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AN ESTIMATE OF THE FIRST DERIVATIVE BY THE LAPLACIAN

Abstract

In this note a particular case of the following general problem is considered: how to control lower order derivatives by higher ones, at least over a sequence of points. The following particular case is proved: if a C^2 negative-valued function $h = h(w)$ depends on one complex variable in the unit disc and $h(1) = h_w(1) = 0$, then the first derivative h_w is controlled by the Laplacian of h over a sequence of points converging to $w = 1$. Such kind of estimates have applications to delicate problems of convexity with respect to various families of functions

1 Introduction

For real functions of one real variable it is a very easy exercise to show that: *If $h \in C^2([0,1])$, $h(1) = h'(1) = 0$, $h(x) < 0$ for $x \in (0,1)$, then there is a sequence $x_n \to 1$ such that $h'(x_n) > 0$, $h''(x_n) < 0$, and $h'(x_n) \leq -\frac{1}{n}h''(x_n)$.* The main goal of this note is to prove a corresponding property for functions of one complex variable.

Theorem I. *Let $D = \{w \in \mathbb{C}; |w| < 1\}$, $h\ \overline{D} \to \mathbb{R}$, $h \in C^2(\overline{D})$, $h(w) < 0$ for $w \in D$, and $h(1) = h_w(1) = 0$. Then there is a sequence $\{w_n\}_{n=1}^{\infty} \subset D$, $\lim_{n\to\infty} w_n = 1$, such that $\Delta h(w_n) < 0$, $|h_w(w_n)| < -\frac{1}{n}\Delta h(w_n)$ for $n = 1, 2, \ldots$, where Δ denotes the Laplacian.*

A motivation to consider such question came from complex analysis, harmonic analysis, and the theory of convex functions, especially dealing with

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pseudoconvexity and plurisubharmonic functions (see [1], [2], [3] where some applications can be found). The theorem holds under weaker assumptions, but then the formulation is more technical.

2 Notation and Formulation of a More General Theorem

Let D be the unit disc in the complex plane $\mathbb{C}$; i.e., $D = \{w \in \mathbb{C}; |w| < 1\}$, and let

$$h : \overline{D} \to \mathbb{R},\ h \in C^2(\overline{D}),\ h(w) < 0 \text{ for } w \in D,$$

and such that

$$S = \{w \in \partial D;\, h(w) = 0\} = \{1\},\ h_w(1) = 0.$$

For convenience, we write $h(r, \theta) = h(re^{i\theta})$ and we let

$$\vartheta_r = \{\theta \in [0, 2\pi];\, h(r, \theta) = \sup_{0 \le t \le 2\pi} h(r, t)\},$$

$$\varphi(r) = h(r, \vartheta_r) = \sup_{0 \le t \le 2\pi} h(r, t) \text{ for } 0 \le r \le 1,$$

$$h_r(r, \vartheta_r) = \sup\{h_r(r, \theta);\ \theta \in \vartheta_r\},\quad h_r(r, \theta_r) = [h(\rho, \theta_r)]'_{\rho = r}, \quad \text{if } \theta_r \in \vartheta_r.$$

Theorem II. *With the above assumptions and notation, there exist a sequence $r_n \nearrow 1$ and a sequence $\theta_{r_n} \in \vartheta_{r_n}$ such that*

$$h_{rr}(r_n, \theta_{r_n}) < 0 \text{ and } 0 < h_r(r_n, \theta_{r_n}) \le -\frac{1}{n} h_{rr}(r_n, \theta_{r_n}). \tag{2.1}$$

We note that Theorem I immediately follows from Theorem II because at points $(r, \theta), \theta \in \vartheta_r$, we have $h_\theta(r, \theta) = 0$, $h_{\theta\theta}(r, \theta) \le 0$, $h_w(w) = \frac{1}{2} e^{-i\theta} h_r(r, \theta)$, and (2.1) immediately yields the estimate from Theorem 1 when we rewrite the Laplacian in the polar coordinates:

$$\Delta h(w) = \Delta h(r, \theta) = h_{rr}(r, \theta) + \frac{1}{r} h_r(r, \theta) + \frac{1}{r^2} h_{\theta\theta}(r, \theta).$$

3 Proof of Theorem II

We divide the proof of Theorem II into four lemmas. Before we formulate and prove the lemmas, we need more notation.

