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ON THE SUMS OF FUNCTIONS SATISFYING THE CONDITION (s_1)

Abstract

A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition (s_1) if for each real $r > 0$, for each x, and for each set $U \ni x$ belonging to the density topology there is an open interval I such that $C(f) \supset I \cap U \neq \emptyset$ and $f(U \cap I) \subset (f(x) - r, f(x) + r)$. ($C(f)$ denotes the set of all continuity points of f). In this article we investigate the sums of two Darboux functions satisfying the condition (s_1).

Let $\mathbb{R}$ be the set of all reals. Let μ denote Lebesgue measure on $\mathbb{R}$ and let μ_e denote Lebesgue outer measure on $\mathbb{R}$. For a set $A \subset \mathbb{R}$ and a point x we define the upper (lower) outer density $D_u(A, x)$ ($D_l(A, x)$) of the set A at the point x as

$$\limsup_{h \to 0^+} \frac{\mu_e(A \cap [x - h, x + h])}{2h}$$
$$\liminf_{h \to 0^+} \frac{\mu_e(A \cap [x - h, x + h])}{2h} \quad \text{respectively.}$$

A point x is said to be an outer density point (a density point) of a set A if $D_l(A, x) = 1$ (if there is a Lebesgue measurable set $B \subset A$ such that $D_l(B, x) = 1$). The family T_d of all sets A for which the implication $x \in A \Longrightarrow x$ is a density point of A holds, is a topology called the density topology ([1, 3]). The sets $A \in T_d$ are Lebesgue measurable [1, 3].

In [2] the following properties are investigated.

A function $f : \mathbb{R} \to \mathbb{R}$ has property (s_1) at a point x ($f \in s_1(x)$) if for each positive real r and for each set $U \in T_d$ containing x there is an open interval I such that $\emptyset \neq I \cap U \subset C(f)$, where $C(f)$ denotes the set of all continuity points of f, and $|f(t) - f(x)| < r$ for all points $t \in I \cap U$. A function f has property (s_1) if $f \in s_1(x)$ for every point $x \in \mathbb{R}$.

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A function $f : \mathbb{R} \to \mathbb{R}$ has property (s_2) if for each nonempty set $U \in T_d$ there is an open interval I such that $\emptyset \neq I \cap U \subset C(f)$.

Evidently each function f having property (s_1) also has property (s_2) and for each function f having property (s_2) the set $D(f) = \mathbb{R} \setminus C(f)$ is nowhere dense and of Lebesgue measure 0. But the closure $\mathrm{cl}(D(f))$ for some functions f having property (s_1) may be of positive measure. For example, if $A \subset [0, 1]$ is a Cantor set of positive measure and (I_n) is a sequence of the components of the set $\mathbb{R} \setminus A$ such that $I_n \neq I_m$ for $n \neq m$, then the function

$$f(x) = \begin{cases} \frac{1}{n} & \text{for } x \in \mathrm{cl}(I_n) \text{ for } n = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

has property (s_1) but $\mu(\mathrm{cl}(D(f))) > 0$.

Obviously the sum of two functions having property (s_2) also has property (s_2) and the sum of a continuous function and a function having property (s_2) has property (s_2). We will prove that every function having property (s_2) is the sum of two functions having property (s_1).

Theorem 1. *If a function $f : \mathbb{R} \to \mathbb{R}$ is the sum of two functions g and h having property (s_1), then there are two Darboux functions ϕ and ψ having property (s_1) such that $f = \phi + \psi$.*

PROOF. The sets $D(g)$ and $D(h)$ of all discontinuity points of g and h respectively are no where dense and of measure zero; so the union $A = D(g) \cup D(h)$ is the same. Without loss of the generality we can suppose that the set A is nonempty.

We start from the case where $\mu(\mathrm{cl}(A)) = 0$. If (a, b), $a, b \in \mathbb{R}$, is a bounded component of the complement $\mathbb{R} \setminus \mathrm{cl}(A)$, then we find two monotone sequences of points

$$a < \cdots < a_{n+1} < a_n < \cdots a_1 < b_1 < \cdots < b_n < b_{n+1} < \cdots < b$$

such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, and

$$\lim_{n\to\infty} \frac{b_{n+1} - b_n}{b - b_{n+1}} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{a - a_{n+1}} = 0. \tag{1}$$

In each interval (a_{n+1}, a_n) ((b_n, b_{n+1})) we find disjoint nondegenerate closed intervals $I_{n,1}, I_{n,2} \subset (a_{n+1}, a_n)$ ($J_{n,1}, J_{n,2} \subset (b_n, b_{n+1})$) such that for $i = 1, 2$ we have

$$\frac{d(I_{n,i})}{a_n - a_{n+1}} < \frac{1}{2n} \quad \Big(\frac{d(J_{n,i})}{b_{n+1} - b_n} < \frac{1}{2n}\Big), \tag{2}$$

and

$$\mu\Big(\cup_{n=1}^{\infty}\cup_{i=1}^{2}\big(I_{n,i}\cup J_{n,i}\big)\Big)<(b-a)^2 \tag{3}$$

where $d(I_{n,i})$ denotes the length of $I_{n,i}$. If (a,b) is an unbounded component of the complement $\mathbb{R}\setminus \mathrm{cl}(A)$; i.e., $a=-\infty$ or $b=\infty$, then we find only two sequences $(J_{n,i})$, $i=1,2$ or respectively $(I_{n,i})$ satisfying the above conditions.

