\mathcal{M}, d_1)$,

(b) the set $\mathcal{M}_0 = \{\mu \in \mathcal{M} : \underline{\dim}_c \mu = 0\}$ *is residual in the space* $(\mathcal{M}, d_2)$,

(c) The set $\mathcal{M}_\alpha^\beta = \{\mu \in \mathcal{M} : \alpha \leq \overline{\dim}_c \mu \leq \beta\}$ *is residual in the space* $(\mathcal{M}, d_2)$.

The proof of Theorem 1 is given in the next section. In the last section we give an example of a space (X,ρ) such that the set $\{\mu \in \mathcal{M} : \overline{\dim}_c \mu = \beta\}$ is residual in $(\mathcal{M}, d_2)$ and an example of a space (X,ρ) for which $\alpha < \beta$ and the set $\{\mu \in \mathcal{M} : \overline{\dim}_c \mu > \alpha\}$ is nowhere dense in $(\mathcal{M}, d_2)$. These examples show that the estimation $\alpha \leq \overline{\dim}_c \mu \leq \beta$ in Theorem 1 cannot be improved.

3 Proofs

We split the proof of Theorem 1 into a sequence of lemmas.

Let (ε_n) and (δ_n) be sequences of positive numbers convergent to zero. Let

$$\mathcal{N}_n = \{\nu \in \mathcal{M} : \nu(\{x_0\}) \geq \varepsilon_n \text{ for some } x_0 \in X\},$$

$$\mathcal{G}_n^i = \bigcup_{\nu \in \mathcal{N}_n} \{\mu \in \mathcal{M} : d_i(\mu,\nu) < \delta_n\}, \text{ and } \mathcal{H}_i = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \mathcal{G}_n^i.$$

for $i = 1, 2$ and $n \in \mathbb{N}$.

Lemma 1. *The set $\mathcal{H}_i$ is residual in the space $(\mathcal{M}, d_i)$ for $i = 1, 2$.*

PROOF. Since for each $m \in \mathbb{N}$ the set $\bigcup_{n=m}^{\infty} \mathcal{N}_n$ is dense in $\mathcal{M}$, the set $\bigcup_{n=m}^{\infty} \mathcal{G}_n^i$ is dense and open in $\mathcal{M}$. This implies that the set $\mathcal{H}_i$ is residual in the space $(\mathcal{M}, d_i)$ for $i = 1, 2$. □

Lemma 2. *The set $\mathcal{M}^0 = \{\mu \in \mathcal{M} : \overline{\dim}_c \, \mu = 0\}$ is residual in the space $(\mathcal{M}, d_1)$.*

PROOF. Let $\varepsilon_n = \frac{1}{n}$ and $\delta_n = \frac{1}{2n}$ for $n \in \mathbb{N}$. According to Lemma 1 it is sufficient to check that if $\mu \in \mathcal{H}_1$, then $\overline{\dim}_c \, \mu = 0$. Let $\mu \in \mathcal{H}_1$. For every $m \in \mathbb{N}$ there are $n \geq m$ and $\nu \in \mathcal{N}_n$ such that $d_1(\mu, \nu) < \delta_n$. Since $\nu \in \mathcal{N}_n$ there is a point x_0 such that $\nu(\{x_0\}) \geq \varepsilon_n$. Consequently,

$$\mu(\{x_0\}) > \nu(\{x_0\}) - \delta_n \geq \varepsilon_n - \delta_n = \delta_n$$

and $\overline{\dim}_c \, \mu = 0$. □

Lemma 3. *The set $\mathcal{M}_0 = \{\mu \in \mathcal{M} : \underline{\dim}_c \, \mu = 0\}$ is residual in the space $(\mathcal{M}, d_2)$.*

PROOF. Let $\varepsilon_n = \frac{1}{n}$, $r_n = (\varepsilon_n)^n$ and $\delta_n = \frac{1}{9}\varepsilon_n r_n$ for $n \in \mathbb{N}$. According to Lemma 1 it is sufficient to check that if $\mu \in \mathcal{H}_2$ then $\underline{\dim}_c \, \mu = 0$. Let $\mu \in \mathcal{H}_2$. For every $m \in \mathbb{N}$ there is $n \geq m$ and $\nu \in \mathcal{N}_n$ such that $d_2(\mu, \nu) < \delta_n$. Since $\nu \in \mathcal{N}_n$, there is a point x_0 such that $\nu(\{x_0\}) \geq \varepsilon_n$. Fix $r \in (0, 1]$ and consider the function $f : X \to [0, \infty)$ given by

$$f(x) = \begin{cases} r & \text{if } \rho(x, x_0) \leq r \\ r - t & \text{if } \rho(x, x_0) = r + t,\ 0 < t < r \\ 0 & \text{if } \rho(x, x_0) \geq 2r. \end{cases} \tag{1}$$