The image of $\overline{D}$ under h is an interval $[-a, 0]$ for some $a > 0$. We denote by $\mathfrak{C} = \mathfrak{C}^h$ the set of critical points of h,

$$\mathfrak{C} = \mathfrak{C}^h = \{w \in \overline{D};\, h_w(w) = 0\}.$$

By Sard's theorem (see e.g. [4]) we have $\mathrm{meas}(h(\mathfrak{C})) = 0$. We put

$$B = [-a, 0] \setminus h(\mathfrak{C}) = (-a, 0) \setminus h(\mathfrak{C}). \tag{3.1}$$

Obviously, the sets $\mathfrak{C}$ and $h(\mathfrak{C})$ are compact. Therefore B is open in $\mathbb{R}$.

Remark 1. In order to apply Sard's theorem for h, the minimum differentiability assumption is class C^2, which follows, for instance, from the remarks in [4], p. 20.

Lemma 1. *Let $H : \overline{D} \longrightarrow \mathbb{R}$, $H \in C^2(\overline{D})$. We can define the corresponding sets ϑ_r for H and also use the other notation. If $H_r(r, \vartheta_r) \leq 0$ for $0 < r < 1$, then the function $r \longrightarrow H(r, \vartheta_r)$ decreases.*

PROOF. We put

$$E = \{w = re^{i\theta} \in \overline{D};\ \ H_r(r, \theta) = 0,\ \forall \theta \in \vartheta_r\} \subset \mathfrak{C}^H,\ \ F = H(\mathfrak{C}^H),$$

and we have $0 \leq \mathrm{meas}(H(E)) \leq \mathrm{meas}(F) = 0$. We note that the function $\phi(r) = H(r, \vartheta_r)$ is continuous for $r \in [0, 1]$ and its image is an interval $I \subset \mathbb{R}$. If the interval I is degenerate; i.e., contains only a point, then the lemma is obvious. So we can assume that I is not just one point. Take $\phi^{-1}(I \setminus F)$, which is open and nonempty in $[0, 1]$. It is enough to show that ϕ decreases on this set. If $r \in \phi^{-1}(I \setminus F)$, then there exists $\theta_0 \in \vartheta_r$ such that $H_r(r, \theta_0) < 0$, which gives the inequalities

$$\phi(\rho) = H(\rho, \vartheta_\rho) \geq H(\rho, \theta_0) > H(r, \theta_0) = \phi(r) \quad \text{for } \rho < r \ (\rho \text{ close to } r).$$

This means that ϕ strictly decreases on each component of $\phi^{-1}(I \setminus F)$, and consequently, decreases on $[0, 1]$. □

Remark 2. We use Lemma 1 several times later in this section. Each time we need a slightly different version of the lemma, which follows easily from the version given above. Adjusting Lemma 1 to each individual case is left to the reader.

Lemma 2. *With the notation from §2, there exist $0 < r < 1$ and $\theta_r \in \vartheta_r$ such that $h_r(r, \theta_r) > 0$.*

PROOF. Assume to the contrary that $\forall_{0<r<1} \forall_{\theta_r \in \vartheta_r}\ h_r(r, \theta_r) \leq 0$. Then, by Lemma 1, the function $r \rightarrow h(r, \vartheta_r)$ decreases, and we get a contradiction $-a = h(0) \geq h(1, \vartheta_1) = 0$, $0 < a \leq 0$. □

Up to the end of this section, we fix $r_0 \in \varphi^{-1}(B)$ (φ is defined in Section 2 and B is defined in (3.1)) such that $h_r(r_0, \vartheta_{r_0}) > 0$, and define

$$r^0 = \inf\{\rho > r_0;\, h_r(\rho, \vartheta_\rho) \le 0\} = \inf\{\rho > r_0;\, h_r(\rho, \vartheta_\rho) = 0\}.$$

Since the set under inf is nonempty, $r^0 \le 1$.

Lemma 3. *With the above notation, we have* $r_0 < r^0$.

