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Harvey Rosen, Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487. email: hrosen@gp.as.ua.edu

SIERPINSKI-ZYGMUND UNIFORM LIMITS OF EXTENDABLE CONNECTIVITY FUNCTIONS

Abstract

We show that the class SZ of Sierpinski-Zygmund functions has a nonempty intersection with the class $\overline{Ext}$ of all uniform limits of sequences of extendable connectivity functions $f_n : \mathbb{R} \to \mathbb{R}$. We reconsider the idea of f-negligible sets this time with respect to $f \in \overline{Ext}$. We also show that under MA, $SZ \cap \overline{Ext}$ cannot be characterized by preimages of sets.

1 Introduction

The class SZ of Sierpinski-Zygmund functions consists of all functions $g : \mathbb{R} \to \mathbb{R}$ which are discontinuous on each subset of $\mathbb{R}$ of cardinality of the continuum. A member $f : \mathbb{R} \to \mathbb{R}$ of the class Ext is called an *extendable connectivity* (or *extendable*) function, which means that there exists $F : \mathbb{R} \times [0, 1] \to \mathbb{R}$ such that $F(x, 0) = f(x)$ for all $x \in \mathbb{R}$ and $F\restriction_J$ is connected for all connected subsets J of $\mathbb{R} \times [0, 1]$. As was pointed out in [5] and [1], $SZ \cap Ext = \emptyset$ because by [11], $f \in Ext \Rightarrow \exists$ a Cantor set C such that $F\restriction_C$ is continuous. However, we show that $SZ \cap \overline{Ext} \neq \emptyset$

For $\mathcal{A}$, $\mathcal{B} \subset \mathcal{P}(\mathbb{R})$, which is the power set of $\mathbb{R}$, define

$$\mathcal{C}_{\mathcal{A},\mathcal{B}} = \{f \in \mathbb{R}^{\mathbb{R}} : \forall A \in \mathcal{A},\ f(A) \in \mathcal{B}\} \text{ and}$$
$$\mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}} = \{f \in \mathbb{R}^{\mathbb{R}} : \forall B \in \mathcal{B},\ f^{-1}(B) \in \mathcal{A}\}$$

A family $\mathcal{F}$ of real functions can be *characterized by images (preimages)* of sets if $\mathcal{F} = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ ($\mathcal{F} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$) for some $\mathcal{A}$, $\mathcal{B} \subset \mathcal{P}(\mathbb{R})$. In [5], Ciesielski and Natkaniec show that SZ can be characterized by neither images nor preimages

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of sets. We show that under Martin's Axiom, the same is still true for $SZ \cap \overline{Ext}$ with regard to preimages. With D being the class of Darboux functions, they ask in [5] whether $SZ \cap D$, under the assumption it is nonempty, can be characterized by images or preimages. The following example illustrates the definition about images and preimages.

Example 1. According to [3], the class $\overline{D}$ of all uniform limits of sequences of real Darboux functions is the same as the class $\mathcal{U}$ of functions $f : \mathbb{R} \to \mathbb{R}$ with the property that for every interval $J = [a, b]$ and every set F of cardinality $< \mathfrak{c}$, $f(J \setminus F)$ is dense in the (possibly degenerate) closed interval with endpoints $f(a)$ and $f(b)$. It easily follows that $\overline{D}$ is characterized by images of sets with

$$\mathcal{A} = \{A \subset \mathbb{R} : \forall a,\, b \in A \ni a < b, \operatorname{card}([a, b] \setminus A) < \mathfrak{c}\}$$

and

$$\mathcal{B} = \{B \subset \mathbb{R} : \forall a,\, b \in B \ni a < b, B \text{ is dense in } [a, b]\}.$$

The proof given for case 1 of Theorem 2.2 in [5] shows that $\overline{D}$ is not characterized by preimages.

2 Sierpinski-Zygmund Functions

We modify the transfinite construction Sierpinski and Zygmund give in [12] and use their result that if $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$ is continuous, then there exist a G_δ set Γ containing E and a continuous function $g : \Gamma \to \mathbb{R}$ such that $f(x) = g(x)$ for all $x \in E$.