Clearly $f \in \mathcal{L}$. From the definition of the function f and inequality $d_2(\mu, \nu) < \delta_n$ it follows that for every $y \in B(x_0, r)$ we have

$$r\mu(B(y, 3r)) \geq \int_X f(x)\, d\mu(x) \geq -\delta_n + \int_X f(x)\, d\nu(x).$$

Since $f(x_0) = r$ and $\nu(\{x_0\}) \geq \varepsilon_n$, the last inequality implies

$$\mu(B(y, 3r)) \geq -\frac{\delta_n}{r} + \varepsilon_n.$$

By a similar calculation, using a function f given by (1) with $r/2$ in the place of r, we can show that $\mu(B(x_0,r)) \geq \frac{-2\delta_n}{r} + \varepsilon_n$. Substituting $r = r_n/3$ we obtain

$$\mu(B(y,r_n)) \geq -\frac{3\delta_n}{r_n} + \varepsilon_n = \frac{2\varepsilon_n}{3} \text{ for every } y \in B(x_0, r_n/3) \tag{2}$$

and

$$\mu(B(x_0, r_n/3)) \geq \frac{\varepsilon_n}{3}. \tag{3}$$

Using (2) and (3) we have

$$\int_X \mu(B(y,r_n))\,d\mu(y) \geq \int_{B(x_0,r_n/3)} \mu(B(y,r_n))\,d\mu(y) \geq \frac{2\varepsilon_n^2}{9}.$$

Hence

$$\begin{aligned}\underline{\dim}_c\,\mu &\leq \lim_{n\to\infty} \frac{1}{\log r_n} \log \int_X \mu(B(y,r_n))\,d\mu(y)\\ &\leq \lim_{n\to\infty} \frac{\log(2\varepsilon_n^2/9)}{\log r_n} = \lim_{n\to\infty} \frac{2\log n - \log(2/9)}{n \log n} = 0,\end{aligned}$$ □

Recall that for given $\mu \in \mathcal{M}$ we define the support of μ by the formula

$$\operatorname{supp}\mu = \{x \in X : \mu(B(x,r)) > 0 \text{ for every } r > 0\}.$$

Lemma 4. *Assume that $\underline{\dim}(B(x_0,a)) > d$ for some point $x_0 \in X$ and some constants $a, d > 0$. Then there exists $C > 0$ such that for every $r > 0$ there exists a measure μ_r with $\operatorname{supp}\mu_r \subset B(x_0,a)$ such that*

$$\mu_r(B(x,r)) \leq Cr^d \text{ for every } x \in X. \tag{4}$$

Proof. Since $\underline{\dim}\,B(x_0,a) > d$, by virtue of Remark 1, there is $0 < r_0 < 1$ such that $M(B(x_0,a),r) > r^{-d}$ for every $0 < r \leq r_0$. Put $C = 2^d/r_0^d$. If $r \geq r_0/2$ then $Cr^d \geq 1$ and (4) is obviously true for every measure $\mu \in \mathcal{M}$.

Suppose now that $0 < r < r_0/2$. Let m be an integer such that

$$M(B(x_0,a),2r) \geq m > (2r)^{-d}.$$

By the definition of $M(B(x_0,a),2r)$ we can find in the ball $B(x_0,a)$ the points $x_1,\dots,x_m$ such that $\rho(x_i,x_j) \geq 2r$ for $i,j \in \{1,\dots,m\}$, $i \neq j$. Set $\mu_r = \frac{1}{m}\sum_{i=1}^m \delta_{x_i}$, where δ_{x_i} denotes the delta Dirac measure supported at point x_i. Since for arbitrary $x \in X$ the ball $B(x,r)$ contains at most one point from the set $\{x_1,\dots,x_m\}$, we have $\mu_r(B(x,r)) \leq \frac{1}{m} < (2r)^d \leq Cr^d$. □