For $i=1,2$ let $g_{n,i}:I_{n,i}\to\mathbb{R}$ and $h_{n,i}:J_{n,i}\to\mathbb{R}$ be continuous functions such that $g_{n,i}(x)=0$ if x is an endpoint of $I_{n,i}$, $h_{n,i}(y)=0$ if y is an endpoint of $J_{n,i}$ and

$$(g+g_{n,1})(I_{n,1})\cap(h+h_{n,1})(J_{n,1})\cap(g+h_{n,2})(J_{n,2})\cap(h+g_{n,2})(I_{n,2})\supset[-n,n].$$

If (a,b) is a bounded component of the complement $\mathbb{R}\setminus\mathrm{cl}(A)$, then we put

$$g_{(a,b)}(x)=\begin{cases} g(x)+g_{n,1}(x) & \text{for } x\in I_{n,1}\\ g(x)+h_{n,2}(x) & \text{for } x\in J_{n,2}\\ g(x)-h_{n,1}(x) & \text{for } x\in J_{n,1}\\ g(x)-g_{n,2}(x) & \text{for } x\in I_{n,2}\\ g(x) & \text{otherwise on } (a,b)\end{cases}$$

and

$$h_{(a,b)}(x)=\begin{cases} h(x)+h_{n,1}(x) & \text{for } x\in J_{n,1}\\ h(x)+g_{n,2}(x) & \text{for } x\in I_{n,2}\\ h(x)-g_{n,1}(x) & \text{for } x\in I_{n,1}\\ h(x)-h_{n,2}(x) & \text{for } x\in J_{n,2}\\ h(x) & \text{otherwise on } (a,b).\end{cases}$$

Similarly we define the functions $g_{(a,b)}$ and $h_{(a,b)}$ on unbounded components (a,b) of the set $\mathbb{R}\setminus\mathrm{cl}(A)$.

Putting $\phi(x)=g_{(a,b)}(x)$ and $\psi(x)=h_{(a,b)}(x)$ on every component (a,b) of the complement $\mathbb{R}\setminus\mathrm{cl}(A)$ and $\phi(x)=g(x)$ and $\psi(x)=h(x)$ on $\mathrm{cl}(A)$ we obtain Darboux functions ϕ and ψ with $\phi+\psi=g+h=f$.

If a point $x\in\mathbb{R}\setminus\mathrm{cl}(A)$, then the functions ϕ and ψ are continuous at some open neighborhood of x and consequently they have property (s_1) at the point x. So we fix a point $x\in\mathrm{cl}(A)$, a set $U\in T_d$ containing x and a positive real r. Since the function g has property (s_1) and $\phi(x)=g(x)$ and by (2) and (3) the upper density $D_u(\{u\in\mathbb{R};\phi(u)\neq g(u)\},x)=0$, there is an open interval $I\subset\mathbb{R}\setminus\mathrm{cl}(A)\setminus\{u;\phi(u)\neq g(u)\}$ such that

$$\emptyset\neq I\cap U \text{ and } \phi(I\cap U)=g(I\cap U)\subset(g(x)-r,g(x)+r)=(\phi(x)-r,\phi(x)+r).$$

So the function ϕ has property (s_1) at x. Similarly we can show that the function ψ has property (s_1) at x.

Now we will consider the case, where $\mu(\mathrm{cl}(A)) > 0$. In this case there are positive numbers $c_1 > c_2 > \cdots > c_n \to 0^+$ such that $\sum_n c_n < \infty$ and the sets $E_1 = \{x; \operatorname{osc} g(x) \geq c_1\} \cup \{x; \operatorname{osc} h(x) \geq c_1\}$ and $E_{n+1} = \{x; c_n > \operatorname{osc} g(x) \geq c_{n+1}\} \cup \{x; c_n > h(x) \geq c_{n+1}\}$ are nonempty for $n \geq 1$.

In the first step, as in the case $\mu(\mathrm{cl}(A)) = 0$, we construct functions ϕ_1, ψ_1 continuous at each point $x \in \mathbb{R} \setminus \mathrm{cl}(A)$, having property (s_1) at each point $x \in E_1$ and such that $\phi_1(x) = g(x)$ and $\psi_1(x) = h(x)$ for $x \in \mathrm{cl}(A)$, $\phi_1 + \psi_1 = g + h$ everywhere on $\mathbb{R}$, the sets $H_1 = \{x; \phi_1(x) \neq g(x)\}$ and $M_1 = \{x; \psi_1(x) \neq h(x)\}$ are contained in countable unions of pairwise disjoint nondegenerate closed intervals $I_{1,k}$ and respectively $J_{1,k}$, $k \geq 1$,

$$\mathrm{cl}(H_1 \cup M_1) = \bigcup_k (I_{1,k} \cup J_{1,k}) \cup E_1,$$

and for each point $x \in E_1$, each nondegenerate closed interval $I \ni x$ and each positive integer m, the density $D_u(\mathrm{cl}(H_1 \cup M_1), x) = 0$, and there are intervals I_{1,k_1} and J_{1,k_2} such that $\phi_1(I_{1,k_1}) \cap \psi_1(J_{1,k_2}) \supset [-m, m]$. For the construction of such functions ϕ_1, ψ_1 and intervals $I_{1,k}$ and $J_{1,k}$ we consider the components (a, b) of the set $\mathbb{R} \setminus \mathrm{cl}(A)$ and repeat the reasoning from the case $\mu(\mathrm{cl}(A)) = 0$ for the set E_1.