Theorem 1. *$SZ \cap \overline{Ext} \neq \emptyset$.*

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ be an extendable function whose graph is dense in $\mathbb{R}^2$ [6], [10]. By [9], there exists a dense G_δ subset A_0 of $\mathbb{R}$ that is h-negligible with respect to Ext. This means that every real-valued function on $\mathbb{R}$ that agrees with h off of A_0 must also be extendable. Then $\mathbb{R} \setminus A_0 = A_1 \cup A_2 \cup \ldots$, where the sets A_n, $n \geq 1$, are pairwise disjoint and nowhere dense in $\mathbb{R}$. Let $\{x_\alpha : \alpha < \mathfrak{c}\}$ be a one-to-one enumeration of $\mathbb{R}$ and $\{g_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all continuous functions defined on G_δ subsets of $\mathbb{R}$. Define a function $f : \mathbb{R} \to \mathbb{R}$ by induction on $\alpha < \mathfrak{c}$ this way: $f(x_\alpha) \in \mathbb{R} \setminus \{g_\xi(x_\alpha) : \xi \leq \alpha\}$, and whenever $x_\alpha \in A_n$ for some $n \geq 0$, we can require $|f(x_\alpha) - h(x_\alpha)| < \dfrac{1}{n+1}$. Then $h_0 = \begin{cases} f & \text{on } A_0 \\ h & \text{on } \mathbb{R} \setminus A_0 \end{cases}$ belongs to Ext because A_0 is h-negligible. For $n \geq 1$,

$h_n = \begin{cases} f & \text{on } A_n \\ h_{n-1} & \text{on } \mathbb{R} \setminus A_n \end{cases}$ belongs to Ext because the nowhere dense set A_n is h_{n-1}-negligible. Therefore $f \in \overline{Ext}$ since f is the uniform limit of h_n. Also $f \in SZ$ since for each $\xi < \mathfrak{c}$, $\{x : f(x) = g_\xi(x)\} \subset \{x_\alpha : \alpha < \xi\}$, which has cardinality $< \mathfrak{c}$. □

3 Negligible Sets

Negligibility of sets has been studied for the classes of connectivity functions [2], almost continuous functions [7], and extendable functions [9]. Now we consider it for the class $\overline{Ext}$. Given a class $\mathcal{F}$ of real-valued functions on $\mathbb{R}$ and given $f \in \mathcal{F}$, we say a subset A of $\mathbb{R}$ is *f-negligible with respect to $\mathcal{F}$* if whenever $g : \mathbb{R} \to \mathbb{R}$ and $f = g$ on $\mathbb{R} \setminus A$, then $g \in \mathcal{F}$, too [2].

Theorem 2. *If $f : \mathbb{R} \to \mathbb{R}$ has a graph dense in $\mathbb{R}^2$ and $f \in \overline{Ext}$, then there exists a dense G_δ subset A of $\mathbb{R}$ that is f-negligible with respect to $\overline{Ext}$. Moreover, every nowhere dense subset M of $\mathbb{R}$ is f-negligible with respect to $\overline{Ext}$.*

Proof. Since$f \in \overline{Ext}$ and it has a dense graph in $\mathbb{R}^2$, f is the uniform limit of a sequence of extendable functions $f_n : \mathbb{R} \to \mathbb{R}$ with dense graphs in $\mathbb{R}^2$. According to [9], for each n, there is a dense G_δ subset A_n of $\mathbb{R}$ that is f_n-negligible with respect to Ext; moreover, every nowhere dense subset M of $\mathbb{R}$ is f_n-negligible with respect to Ext. By the Baire Category Theorem, $A = \cap_{n=1}^{\infty} A_n$ is a dense G_δ subset of $\mathbb{R}$. Let $B \in \{A, M\}$, and suppose $g : \mathbb{R} \to \mathbb{R}$ and $g = f$ on $\mathbb{R} \setminus B$. For each n, define $g_n = \begin{cases} g & \text{on } B \\ f_n & \text{on } \mathbb{R} \setminus B. \end{cases}$ Then g is the uniform limit of g_n, and each $g_n \in Ext$ because B is f_n-negligible with respect to Ext. Therefore $g \in \overline{Ext}$, and so B is f-negligible with respect to $\overline{Ext}$. □

4 Preimages

Our next result assumes that if $A \subset \mathbb{R}$ and card $A < \mathfrak{c}$, then A is first category. Martin's Axiom implies this assumption according to Theorem 8.2.6 in [4]. We show how to extend the argument Ciesielski and Natkaniec use in [5, Thm 3.1] when they prove SZ is not characterized by preimages of sets.