Lemma 5. *Assume that there is a constant $d > 0$ such that $\underline{\dim}\, B(x,a) > d$ for every $x \in X$ and every $a > 0$. Then the set $\mathcal{M}_d = \{\mu \in \mathcal{M} : \overline{\dim}_c\, \mu \geq d\}$ is residual in the space $(\mathcal{M}, d_2)$.*

PROOF. Let $\{x_1, x_2, \dots\}$ be a dense subset of X. Fix $n \in \mathbb{N}$ and define

$$a_n = \min\left\{\frac{1}{3}\rho(x_i, x_j) : \quad \text{for } 1 \leq i < j \leq n\right\}.$$

According to Lemma 4, for every $i \in \{1, \dots, n\}$ there exists a constant C_i such that for every $r > 0$ there exists a measure $\mu_{r,i}$ with $\operatorname{supp} \mu_{r,i} \subset B(x_i, a_n)$ such that $\mu_{r,i}(B(x,r)) \leq C_i r^d$ for every $x \in X$. Set

$$\bar{C}_n = \max\{n, C_1, \dots, C_n\},\ r_n = 2^{-\bar{C}_n} \text{ and } \delta_n = r_n^{d+1}.$$

Now fix $r = 2r_n$ and denote by $\mathcal{N}_n$ the set of all measures of the form

$$\nu = p_1\mu_{r,1} + \dots + p_n\mu_{r,n},$$

where $(p_1, \dots, p_n)$ is any sequence of non-negative numbers such that $p_1 + \dots + p_n = 1$. Clearly $\nu(B(x, 2r_n)) \leq 2^d \bar{C}_n r_n^d$ for every $\nu \in \mathcal{N}_n$ and $x \in X$. Let

$$\mathcal{G}_n = \bigcup_{\nu \in \mathcal{N}_n} \{\mu \in \mathcal{M} :\ d_2(\mu, \nu) < \delta_n\}.$$

Suppose that the sets $\mathcal{G}_n$ are constructed for every $n \in \mathbb{N}$ and define $\mathcal{H} = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \mathcal{G}_n$. Clearly $\mathcal{H}$ is a residual subset of $(\mathcal{M}, d_2)$. Let $\mu \in \mathcal{H}$. For every $m \in \mathbb{N}$ there are $n \geq m$ and $\nu \in \mathcal{N}_n$ such that $d_2(\mu, \nu) < \delta_n$. Fix a point $y \in X$ and let f be the function given by (1) with y in the place of x_0. Since $d_2(\mu, \nu) < \delta_n$ we have

$$r\mu(B(y,r)) \leq \int_X f(x)\, d\mu(x) \leq \delta_n + \int_X f(x)\, d\nu(x) \leq \delta_n + r\nu(B(y, 2r)). \quad (5)$$

Substituting $r = r_n$ in (5) we obtain

$$\mu(B(y, r_n)) \leq \frac{\delta_n}{r_n} + \nu(B(y, 2r_n)) \leq r_n^d + 2^d \bar{C}_n r_n^d < 2^{d+1} \bar{C}_n r_n^d = 2^{d+1} \bar{C}_n 2^{-\bar{C}_n d}.$$

This implies that

$$\limsup_{n\to\infty} \frac{1}{\log r_n} \log \int_X \mu(B(x, r_n))\, d\mu(x) \geq \limsup_{n\to\infty} \frac{d + 1 + \log_2 \bar{C}_n - \bar{C}_n d}{-\bar{C}_n} = d,$$

which completes the proof. □

Remark 2. Assume there exist sequences (a_n) and (r_n) of positive numbers convergent to zero such that for each $x \in X$ we have $M(B(x, a_n), r_n) \geq r_n^{-d}$ for $n \in \mathbb{N}$. An argument similar to that of the proofs of Lemmas 4 and 5 shows that the set $\mathcal{M}_d = \{\mu \in \mathcal{M} : \overline{\dim}_c \mu \geq d\}$ is residual in the space $(\mathcal{M}, d_2)$.