PROOF. Assume to the contrary that $r_0 = r^0$. The point r_0 belongs to $\varphi^{-1}(B)$. Therefore for r from a small neighborhood of r_0 we have $h_r(r, \theta) \ne 0$, $\theta \in \vartheta_r$. Consequently,

$$r_0 = r^0 = \inf\{\rho > r_0;\, h_r(\rho, \vartheta_\rho) < 0\}. \tag{3.2}$$

Since $h_r(r_0, \theta_{r_0}) > 0$ for some $\theta_{r_0} \in \vartheta_{r_0}$, $\varphi(r) > \varphi(r_0)$ for $r > r_0$ close to r_0. From (3.2) and the last argument, we get that there exist points $r_n > r_0$, arbitrarily close to r_0, where the function φ attains local maxima. But at these points we have $h_r(r_n, \theta) = 0$, $\theta \in \vartheta_{r_n}$, which contradicts the choice of r_0 and, consequently, proves the lemma. □

Lemma 4. *The function* $h(r, \theta)$ *has the property*

$$\forall_{C>0}\ \exists_{r_0 \le r < r^0}\ \exists_{\theta_r \in \vartheta_r}\ \ h_{rr}(r, \theta_r) < 0 \ \text{and}\ 0 < h_r(r, \theta_r) \le -C h_{rr}(r, \theta_r).$$

BEGINNING OF THE PROOF LEMMA 4. Assume to the contrary that

$$\exists_{C>0}\ \forall_{r_0 \le r < r^0}\ \forall_{\theta_r \in \vartheta_r}\ h_{rr}(r, \theta_r) \ge 0 \ \text{or}\ h_r(r, \theta_r) > -C h_{rr}(r, \theta_r). \tag{3.3}$$

Condition (3.3) implies one of the following three cases:

$$h_{rr}(r, \theta_r) > 0 \ \text{for some}\ \theta_r \in \vartheta_r, \tag{3.4}$$

$$h_r(r, \theta_r) > -C h_{rr}(r, \theta_r) \ \text{and}\ h_{rr}(r, \theta_r) < 0 \ \text{for some}\ \theta_r \in \vartheta_r, \tag{3.5}$$

$$h_{rr}(r, \theta_r) = 0 \ \text{for some}\ \theta_r \in \vartheta_r. \tag{3.6}$$

□

Now we consider these three cases in the subsequent three sublemmas.

Sub Lemma 4-1. *If* (3.4) *is satisfied, then*

$$\forall_{\varepsilon>0}\ \exists_{r<\rho<r+\varepsilon}\ \exists_{\theta_\rho \in \vartheta_\rho}\ h_r(\rho, \theta_\rho) > h_r(r, \theta_r).$$

Proof of Sublemma 4-1. Assume to the contrary that there exists ε_0 such that for any $r < \rho < r + \varepsilon_0$ we have $h_r(\rho, \theta_\rho) \le h_r(r, \theta_r)$ for any $\theta_\rho \in \vartheta_\rho$. We define the function $H(\rho, \theta) = h(\rho, \theta) - \rho h_r(r, \theta_r)$. Obviously

$$\forall_{\theta_\rho \in \vartheta_\rho} \; H_r(\rho, \theta_\rho) = h_r(\rho, \theta_\rho) - h_r(r, \theta_r) \le 0.$$

We can apply Lemma 1 to the function $H(\rho, \theta)$, $r < \rho < r + \varepsilon_0$, $\theta \in [0, 2\pi]$, and obtain that the function $(r, r + \varepsilon_0) \ni \rho \to H(\rho, \vartheta_\rho)$ decreases, which gives

$$h(\rho, \vartheta_\rho) \le h(r, \vartheta_r) + (\rho - r) h_r(r, \theta_r). \tag{3.7}$$

On the other hand, by (3.4), we have

$$h(\rho, \vartheta_\rho) \ge h(\rho, \theta_r) \ge h(r, \theta_r) + (\rho - r)[h_r(r, \theta_r) + \delta],$$

where $\delta = \delta(r, \rho) > 0$, and hence

$$h(\rho, \vartheta_\rho) \ge h(r, \vartheta_r) + (\rho - r) h_r(r, \theta_r) + \delta(\rho - r),$$

which contradicts (3.7). □

Sub Lemma 4-2. *If* (3.5) *holds, then*

$$\forall_{\varepsilon > 0} \; \exists_{r < \rho < r + \varepsilon} \; \exists_{\theta_\rho \in \vartheta_\rho} \ln h_r(\rho, \theta_\rho) \ge \ln h_r(r, \theta_r) - \frac{2}{C}(\rho - r). \tag{3.8}$$