Theorem 3. *Under MA, $SZ \cap \overline{Ext}$ cannot be characterized by preimages of sets.*

PROOF. Assume $SZ \cap \overline{Ext} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$ for some $\mathcal{A}$, $\mathcal{B} \subset \mathcal{P}(\mathbb{R})$. As in the proof of Theorem 1, let $h : \mathbb{R} \to \mathbb{R}$ be an extendable function with dense graph in $\mathbb{R}^2$, and let G_0 be a dense G_δ, h-negligible subset of $\mathbb{R}$. Then $\mathbb{R} \setminus G_0$ is meager in $\mathbb{R}$. Also let $\{x_\alpha : \alpha < \mathfrak{c}\}$ be a one-to-one enumeration of $\mathbb{R}$ and $\{g_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all continuous functions on G_δ subsets of $\mathbb{R}$. As shown in [5], $\mathcal{B} \not\subset \{\emptyset, \mathbb{R}\}$. Let $B \in \mathcal{B} \setminus \{\emptyset\}$ and pick $x \in B$. In showing $B \in \mathcal{A}$, we consider four cases for B.

Case 1: Suppose $\operatorname{card} B < \mathfrak{c}$.

We claim in this case that $\mathcal{A}$ contains each subset A of cardinality $< \mathfrak{c}$. Suppose A has cardinality $< \mathfrak{c}$, and so, under MA, A is first category. It follows from Lemma 3 of [8] that there exists a homeomorphism $h_0 : \mathbb{R} \to \mathbb{R}$ such that $(\mathbb{R} \setminus G_0) \cap h_0(A) = \emptyset$; i.e., $A \subset h_0^{-1}(G_0)$. According to Corollary 1 and Lemma 2 (which hold for $\mathbb{R}$ in place of intervals I and J there) in [8], $h \circ h_0$ is extendable and $h_0^{-1}(G_0)$ is $h \circ h_0$-negligible. So A is $h \circ h_0$-negligible, too. Therefore $f_0 = \begin{cases} x & \text{on } A \\ h \circ h_0 & \text{on } \mathbb{R} \setminus A \end{cases}$ is in Ext. Let A_0 be a dense G_δ, f_0-negligible set and $\mathbb{R} \setminus A_0 = A_1 \cup A_2 \cup \ldots$, where the A_n, $n \geq 1$, are pairwise disjoint, nowhere dense subsets of $\mathbb{R}$. Define the values $f(x_\alpha)$ by induction on $\alpha < \mathfrak{c}$ by letting $f(A) = \{x\}$ and if $x_\alpha \in \mathbb{R} \setminus A$ by choosing $f(x_\alpha) \in \mathbb{R} \setminus (B \cup \{g_\xi(x_\alpha) : \xi \leq \alpha\})$, *extendable* and whenever $x_\alpha \in A_n$ for some $n \geq 0$, we can require $|f(x_\alpha) - f_0(x_\alpha)| < \dfrac{1}{n+1}$. For $n \geq 1$, $f_n = \begin{cases} f & \text{on } A_{n-1} \\ f_{n-1} & \text{on } \mathbb{R} \setminus A_{n-1} \end{cases}$ is extendable. Then $f \in SZ \cap \overline{Ext}$ and $A = f^{-1}(B) \in \mathcal{A}$. Now if every nonempty $B \in \mathcal{B}$ has card $< \mathfrak{c}$, then the identity i obeys $i^{-1}(B) = B \in \mathcal{A}$. Therefore $i \in \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$, a contradiction. So there exists a member of $\mathcal{B}$ of cardinality $\mathfrak{c}$.

Case 2: Suppose $B = \{b_\alpha : \alpha < \mathfrak{c}\}$ is nowhere $\mathfrak{c}$-dense in $\mathbb{R}$.