Lemma 6. *Let K be a subset of X and let μ be a probability Borel measure on X such that $\mu(K) > 0$. Then $\overline{\dim}_c \mu \leq \overline{\dim} K$.*

PROOF. Suppose K is relatively compact. (Otherwise there is nothing to prove.) First assume that $\operatorname{supp} \mu \subset K$. Given an $r > 0$ we denote by $N = N(K, r)$ the least number of balls of radius r which cover the set K. Now, denote by $x_1, \ldots, x_N$ the centers of the balls of such a covering. Let $A_1, \ldots, A_N$ be a pairwise disjoint measurable covering of K such that $A_i \subset B(x_i, r)$ for $i = 1, \ldots, N$. If $x \in A_i$, then $A_i \subset B(x, 2r)$. Consequently

$$\int_X \mu(B(x, 2r))\, d\mu(x) = \sum_{i=1}^{N} \int_{A_i} \mu(B(x, 2r))\, d\mu(x) \geq \sum_{i=1}^{N} \mu(A_i)^2.$$

Using the Buniakowski-Schwarz inequality

$$\left(\sum_{i=1}^{N} \mu(A_i)^2 \right) \left(\sum_{i=1}^{N} 1^2 \right) \geq \left(\sum_{i=1}^{N} \mu(A_i) \right)^2$$

we obtain

$$\int_X \mu(B(x, 2r))\, d\mu(x) \geq \frac{1}{N}. \tag{6}$$

Let (r_n) be a sequence of positive numbers convergent to zero. Then from (6) it follows that

$$\limsup_{n \to \infty} \frac{1}{\log 2r_n} \log \int_X \mu(B(x, 2r_n))\, d\mu(x) \leq \limsup_{n \to \infty} \frac{\log N(K, r_n)}{\log(1/r_n)} \leq \overline{\dim} K.$$

Thus $\overline{\dim}_c \mu \leq \overline{\dim} K$ for every μ such that $\operatorname{supp} \mu \subset K$.

Now take an arbitrary μ in $\mathcal{M}$ such that $\mu(K) > 0$. Set $\nu(A) = \frac{\mu(A \cap K)}{\mu(K)}$, $A \in \mathcal{B}$. Since $\mu(A) \geq \mu(K)\nu(A)$, we have

$$\int_X \mu(B(x, r))\, d\mu(x) \geq \mu^2(K) \int_X \nu(B(x, r))\, d\nu(x). \tag{7}$$

By (7) and the fact that $\operatorname{supp} \nu \subset K$ we have

$$\overline{\dim}_c \mu \leq \limsup_{r \to 0} \frac{\log \mu^2(K)}{\log r} + \overline{\dim}_c \nu = \overline{\dim}_c \nu \leq \overline{\dim} K. \qquad \square$$

Lemma 7. *Assume that $\overline{\dim}\, B(x_0,a) < d$ for some point $x_0 \in X$ and some constants $a, d > 0$. Then the set $\mathcal{M}^d = \{\mu \in \mathcal{M} : \overline{\dim}_c\, \mu \geq d\}$ is nowhere dense in the space $(\mathcal{M}, d_2)$.*

PROOF. Let $\mathcal{D}$ be the set of all probability measures ν such that $\nu(B(x_0, \frac{a}{2})) > 0$. Clearly the set $\mathcal{D}$ is dense in $\mathcal{M}$. If $\nu \in \mathcal{D}$, then let $\delta(\nu) = \frac{1}{4}a\nu(B(x_0, \frac{a}{2}))$. The set

$$\mathcal{G} = \bigcup_{\nu\in\mathcal{D}} \{\mu \in \mathcal{M} :\, d_2(\mu,\nu) < \delta(\nu)\}$$

is open and dense in $\mathcal{M}$. We claim that $\overline{\dim}_c\, \mu < d$ for every $\mu \in \mathcal{G}$. Indeed, let $\mu \in \mathcal{G}$ and $\nu \in \mathcal{D}$ be such that $d_2(\mu,\nu) < \delta(\nu)$. Taking a function f, defined by (1) with $r = a/2$, we have

$$\frac{a}{2}\mu(B(x_0,a)) \geq \int_X f\, d\mu \geq \int_X f\, d\nu - \delta(\nu) \geq \frac{a}{2}\nu(B(x_0, \tfrac{a}{2})) - \delta(\nu) > 0.$$

According to Lemma 6 we have $\overline{\dim}_c\, \mu < d$. This implies that the set $\mathcal{M}^d$ is nowhere dense in the space $(\mathcal{M}, d_2)$. □