As in the second step above, we find pairwise disjoint nondegenerate closed intervals

$$I_{2,k,i} \subset \mathbb{R} \setminus \mathrm{cl}(A) \setminus \bigcup_k (I_{1,k} \cup J_{1,k}), \quad i = 1, 2 \text{ and } k \geq 1,$$

such that $\max(\operatorname{osc}_{I_{2,k,i}} g, \operatorname{osc}_{I_{2,k,i}} h) < c_1$, $\lim_{k\to\infty} \mathrm{dist}(I_{2,k,i}, E_2) = 0$, for $i = 1, 2$, where $\mathrm{dist}(I_{2,k,i}, E_2) = \inf\{|x - y|; x \in I_{2,k,i}, \ y \in E_2\}$,

$$E_2 \subset \mathrm{cl}(\bigcup_k I_{2,k,i}) \subset \bigcup_k I_{2,k,i} \cup E_1 \cup E_2, \text{ for } i = 1, 2,$$

$$D_u(\bigcup_{k,i} I_{2,k,i}, x) = 0 \text{ for } x \in E_2,$$

for each point $x \in E_2$ and for each nondegenerate closed interval $I \ni x$ there are intervals $I_{2,k_1,i}$, $i = 1, 2$, contained in I and such that $\operatorname{osc}_{I_{2,k_1,i}} g < c_1$ and $\operatorname{osc}_{I_{2,k_1,i}} h < c_1$ for $i = 1, 2$. For each pair (k, i), where $k \geq 1$ and $i = 1, 2$ denote by $K_{2,k,i}$ ($L_{2,k,i}$) the closed interval of the length $3c_1$ and having the same center as $g(I_{2,k,i})$ ($h(I_{2,k,i})$) and define continuous functions

$$g_{2,k,1} : I_{2,k,1} \to K_{2,k,1} \text{ and } h_{2,k,2} : I_{2,k,2} \to L_{2,k,2}$$

such that $g_{2,k,1}(I_{2,k,1}) = K_{2,k,1}$, $h_{2,k,2}(I_{2,k,2}) = L_{2,k,2}$ and $g_{2,k,1}(x) = g(x)$ at the endpoints of $I_{2,k,1}$ and $h_{2,k,2}(x) = h(x)$ at the endpoints of $I_{2,k,2}$.

Let

$$\phi_2(x) = \begin{cases} g_{2,k,1}(x) & \text{for } x \in I_{2,k,1} \\ g(x) - h_{2,k,2}(x) + h(x) & \text{for } x \in I_{2,k,2} \\ \phi_1(x) & \text{otherwise on } \mathbb{R} \end{cases}$$

and

$$\psi_2(x) = \begin{cases} h(x) - g_{2,k,1}(x) + g(x) & \text{for } x \in I_{2,k,1} \\ h_{2,k,2}(x) & \text{for } x \in I_{2,k,2}, \\ \psi_1(x) & \text{otherwise on } \mathbb{R}. \end{cases}$$

Then $|\phi_2 - \phi_1| \le 2c_1$, $|\psi_2 - \psi_1| \le 2c_1$ and $\phi_2 + \psi_2 = \phi_1 + \psi_1 = g + h$.

For the construction of such functions ϕ_2, ψ_2 and of intervals $I_{2,k,i}$ we consider the components (a, b) of the set

$$(\mathbb{R} \setminus \mathrm{cl}(A)) \setminus \bigcup_k (I_{1,k} \cup J_{1,k})$$

and analogously as in the proof of the case $\mu(\mathrm{cl}(A)) = 0$ we take intervals $I_{2,k,i}$ satisfying all requirements. Next we repeat the reasoning from the case $\mu(\mathrm{cl}(A)) = 0$ and construct functions ϕ_2 and ψ_2 satisfying our required conditions.

In step $m > 2$ we construct similarly two functions ϕ_m, ψ_m which are continuous on $\mathbb{R} \setminus \mathrm{cl}(A)$ and such that $\mathrm{cl}(\{x; \phi_m(x) \ne g(x)\}) \cap \mathrm{cl}(\{x; \phi_i(x) \ne g(x)\}) \subset E_1 \cup \ldots \cup E_m$ and $\mathrm{cl}(\{x; \psi_m(x) \ne h(x)\}) \cap \mathrm{cl}(\{x; \psi_i(x) \ne h(x)\}) \subset E_1 \cup \ldots \cup E_m$ for $i < m$, $\phi_m + \psi_m = g + h$ and $\max(|\phi_m - \phi_{m-1}|, |\psi_m - \psi_{m-1}|) \le 2c_{m-1}$, the sets $H_m = \{x; \phi_m(x) \ne \phi_{m-1}(x)\}$ and $M_m = \{x; \psi_m(x) \ne \psi_{m-1}(x)\}$ are contained in the countable unions of pairwise disjoint nondegenerate closed intervals $I_{m,n}$ and respectively $J_{m,n}$, $n \ge 1$, contained in

$$\mathbb{R} \setminus \mathrm{cl}(A) \setminus \bigcup_{k<m} \bigcup_n (I_{k,n} \cup J_{k,n})$$

for which $\max(\mathrm{osc}_{I_{m,n}} g, \mathrm{osc}_{I_{m,n}} h, \mathrm{osc}_{J_{m,n}} g, \mathrm{osc}_{J_{m,n}} h) < c_{m-1}$,

$$E_m \subset \mathrm{cl}(\bigcup_{m,n} (I_{m,n} \cup J_{m,n})) \subset \bigcup_{m,n} (I_{m,n} \cup J_{m,n}) \cup E_1 \cup \ldots \cup E_m,$$

$$\lim_{n\to\infty} \mathrm{dist}(I_{m,n}, E_m) = 0, \lim_{n\to\infty} \mathrm{dist}(J_{m,n}, E_m) = 0,$$

$$D_u(\bigcup_n (I_{m,n} \cup J_{m,n}), x) = 0 \text{ for each } x \in E_m$$

and for each point $x \in E_m$ and each nondegenerate closed interval $I \ni x$ there are intervals $I_{m,n_1}, J_{m,n_2} \subset I \setminus \mathrm{cl}(A)$ for which $\max(d(g(I_{m,n_1})), d(h(J_{m,n_2}))) < c_{m-1}$ and $g(I_{m,n_1}) \subset \phi_m(I_{m,n_1})$, $h(J_{m,n_2}) \subset \psi_m(J_{m,n_2})$ and $d(\phi_m(I_{m,n_1})) = d(\psi_m(J_{m,n_2}) = 3c_{m-1}$.