This means that if $\overline{B}$ contains an interval, then some subinterval meets B in less than $\mathfrak{c}$-many points. Then B is first category and $\mathbb{R} \setminus B$ is $\mathfrak{c}$-dense in $\mathbb{R}$. As shown above, if $h_1 : \mathbb{R} \to \mathbb{R}$ is a homeomorphism such that $(\mathbb{R} \setminus G_0) \cap h_1(B) = \emptyset$, then B is $h \circ h_1$-negligible. Define a dense extendable function $f_0 : \mathbb{R} \to \mathbb{R}$ by $f_0 = h \circ h_1$ on $\mathbb{R} \setminus B$ and $f_0(b_\alpha) \in B \setminus \{g_\xi(b_\alpha) : \xi \leq \alpha\}$ on B. Let A_0 be a dense G_δ, f_0-negligible set and $\mathbb{R} \setminus A_0 = A_1 \cup A_2 \cup \ldots$, where the A_n, $n \geq 1$, are pairwise disjoint and nowhere dense in $\mathbb{R}$. Define $f(x_\alpha) = f_0(x_\alpha)$ if $x_\alpha \in B$, and define $f(x_\alpha) \in \mathbb{R} \setminus \{g_\xi(x_\alpha) : \xi \leq \alpha\}$ if $x_\alpha \in \mathbb{R} \setminus B$, but whenever $x_\alpha \in A_n \setminus B$ for some $n \geq 0$, we can require $f(x_\alpha) \in \mathbb{R} \setminus B$ and $|f(x_\alpha) - f_0(x_\alpha)| < \dfrac{1}{n+1}$ because $\mathbb{R} \setminus B$ is $\mathfrak{c}$-dense in $\mathbb{R}$. It follows that $f \in \overline{Ext}$, and $f \in SZ$ because $\{x : f(x) = g_\xi(x)\} \subset \{x_\alpha : \alpha < \xi\} \cup \{b_\alpha : \alpha < \xi\}$ for each $\xi < \mathfrak{c}$. Therefore $B = f^{-1}(B) \in \mathcal{A}$. Similarly, one can show $\emptyset = g^{-1}(B) \in \mathcal{A}$, where $g \in SZ \cap \overline{Ext}$ is such that $g(\mathbb{R}) \subset \mathbb{R} \setminus B$.

Case 3: Suppose each of B and $\mathbb{R} \setminus B$ is somewhere $\mathfrak{c}$-dense in $\mathbb{R}$.

Then $\overline{B}$ contains an interval (a, b) such that $B \cap (a, b)$ is $\mathfrak{c}$-dense in (a, b), and $\overline{\mathbb{R} \setminus B}$ contains an interval (c, d) such that $(\mathbb{R} \setminus B) \cap (c, d)$ is $\mathfrak{c}$-dense in (c, d). It can be shown there exists $\phi \in SZ \cap \overline{Ext}$ such that $\phi(\mathbb{R}) \subset B \cap (a, b) \subset B$. Therefore $\mathbb{R} = \phi^{-1}(B) \in \mathcal{A}$. Also it can be shown there exists $\psi \in SZ \cap \overline{Ext}$ such that $\psi(\mathbb{R}) \subset (\mathbb{R} \setminus B) \cap (c, d) \subset \mathbb{R} \setminus B$. Therefore $\emptyset = \psi^{-1}(B) \in \mathcal{A}$. But $\emptyset, \mathbb{R} \in \mathcal{A}$ implies the constant functions belong to $SZ \cap \overline{Ext} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$, a contradiction. Therefore this case cannot occur.

Case 4: Suppose B is somewhere $\mathfrak{c}$-dense and $\mathbb{R} \setminus B$ is nowhere $\mathfrak{c}$-dense in $\mathbb{R}$. Then based on Case 2, there exists a function $f \in SZ \cap \overline{Ext}$ such that $f^{-1}(\mathbb{R} \setminus B) = \mathbb{R} \setminus B$. But then $B = f^{-1}(B) \in \mathcal{A}$.

According to the above cases, $\mathcal{B} \setminus \{\emptyset\} \subset \mathcal{A}$, and so the identity i obeys $i^{-1}(B) = B \in \mathcal{A}$ for every nonempty set $B \in \mathcal{B}$. Assume $\emptyset \in \mathcal{B}$. Then for any $f \in SZ \cap \overline{Ext}$, $\emptyset = f^{-1}(\emptyset) \in \mathcal{A}$, and so i obeys $i^{-1}(\emptyset) = \emptyset \in \mathcal{A}$, too. Therefore $i \in \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$, a contradiction. Finally, assume $\emptyset \notin \mathcal{B}$. Then this same contradiction that $i \in \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$ is reached because according to the definition of $\mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$, when $\emptyset \notin \mathcal{B}$, $i^{-1}(\emptyset) = \emptyset$ is not required to belong to $\mathcal{A}$ in order for i to belong to $\mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$. $\square$

Problem 1. Can $SZ \cap \overline{Ext}$ be characterized by images of sets?

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