PROOF OF THEOREM 1. The statement (a) of Theorem 1 follows from Lemma 2. The statement (b) follows from Lemma 3. According to Lemma 5 and Lemma 7, for every $n \in \mathbb{N}$ the sets

$$\mathcal{M}_{\alpha-\frac{1}{n}} = \{\mu \in \mathcal{M} : \overline{\dim}_c\, \mu \geq \alpha - \tfrac{1}{n}\} \text{ and } \mathcal{M}^{\beta+\frac{1}{n}} = \{\mu \in \mathcal{M} : \overline{\dim}_c\, \mu < \beta + \tfrac{1}{n}\}$$

are residual in the space $(\mathcal{M}, d_2)$. From this and the equality

$$\mathcal{M}_\alpha^\beta = \bigcap_{n=1}^{\infty} \left(\mathcal{M}_{\alpha-\frac{1}{n}} \cap \mathcal{M}^{\beta+\frac{1}{n}}\right)$$

the statement (c) follows. The proof of Theorem 1 is completed. □

4 Examples

Example 1. We construct a Cantor-like set C such that $\underline{\dim}\, C = 0$ and $\overline{\dim}\, C = 1$ and such that the set $\mathcal{M}^1 = \{\mu \in \mathcal{M} : \overline{\dim}_c\, \mu = 1\}$ is residual in the space $(\mathcal{M}, d_2)$. Let (k_n) be a strictly increasing sequence of positive integers such that $\liminf_{n\to\infty} \frac{k_n}{n} = 1$ and $\limsup_{n\to\infty} \frac{k_n}{n} = \infty$. Let $h_0 = 1$ and $h_n = 2^{-k_n}$ for $n \in \mathbb{N}$. We define a sequence of sets (C_n) by induction. Let $C_0 = [0,1]$ and if $C_n = \bigcup_{i=1}^{2^n} [\alpha_i^n, \beta_i^n]$, where $\beta_i^n = \alpha_i^n + h_n \leq \alpha_{i+1}^n$, then $C_{n+1} = \bigcup_{i=1}^{2^{n+1}} [\alpha_i^{n+1}, \beta_i^{n+1}]$, where $\alpha_{2i-1}^{n+1} = \alpha_i^n$, $\beta_{2i-1}^{n+1} = \alpha_i^n + h_{n+1}$, $\alpha_{2i}^{n+1} = \beta_i^n - h_{n+1}$ and $\beta_{2i}^{n+1} = \beta_i^n$. Let $C = \bigcap_{n=0}^{\infty} C_n$. From the definitions of entropy dimensions it follows easily that $\underline{\dim}\, C = 0$ and $\overline{\dim}\, C = 1$. Fix $\varepsilon > 0$ and $n \in \mathbb{N}$. Let $m \in \mathbb{N}$ be such that $k_m \le \frac{m-1-n}{1-\varepsilon}$. Set $a_n = h_n$ and $r_n = h_m$. Then, for each $n \in \mathbb{N}$, we have

$$M(B(x, a_n), r_n) \ge 2^{m-1-n} \ge 2^{(1-\varepsilon)k_m} = r_n^{\varepsilon - 1}.$$

According to Remark 2 the set $\mathcal{M}^{1-\varepsilon} = \{\mu \in \mathcal{M} : \overline{\dim}_c\, \mu \ge 1 - \varepsilon\}$ is residual. Since $\varepsilon > 0$ is arbitrary and $\mathcal{M}^1 = \bigcap_{n=1}^{\infty} \mathcal{M}^{1-1/n}$, it follows that the set $\mathcal{M}^1$ is residual.

Example 2. Now we construct a set $X \subset \mathbb{R}$ such that $\overline{\dim}\, B(x, r) = 1$ for all $x \in X$ and $r > 0$ but $\overline{\dim}_c\, \mu = 0$ for μ from some open and dense subset $\mathcal{G}$ of $\mathcal{M}$. Let (k_n) and (k'_n) be two strictly increasing sequences of positive integers such that

$$\liminf_{n\to\infty} \frac{k_n}{n} = 1, \quad \liminf_{n\to\infty} \frac{k'_n}{n} = 1, \tag{8}$$

and

$$\lim_{n\to\infty} \frac{\max(k_n, k'_n)}{n} = \infty. \tag{9}$$

As in Example 1 we construct Cantor-like sets C and C' corresponding to the sequences (k_n) and (k'_n), respectively. Let $X = C \cup (C' + 2)$, where $C' + 2 = \{x + 2 : x \in C'\}$. From (8) it follows that $\overline{\dim}\, B(x, r) = 1$ for all $x \in X$ and $r > 0$. Set

$$\mathcal{G} = \{\mu \in \mathcal{M} : \mu(C) > 0 \text{ and } \mu(C' + 2) > 0\}.$$