The sequences (ϕ_n) and (ψ_n) uniformly converge to functions ϕ and ψ respectively. Observe that $\phi + \psi = \lim_{n\to\infty}(\phi_n + \psi_n) = g + f$. As the uniform limits the functions ϕ and ψ are continuous at each point of the set $\mathbb{R} \setminus A$. So they have property (s_1) at the points $x \in \mathbb{R} \setminus \mathrm{cl}(A)$. We will prove that these functions have also property (s_1) at other points. For this fix a point $x \in \mathrm{cl}(A)$, a real $r > 0$ and a set $U \in T_d$ containing x. Let j be an integer such that $|\phi_j - \phi| < \frac{r}{3}$. Since the function g has property (s_1) and $D_u(\{u; \phi_j(u) \neq g(u)\}, x) = 0$, there is an open interval $I \subset \{u; \phi_j(u) = g(u)\}$ such that

$$I \cap U \neq \emptyset \text{ and } g(I \cap U) = \phi_j(I \cap U) \subset (g(x) - \frac{r}{3}, g(x) + \frac{r}{3}).$$

Consequently, for $u \in I \cap U$ we have

$$|\phi(u) - \phi(x)| \leq |\phi(u) - \phi_j(u)| + |\phi_j(u) - \phi_j(x)| + |\phi_j(x) - \phi(x)| < \frac{r}{3} + \frac{r}{3} + \frac{r}{3} = r.$$

So the function ϕ has property (s_1). The same we can check that ψ has property (s_1).

Now we will prove that the function ϕ has property of Darboux. Assume to the contrary that it has not the Darboux property. Then there are points a, b with $a < b$ and $\phi(a) \neq \phi(b)$ and a real $c \in K = (\min(\phi(a), \phi(b)), \max(\phi(a), \phi(b))$ such that $\phi^{-1}(c) \cap [a, b] \neq \emptyset$. If there is a point $x \in E_1 \cap [a, b]$, there is a nondegenerate closed interval $I \subset [a, b]$ with $\phi(I) = \phi_1(I) \supset K \ni c$, a contradiction. So $E_1 \cap [a, b] = \emptyset$. Fix a point

$$z \in [a, b] \cap \mathrm{cl}(\{u; \phi(u) < c\}) \cap \mathrm{cl}(\{u; \phi(u) > c\}).$$

Then $z \in A$ and there is an integer $m > 1$ with $z \in E_m$. So $\mathrm{osc}\, g(z) < c_{m-1}$ and there is an open interval $I \ni z$ such that $\mathrm{osc}_I\, g < c_{m-1}$. We have either $\phi(z) = g(z) < c$ or $\phi(z) = g(z) > c$. Suppose that $g(z) < z$. Then there is a point $t \in I \cap [a, b]$ with $g(t) > c$. Since $t \in I$, we have $g(t) - g(z) < c_{m-1}$ and consequently $c - g(z) < c_{m-1}$. From the construction of ϕ_m follows that there is a nondegenerate closed interval $J \subset [a, b] \cap I$ such that $\phi_m(J) \supset [g(z), g(t)] \ni c$, a contradiction. So the function ϕ has Darboux property. If $g(z) > c$ the reasoning is similar. The same we can show that the function ψ has Darboux property. □

Lemma 1. *If $A \subset \mathbb{R}$ is a nonempty compact set of Lebesgue measure zero, $U \supset A$ is a bounded open set and $E \subset U \setminus A$ is a dense set in U, then there is a* *family* $K_{i,j}$, $i,j = 1,2,\ldots$, *of pairwise disjoint nondegenerate closed intervals* $K_{i,j} \subset U \setminus A$ *with the endpoints belonging to* E *such that for each positive integer* i *and each point* $x \in A$ *the upper density* $D_u(\bigcup_{j=1}^{\infty} K_{i,j}, x) = 1$ *and for each positive real* r *the set of pairs* (i,j) *for which* $\operatorname{dist}(K_{i,j}, A) \geq r$ *is empty or finite.*

Proof. Since the set A is compact, there are pairwise disjoint open intervals

$$I_{1,1}, I_{1,2}, \ldots, I_{1,i(1)} \subset U \cap \bigcup_{x \in A} (x-1, x+1)$$

such that $A \subset U_1 = I_{1,1} \cup \ldots \cup I_{1,i(1)}$ and $I_{1,i} \cap A \neq \emptyset$ for $i \leq i(1)$. There are pairwise disjoint nondegenerate closed intervals $L_{1,1}, \ldots, L_{1,k(1)} \subset U_1 \setminus A$ with the endpoints belonging to E such that for every positive integer $j \leq i(1)$ the inequality

$$\frac{\mu(I_{1,j} \cap \bigcup_{i \leq k(1)} L_{1,i})}{\mu(I_{1,j})} > \frac{1}{2}$$

is true.

In the second step put

$$r_2 = \frac{\inf\{|x-y|; x \in A,\ y \in \bigcup_{i \leq k(1)} L_{1,i}\}}{2}.$$

There are pairwise disjoint open intervals

$$I_{2,1}, I_{2,2}, \ldots, I_{2,i(2)} \subset U \cap \bigcup_{x \in A} (x - r_2, x + r_2)$$

such that $A \subset U_2 = I_{2,1} \cup \ldots \cup I_{2,i(2)}$ and $I_{2,j} \cap A \neq \emptyset$ for $j \leq i(2)$. Now we find pairwise disjoint nondegenerate closed intervals $L_{2,1}, \ldots, L_{2,k(2)} \subset U_2 \setminus A$ with the endpoints belonging to E such that for every positive integer $j \leq i(2)$ the inequality

$$\frac{\mu(I_{2,j} \cap \bigcup_{i \leq k(2)} L_{2,i})}{\mu(I_{2,j})} > 1 - \frac{1}{2^2}$$

is true.