Proof of Sublemma 4-2. By the assumptions of the sublemma we have $-\dfrac{1}{C} < \dfrac{h_{rr}(r, \theta_r)}{h_r(r, \theta_r)} < 0$, and from this we get $-\frac{1}{C} < [\ln h_r(\rho, \theta_r)]_\rho' < 0$ for ρ close to r. After integration with respect to ρ, we obtain

$$-\frac{1}{C}(\rho - r) + \ln h_r(r, \theta_r) < \ln h_r(\rho, \theta_r) < \ln h_r(r, \theta_r) \text{ for } \rho > r, \; \rho \text{ close to } r.$$

Now we take the exponential of the expressions at the left inequality, and we get $h_r(\rho, \theta_r) > e^{[-\frac{1}{C}(\rho - r)]} h_r(r, \theta_r)$ for $\rho > r$, ρ close to r. Again integrating with respect to ρ, we obtain

$$h(\rho, \theta_r) - h(r, \theta_r) > C \left[1 - e^{[-\frac{1}{C}(\rho - r)]}\right] h_r(r, \theta_r),$$

and from this

$$h(\rho, \vartheta_\rho) \ge h(\rho, \theta_r) > h(r, \theta_r) - C h_r(r, \theta_r) \, e^{[-\frac{1}{C}(\rho - r)]} + C h_r(r, \theta_r). \tag{3.9}$$

Assume that (3.8) does not hold; i.e.,

$$\exists_{\varepsilon>0}\ \forall_{r<\rho<r+\varepsilon}\ \forall_{\theta_\rho\in\vartheta_\rho}\quad \ln h_r(\rho,\theta_\rho) < \ln h_r(r,\theta_r) - \frac{2}{C}(\rho - r),$$

which gives $h_r(\rho,\theta_\rho) < e^{[-\frac{2}{C}(\rho-r)]}\, h_r(r,\theta_r)$ for ρ close to r. As in the previous sublemma, we introduce the function

$$H(\rho,\theta) = h(\rho,\theta) + \frac{C}{2}\, e^{[-\frac{2}{C}(\rho-r)]}\, h_r(r,\theta_r).$$

Since $H_\rho(\rho,\vartheta_\rho) < 0$, by Lemma 1, the function $\rho \to H(\rho,\vartheta_\rho)$ decreases on the interval $[r, r+\varepsilon]$, which yields

$$h(\rho,\vartheta_\rho) \le h(r,\vartheta_r) + \frac{C}{2}\, h_r(r,\theta_r)\left[1 - e^{[-\frac{2}{C}(\rho-r)]}\right] \text{ for } \rho\in(r,r+\varepsilon). \quad (3.10)$$

Comparing (3.9) and (3.10) we get a contradiction. □

The last case (3.6) can be easily reduced to Sublemma 4-2; so we leave the proof to the reader. We only formulate the following.

Sub Lemma 4-3. *If* (3.6) *is satisfied, then*

$$\forall_{\varepsilon>0}\ \exists_{r<\rho<r+\varepsilon}\ \exists_{\theta_\rho\in\vartheta_\rho}\ \ln h_r(\rho,\theta_\rho) \ge \ln h_r(r,\theta_r) - (\rho - r).$$

We need one more sublemma before finishing the proof of Lemma 4.

Sub Lemma 4-4. *Let a sequence* (r_n,θ_{r_n}), $n = 1, 2, \ldots$, *be given such that* $r_n \to\ r^*$ *and* $\theta_{r_n}\in\vartheta_{r_n}$. *Then* $\limsup\limits_{n\to\infty} h_r(r_n,\theta_{r_n}) \le h_r(r^*,\vartheta_{r^*})$.