Then obviously the set $\mathcal{G}$ is open and dense in $\mathcal{M}$. Let $\mu \in \mathcal{G}$. From (6) applied to the measures $\mu_1(A) = \frac{\mu(A \cap C)}{\mu(C)}$ and $\mu_2(A) = \frac{\mu(A \cap (C'+2))}{\mu(C'+2)}$ it follows that

$$\int_X \mu(B(x, 2r))\, d\mu(x) \ge \frac{\mu^2(C)}{N(C, r)} + \frac{\mu^2(C' + 2)}{N(C' + 2, r)}.$$

This implies that

$$\frac{1}{\log 2r} \log \int_X \mu(B(x, 2r))\, d\mu(x) \le \min\left\{ \frac{\log \mu^2(C) - \log N(C, r)}{\log 2r}, \frac{\log \mu^2(C' + 2) - \log N(C' + 2, r)}{\log 2r} \right\}$$

and consequently

$$\overline{\dim}_c \mu \le \limsup_{r\to 0} \min\left\{\frac{\log N(C,r)}{\log(1/r)}, \frac{\log N(C',r)}{\log(1/r)}\right\}. \tag{10}$$

Now, for given $r \in (0,1)$ we set $n(r) = \min\{n : 2^{-k_n} \le r\}$ and $n'(r) = \min\{n : 2^{-k'_n} \le r\}$. Then $N(C,r) \le 2^{n(r)}$ and $N(C',r) \le 2^{n'(r)}$. Thus, for $\mu \in \mathcal{G}$, we have

$$\overline{\dim}_c \mu \le \limsup_{r\to 0} \frac{\min\{n(r), n'(r)\}}{\log(1/r)}. \tag{11}$$

By the definitions of $n(r)$ and $n'(r)$ we have

$$\log(1/r) \ge \max\{k_{n(r)-1},\ k'_{n'(r)-1}\}. \tag{12}$$

From (11), (12) and (9) it follows that $\overline{\dim}_c \mu = 0$.

References

[1] W. Chin, B. Hunt and J. A. Yorke, *Correlation dimension for iterated function systems,* Trans. Amer. Math. Soc., **349** (1997), 1783–1796.

[2] P. M. Gruber, *Dimension and structure of typical compact sets, continua and curves,* Mh. Math., **108** (1989), 149–164.

[3] C. A. Guerin and M. Holschneider, *On equivalent definitions of the correlation dimension for a probability measure,* J. Statist. Phys., **86** (1997), 707–720.

[4] A. Manning and K. Simon, *A short existence proof for correlation dimension,* J. Statist. Phys., **90** (1998), 1047–1049.

[5] Ya. B. Pesin, *On rigorous mathematical definitions of correlation dimension and generalized spectrum for dimensions,* J. Stat. Phys, **71** (1993), 529–547.

[6] Ya. B. Pesin and A. Tempelman, *Correlation dimension of measures invariant under group actions,* Random Comput. Dynam., **3** (1995), 137–156.

[7] Ya. B. Pesin and H. Weiss, *The multifractal analysis of Gibbs measures: motivation, mathematical foundation, and examples,* Chaos, **7** (1997), 89–106.

[8] Ya. B. Pesin, *Dimension Theory in Dynamical Systems,* Chicago Lecture in Math. Series, Chicago, 1997.

[9] I. Procaccia, P. Grassberger and H.G.E. Hentschel, *On the characterization of chaotic motions,* Dynamical systems and chaos (Sitges/Barcelona, 1982), Lecture Notes in Phys., **179**, Springer, Berlin (1983), 212–222.

[10] T. D. Sauer and J. A. Yorke, *Are the dimensions of a set and its image equal under typical smooth functions,* Ergodic Theory Dynam. Systems, **17** (1997), 941–956.

[11] K. Simon and B. Solomyak, *Correlation dimension for self-similar Cantor sets with overlaps,* Fund. Math., **155** (1998), 293–300.