In general in n^{th} step we define the positive real

$$r_n = \frac{\inf\{|x-y|; x \in A,\ y \in \bigcup_{i \leq k(n-1)} L_{n-1,i}\}}{2},$$

pairwise disjoint open intervals $I_{n,1}, I_{n,2}, \ldots, I_{n,i(n)} \subset U \cap \bigcup_{x \in A} (x - r_n, x + r_n)$ such that $A \subset U_n = I_{n,1} \cup \ldots \cup I_{n,i(n)}$ and $I_{n,j} \cap A \neq \emptyset$ for $j \leq i(n)$, and pairwise disjoint nondegenerate closed intervals $L_{n,1}, \dots, L_{n,k(n)} \subset U_n \setminus A$ with the endpoints belonging to E such that for each positive integer $j \le i(n)$ the inequality

$$\frac{\mu(I_{n,j} \cap \bigcup_{i \le k(n)} L_{n,i})}{\mu(I_{n,j})} > 1 - \frac{1}{2^n}$$

holds.

Let $N_1, N_2, \dots, N_m, \dots$ be a sequence of pairwise disjoint infinite subsets of positive integers and let $N_k = \{n_{k,1}, n_{k,2}, \dots\}$, where $n_{k,i} < n_{k,j}$ for $i < j$. For $i = 1, 2, \dots$ let

$$(K_{i,j})_j = (L_{n_{i,1},1}, \dots, L_{n_{i,1},k(n_{i,1})}, L_{n_{i,2},1}, \dots, L_{n_{i,2},k(n_{i,2})}, \dots).$$

We will prove that the family $\{K_{i,j}; i, j = 1, 2, \dots\}$ satisfies all requirements. From the construction follows immediately that the intervals $K_{i,j} \subset U \setminus A$ are pairwise disjoint and their endpoints belong to E. Fix a positive real r. There is an integer k such that $r_n < r$ for $n \ge k$. Observe that for $L_{n,j}$ with $n \ge k$ we obtain $\operatorname{dist}(A, L_{n,j}) < r_n < r$. So the set of all pairs (i, j) for which $\operatorname{dist}(A, K_{i,j}) \ge r$ is empty or finite. Now fix an integer i and a point $x \in A$. For each integer $n_{i,j}$ there is an open interval $I_{n_{i,j},l_{i,j}} \ni x$, where $l_{i,j} \le i(n_{i,j})$. Evidently $\lim_{j\to\infty} d(I_{n_{i,j},l_{i,j}}) = 0$. Since

$$\frac{\mu(I_{n_{i,j},l_{i,j}} \cap \bigcup_{m \le k(n_{i,j})} L_{n_{i,j},m})}{\mu(I_{n_{i,j},l_{i,j}})} > 1 - \frac{1}{2^{n_{i,j}}}$$

and all intervals $L_{n_{i,j},m}$ occur in the sequence $(K_{i,j})$, we have

$$D_u(\bigcup_{j=1}^{\infty} K_{i,j}, x) = 1. \qquad \square$$

Theorem 2. *If the function $f : \mathbb{R} \to \mathbb{R}$ has property (s_2), then there are functions $g, h : \mathbb{R} \to \mathbb{R}$ having the property (s_1) such that $f = g + h$.*

PROOF. At first suppose that the set $D(f)$ of all discontinuity points is bounded. Then $\operatorname{cl}(D(f))$ is a compact set. If $\mu(\operatorname{cl}(D(f)) = 0$, then by Lemma 1 there is a family $K_{i,j}$ of pairwise disjoint nondegenerate closed intervals $K_{i,j} \subset \mathbb{R} \setminus \operatorname{cl}(D(f))$, $i, j \ge 1$ such that for each positive real r the set of pairs (i, j) for which $\operatorname{dist}(K_{i,j}, \operatorname{cl}(D(f))) \ge r$ is empty or finite and such that for each integer i and each point $x \in \operatorname{cl}(D(f))$ the upper density $D_u(\bigcup_{j=1}^{\infty} K_{i,j}, x) = 1$. Let (w_i) be a sequence of all rationals and let

$$g(x) = \begin{cases} w_i & \text{for } x \in K_{2i-1,j} \\ f(x) - w_i & \text{for } x \in K_{2i,j} \\ f(x) & \text{otherwise on } \mathbb{R} \end{cases}$$

and

$$h(x) = \begin{cases} f(x) - w_i & \text{for } x \in K_{2i-1,j} \\ w_i & \text{for } x \in K_{2i,j} \\ 0 & \text{otherwise on } \mathbb{R}. \end{cases}$$

Evidently, $g+h=f$. If $x \in \mathbb{R} \setminus \mathrm{cl}(D(f))$, then g and h are continuous on some interval $[x, x+s) \subset \mathbb{R} \setminus \mathrm{cl}(D(f))$ or $(x-s, x] \subset \mathbb{R} \setminus \mathrm{cl}(D(f))$, where $s > 0$ and consequently they have property (s_1) at x.

If $x \in \mathrm{cl}(D(f))$, $x \in U \in T_d$ and $r > 0$, then there is an index k with $|f(x) - w_k| < r$. Since $D_u(\bigcup_{j=1}^{\infty} K_{2k-1,j}, x) = 1$, there is an index m such that $\emptyset \neq \mathrm{int}(K_{2k-1,m}) \cap U$. For $u \in K_{2k-1,m} \cap U$ we have $|g(u) - g(x)| = |w_k - f(x)| < r$, thus the function g has property (s_1) at x. Similarly we can check that the function h has property (s_1) at $x \in \mathrm{cl}(D(f))$. So the proof in the case, where $\mu(\mathrm{cl}(D(f)) = 0$ (and $D(f)$ is bounded) is finished.