PROOF OF SUBLEMMA 4-4. Without loss of generality we can assume that $r_n \to r^*$ and $\theta_{r_n} \to \theta^*$. Since

$$\lim_{n\to\infty} h(r_n,\theta_{r_n}) = h(r^*,\theta^*) = \sup_{0\le\theta\le 2\pi} h(r^*,\theta),$$

$\theta^*\in\vartheta_{r^*}$. By smoothness of h we obtain $\lim_{n\to\infty} h_r(r_n,\theta_{r_n}) = h_r(r^*,\theta^*) \le h_r(r^*,\vartheta_{r^*})$, and consequently, $\limsup_{n\to\infty} h_r(r_n,\theta_{r_n}) \le h_r(r^*,\vartheta_{r^*})$. □

END OF THE PROOF OF LEMMA 4. In the beginning of the proof of this lemma, we assumed (3.3). As we already mentioned, (3.3) implies (3.4)–(3.6). Now we shall get a contradiction to the definition of r^0.

Without loss of generality, we can assume that the constant C in (3.5) is smaller than 1/2. We let

$$R = \sup\{\rho\in(r_0,r^0);\ \ln h_r(\rho,\vartheta_\rho) \ge \ln h_r(r_0,\vartheta_{r_0}) - \frac{2}{C}(\rho - r_0)\}.$$

From Sublemmas 4-1–4-3 we have that $R > r_0$. Assume that $R < r^0$. Then there exist sequences $r_n \to R^-$ and $\theta_{r_n} \in \vartheta_{r_n}$ such that

$$\ln h_r(r_n, \theta_{r_n}) \geq \ln h_r(r_0, \vartheta_{r_0}) - \frac{2}{C}(r_n - r_0).$$

By Sublemma 4-4 we get

$$\ln h_r(R, \vartheta_R) \geq \ln h_r(r_0, \vartheta_{r_0}) - \frac{2}{C}(R - r_0).$$

Again applying Sublemmas 4-1–4-3, we obtain that there exists r^*, $r^* > R$, close to R such that

$$\begin{aligned}\ln h_r(r^*, \vartheta_{r^*}) &\geq \ln h_r(R, \vartheta_R) - \frac{2}{C}(r^* - R)\\ &\geq \ln h_r(r_0, \vartheta_{r_0}) - \frac{2}{C}(R - r_0) - \frac{2}{C}(r^* - R)\\ &= \ln h_r(r_0, \vartheta_{r_0}) - \frac{2}{C}(r^* - r_0).\end{aligned}$$

But the above contradicts the definition of R. Therefore $R = r^0$. Consequently, there exist sequences $r_n \to r^{0-}$ and $\theta_{r_n} \in \vartheta_{r_n}$ such that

$$\lim_{n\to\infty} \ln h_r(r_n, \theta_{r_n}) \geq \ln h_r(r_0, \vartheta_{r_0}) - \frac{2}{C}(r^0 - r_0).$$

From Sublemma 4-4 we get

$$h_r(r^0, \vartheta_{r^0}) \geq e^{[-\frac{2}{C}(r^0 - r_0)]}\, h_r(r_0, \vartheta_{r_0}) > 0.$$

On the other hand $h_r(r^0, \vartheta_{r^0}) = 0$, which contradicts the above inequality. This completes the proof of Lemma 4. □

Proof of Theorem II. We have two cases:

1^0 There exists $\varepsilon > 0$ such that $h_r(r, \vartheta_r) > 0$, for $1 - \varepsilon < r < 1$,

2^0 There exists a sequence $r_n \nearrow 1$ such that $h_r(r_n, \vartheta_{r_n}) \leq 0$.

In the first case, we immediately apply Lemma 4, where $r^0 = 1$, and we get the theorem. In the second case, it is very easy to construct a sequence of intervals (σ_n, τ_n), $\sigma_n < \tau_n$, $\sigma_n \to 1$, $\tau_n \to 1$, such that

$$h_r(\sigma_n, \vartheta_{\sigma_n}) > 0,\ h_r(\tau_n, \vartheta_{\tau_n}) = 0,\ h_r(r, \vartheta_r) > 0 \quad \text{for } \sigma_n < r < \tau_n.$$

We apply Lemma 4 to each interval (σ_n, τ_n), and again the theorem follows. □

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