So suppose that $\mu(\mathrm{cl}(D(f)) > 0$. Then there is a sequence (a_n) of positive reals such that $a_{n+1} < a_n$ for ≥ 1 and $\sum_{k=1}^{\infty} a_k < \infty$, and $A_{n+1} \setminus A_n \neq \emptyset$ for $n = 1, 2, \ldots$, where $A_n = \{x; \mathrm{osc}\, f(x) \geq a_n\}$. Every set A_n is closed of measure zero and for the set $D(f)$ of all discontinuity points of f the equality $D(f) = \bigcup_{n=1}^{\infty} A_n$ is true. By Lemma 1 there is a family of pairwise disjoint closed intervals

$$K_{1,i,j} \subset \mathbb{R} \setminus A_1, \quad i, j = 1, 2, \ldots,$$

with the endpoints belonging to $C(f)$ such that for each $i = 1, 2, \ldots$ and for each $x \in A_1$ the upper density $D_u(\bigcup_{j=1}^{\infty} K_{1,i,j}, x) = 1$ and for each positive real r the set of pairs (i, j) for which $\mathrm{dist}(K_{1,i,j}, A_1) \geq r$ is empty or finite.

In the interiors $\mathrm{int}(K_{1,i,j})$ we find closed intervals $I_{1,i,j} \subset \mathrm{int}(K_{1,i,j})$ such that for each point $x \in A_1$ and for each integer $i = 1, 2, \ldots$ the upper density

$$D_u(\bigcup_{j=1}^{\infty} I_{1,i,j}, x) = 1.$$

Let $w_{1,i}$ be a sequence of all rationals and let $g_1, h_1 : \mathbb{R} \to \mathbb{R}$ be defined as follows

$$g_1(x) = \begin{cases} w_{1,i} & \text{for } x \in I_{1,2i,j}, \ \ i, j = 1, 2, \ldots \\ f(x) & \text{for } x \in \mathbb{R} \setminus \bigcup_{i,j=1}^{\infty} \mathrm{int}(K_{1,2i,j}), \end{cases}$$

g_1 is linear on all components of the sets $K_{1,2i.j} \setminus \mathrm{int}(I_{1,2i,j})$, $i, j = 1, 2, \ldots$, and $h_1(x) = f(x) - g_1(x)$ for $x \in \mathbb{R}$.

In the second step we consider the set $A_2 \setminus A_1 = A_2 \cap (\mathbb{R} \setminus A_1)$. There are pairwise disjoint open intervals $P_{2,k} \subset \mathbb{R} \setminus A_1$, $k \geq 1$, with the centers belonging to $C(f)$ such that every set $A_2 \cap P_{2,k}$ is nonempty and compact and

$$A_2 \setminus A_1 = \bigcup_k (A_2 \cap P_{2,k}).$$

A construction of such intervals $P_{2,k}$ may be the following.w We find a bounded open set $G \supset A_2$ and divide each component of the the set $G \setminus A_1$ by points belonging to $C(f)$ into open intervals. As $P_{2,k}$ we take all from the above intervals which have common points with A_2.

If $x \in (A_2 \cap \operatorname{int}(K_{1,2i,j})) \setminus A_1$ for some pair (i, j), then g_1 is continuous at x, and consequently $\operatorname{osc} g_1(x) = 0$ and $\operatorname{osc} h_1(x) = \operatorname{osc} f(x) < a_1$. If

$$x \in A_2 \setminus A_1 \setminus \bigcup_{i,j \geq 1} K_{1,2i,j}$$

then $g_1(t) = f(t)$ and $h_1(t) = 0$ on an open interval containing x and contained in $\mathbb{R} \setminus A_1$. So $\operatorname{osc} g_1(x) = \operatorname{osc} f(x) < a_1$ and $\operatorname{osc} h_1(x) = 0$. Similarly we show that $\max(\operatorname{osc} g_1(x), \operatorname{osc} h_1(x)) < a_1$ if $x \in A_2 \setminus A_1$ is an endpoint of some $K_{1,2i,j}$. So for each integer k and each point $x \in A_2 \cap P_{2,k}$ there is an open interval $J_{2,k}(x) \subset P_{2,k}$ containing x such that on the interval $J_{2,k}(x)$ the oscillation $\operatorname{osc}_{J_{2,k}(x)} g_1 < a_1$ and $\operatorname{osc}_{J_{2,k}(x)} h_1 < a_1$. Since the set $A_2 \cap P_{2,k}$ is compact there are points $x_1, x_2, \ldots, x_{j(k)}$ such that $A_2 \cap P_{2,k} \subset J_{2,k}(x_1) \cup \ldots \cup J_{2,k}(x_{j(k)})$. Without loss of the generality we can assume that the above intervals $J_{2,k}(x_j)$, $j \leq j(k)$, are pairwise disjoint. For each pair of positive integers (i, j) such that $A_2 \cap K_{1,i,j} \neq \emptyset$ we find an open set $U(K_{1,i,j}) \subset \operatorname{int}(K_{1,i,j})$ such that $A_2 \cap K_{1,i,j} \subset U(K_{1,i,j})$ and

$$\frac{\mu(\operatorname{cl}(U(K_{1,i,j})))}{\mu(K_{1,i,j})} < \frac{1}{4^{1+i+j}}.$$

If for some integers i_1, j_1, j_2 the intersection $A_2 \cap \operatorname{int}(K_{1,i_1,j_1}) \cap J_{2,k}(x_{j_2}) \neq \emptyset$ then, by Lemma 1, we find pairwise disjoint nondegenerate closed intervals

$$K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2})) \subset U(K_{1,i_1,j_1}) \cap J_{2,k}(x_{j_2})$$

with the endpoints belonging to $C(f)$ such that for every positive integer i and every point $x \in A_2 \cap J_{2,k}(x_{j_2}) \cap K_{1,i_1,j_1}$ the upper density

$$D_u(\bigcup_{j=1}^{\infty} K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2})), x) = 1$$

and for every positive real r the set of all pairs (i, j) for which

$$\operatorname{dist}(A_2 \cap J_{2,k}(x_{j_2}) \cap K_{1,i_1,j_1}, K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2}))) > r$$

is empty or finite. In every interval $\operatorname{int}(K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2})))$ we find a closed interval $I_{2,i,j}(K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2}))$ such that for every integer i and for every point $x \in A_2 \cap J_{2,k}(x_{j_2}) \cap K_{1,i_1,j_1}$ the upper density

$$D_u(\bigcup_{j=1}^{\infty} I_{2,i,j}(K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2}))), x) = 1. \tag{4}$$

For each positive integer $j \le j(k)$ let $(w_i(x_j))$ be an enumeration of all rationals in $(y_j - \frac{a_1}{2}, y_j + \frac{a_1}{2})$, where y_j is the center of the interval

$$[\inf_{A_2 \cap J_{2,k}(x_j)} g_1, \sup_{A_2 \cap J_{2,k}(x_j)} g_1],$$

and let $(u_i(x_j))$ be an enumeration of all rationals in $(z_j - \frac{a_1}{2}, z_j + \frac{a_1}{2})$, where z_j is the center of the interval $[\inf_{A_2 \cap J_{2,k}(x_j)} h_1, \sup_{A_2 \cap J_{2,k}(x_j)} h_1]$. Put

$$g_2(x) = \begin{cases} w_i(x_{j_2}) & \text{for } x \in I_{2,2i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2})), \\ & j_2 \le j(k), \quad i,j = 1,2,\dots \\ f(x) - h_2(x) & \text{for } x \in I_{2,2i-1,j}(K_{1,i_1,j_1} J_{2,k}(x_{j_2})) \\ & j_2 \le j(k), \quad i,j = 1,2,\dots, \\ g_1(x) & \text{for } x \in K_{1,i_1,j_1} \setminus \bigcup_{j_2 \le j(k)} \bigcup_{i,j=1}^{\infty} K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2})), \end{cases}$$

and

$$h_2(x) = \begin{cases} f(x) - g_2(x) & \text{for } x \in I_{2,2i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2})), \\ & j_2 \le j(k), \quad i,j = 1,2,\dots \\ u_i(x_{j_2}) & \text{for } x \in I_{2,2i-1,j}(K_{1,i_1,j_1} J_{2,k}(x_{j_2})) \\ & j_2 \le j(k), \quad i,j = 1,2,\dots, \\ h_1(x) & \text{for } x \in K_{1,i_1,j_1} \setminus \bigcup_{j_2 \le j(k)} \bigcup_{i,j=1}^{\infty} K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2})), \end{cases}$$

and assume that the function g_2 is linear and $h_2 = f - g_2$ on the components of the sets $K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2})) \setminus I_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2}))$.

Similarly, modifying the values of g_1 and h_1 on respectively constructed closed intervals we define the functions g_2 and h_2 on components $L_{2,m}$ of the set $P_{2,k} \setminus A_1 \setminus \bigcup_{i,j=1}^{\infty} K_{1,i,j}$ for which $L_{2,m} \cap A_2 \neq \emptyset$. Put $g_2(x) = g_1(x)$ and $h_2(x) = h_1(x)$ otherwise on $\mathbb{R}$. Observe that if the function f is continuous at a point x, then from the constructions of g_1 and g_2 follows that $x \in \mathbb{R} \setminus A_2$, and g_1 and g_2 are continuous at x. Consequently the functions h_1 and h_2 as the differences of functions continuous at x are also continuous at this point. So $C(f) \subset C(g_2) \cap C(h_2)$. Observe that

$$|g_2 - g_1| \leq a_1, \ \ |h_2 - h_1| \leq a_1 \ \text{ and } \ g_2 + h_2 = f.$$

We will show that $g_2, h_2 \in s_1(x)$ for $x \in A_2$. For this fix a point $x \in A_2$, a set $U \ni x$ belonging to T_d and a real $r > 0$.

If $x \in A_1$, then we find a rational $w_{1,k}$ with $|g_1(x) - w_{1,k}| < r$. Since

$$D_u(\bigcup_{j=1}^{\infty} I_{1,2k,j}, x) = 1 \text{ and } \frac{\mu(\mathrm{cl}(U(K_{1,2k,j})))}{\mu(K_{1,2k,j})} < \frac{1}{4^{1+2k+j}},$$

we obtain

$$D_u((g_1)^{-1}(w_{1,k}) \cap \bigcup_{j=1}^{\infty} I_{1,2k,j}, x) = 1$$

and consequently there is an integer m and an open interval $I \subset I_{1,2k,m} \setminus \mathrm{cl}(U(K_{1,2k,m}))$ such that $\emptyset \neq I \cap U$. But $g_2(u) = w_{1,k}$ for $u \in I \cap U$; so $I \cap U \subset C(g_2)$. Moreover for $u \in I \cap U$ we have $|g_2(u) - g_2(x)| = |w_{1,k} - g_2(x)| < r$. So $g_2 \in s_1(x)$ for $x \in A_1$. Similarly we show that $h_2 \in s_1(x)$ for $x \in A_1$.

Using (4) by similar reasoning we can show that $g_2, h_2 \in s_1(x)$ for $x \in A_2 \setminus A_1$. Let $(K_{2,i,j})$ be a double sequence of all closed intervals on which we have modified the functions g_1 and h_1 for the obtaining of g_2 and h_2. Similarly in n^{th} step we change the functions g_{n-1} and h_{n-1} on respectively taken closed intervals $K_{n,2i,j}$ and $K_{n,2i-1,j}$ and define functions g_n and h_n such that g_n (and resp. h_n) has constant rational values on respective closed intervals $I_{n,2i,j} \subset \mathrm{int}(K_{n,2i,j})$ (resp. on $I_{n,2i-1,j}$), $C(f) \subset C(g_n) \cap C(h_n)$, $g_n, h_n \in s_1(x)$ for $x \in A_n$,

$$|g_n - g_{n-1}| \leq a_{n-1}, \ \ |h_n - h_{n-1}| \leq a_{n-1}, \ \text{ and } \ g_n + h_n = f.$$

Moreover, we suppose that for every triple (k, i_1, j_1), where $k < n$ and $i_1, j_1 = 1, 2, \ldots$, the inequality

$$\frac{\mu(K_{k,i_1,j_1} \setminus \bigcup_{i,j=1}^{\infty} K_{n,i,j})}{\mu(K_{k,i_1,j_1})} > 1 - \frac{1}{4^{n+i+j}} \tag{5}$$

is true. Let $g = \lim_{n\to\infty} g_n$ and $h = \lim_{n\to\infty} h_n$. Evidently, $g + h = f$. Since the above convergence is uniform, $C(f) \subset C(g) \cap C(h)$, and consequently the functions g, h have property (s_2). We will prove that the functions g, h have property (s_1). For this, fix a positive real r, a point $x \in \mathbb{R}$ and a set $U \in T_d$ containing x. If $x \in C(f)$, then g is continuous at x and there is a positive real s such that $|g(t) - g(x)| < r$ for $t \in (x - s, x + s)$. But g has property (s_2), there are an open interval $J \subset (x-s, x+s)$ such that $C(g) \supset J \cap U \neq \emptyset$. Since $|g(t) - g(x)| < r$ for $t \in J \cap U$, we obtain $g \in s_1(x)$. Similarly we can prove that $h \in s_1(x)$.

In the case where x is a discontinuity point of g the point x is also a discontinuity point of f and there is a positive integer n such that $x \in A_n \setminus A_{n-1}$ (we assume that $A_0 = \emptyset$). Let $k > n$ be a positive integer such that $\sum_{i=k+1}^{\infty} a_i < \frac{r}{3}$. There is a rational value w of the function g_n such that $|g_n(x) - w| < \frac{r}{3}$ and $D_u((g_n)^{-1}(w), x) = 1$. By condition (5) the upper density

$$D_u((g_n)^{-1}(w) \setminus \bigcup_{m>n} \bigcup_{l,j=1}^{\infty} K_{m,l,j}, x) = 1.$$

So

$$D_u((g_n)^{-1}(w) \setminus \bigcup_{m=n+1}^{k-1} \bigcup_{l,j=1}^{\infty} K_{m,l,j}, x) = 1,$$

and by the construction of g_n and $K_{m,l,j}$ also

$$D_u(\mathrm{int}((g_n)^{-1}(w) \setminus \bigcup_{m=n+1}^{k-1} \bigcup_{l,j=1}^{\infty} K_{m,l,j}), x) = 1.$$

Since $x \in U \in T_d$, we have

$$D_u(U \cap \mathrm{int}((g_n)^{-1}(w) \setminus \bigcup_{m=n+1}^{k-1} \bigcup_{l,j=1}^{\infty} K_{m,l,j}), x) = 1.$$

Consequently, there is an open interval

$$I \subset \mathrm{int}((g_n)^{-1}(w) \setminus \bigcup_{m=n+1}^{k-1} \bigcup_{l,j=1}^{\infty} K_{m,l,j}) \setminus A_k$$

such that $I \cap U \neq \emptyset$. Evidently, $\emptyset \neq I \cap U \subset C(f) \subset C(g)$. For $t \in I \cap U$ we obtain $g_n(t) = g_k(t)$ and

$$|g(t) - g(x)| = |g(t) - g_k(t) + w - g_n(x)| \leq \sum_{i=k+1}^{\infty} a_i + \frac{r}{3} < \frac{2r}{3} < r.$$

So $g \in s_1(x)$. The proof that $h \in s_1(x)$ is analogous.

Up to now we have supposed that the set $D(f)$ is bounded. Now we consider the general case. Since the closure $\mathrm{cl}(D(f))$ is a nowhere dense set, there are points $x_k \in \mathbb{R} \setminus \mathrm{cl}(D(f))$, $k = 0, 1, -1, 2, -2, \ldots$ such that

$$\lim_{k\to-\infty} x_k = -\infty, \quad \lim_{k\to\infty} x_k = \infty, \quad \text{and} \quad x_k < x_{k+1} \quad \text{for all integers} \quad k.$$

Then $\mathbb{R} = \bigcup_{k=-\infty}^{\infty}[x_k, x_{k+1}]$. Every restricted function $f_k = f/[x_k, x_{k+1}]$ is the sum of two functions $g_k, h_k : [x_k, x_{k+1}] \to \mathbb{R}$ having property (s_1) and continuous at the points x_k and x_{k+1}. Let

$$g(x) = \begin{cases} g_k(x) - (a_1 + \cdots + a_k) & \text{for } x \in [x_k, x_{k+1}] \\ g_0(x) & \text{for } x \in [0, 1] \\ g_k(x) + (a_0 + a_{-1} + \cdots + a_{k+1}) & \text{for } x \in [x_k, x_{k+1}] \end{cases}$$

where $a_k = g_k(k) - g_{k-1}(k)$ for $k = 0, -1, 1, -2, 2, \ldots$, and $h(x) = f(x) - g(x)$ for $x \in \mathbb{R}$. Observe that the functions g, h have property (s_1) and $f = g + h$. □

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