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PERCENTILES

Abstract

It is shown that, similarly to the median, percentiles of a random variable can be characterized as minima of suitable functionals.

In our previous work, [1], we studied a functional $\xi_F(X)$ which assigns to a random variable X the real variable t at which the expected value $E(F(X-t))$ attains its minimum. Function F was assumed to be convex and even. For specific choices of F, $\xi_F(X)$ corresponds to quantities of significance in statistics – if $F(x)=|x|$, then $\xi_F(X)$ is the median of X, if $F(x)=|x|^k$, $k>1$, then $\xi_F(X)$ is the k-th moment of X. In the present note we consider a special case of F which is not even. For this choice of $F=f_s$ the resulting functional is a percentile of X.

The *r-percentile* of a real valued random variable (r.v.) X is defined as any number (or set of numbers) m such that $P(X\le m)\ge r$ and $P(X\ge m)\ge 1-r$. We denote this percentile by $m_r(X)$. For $r=\frac{1}{2}$ this is the median of X. Note that $m_r(X)$ may be multivalued.

For a real number $s>0$ let $F_s(x)=x$ for $x\ge 0$ and $F_s(x)=-sx$ for $x\le 0$. In particular, $F_1(x)=|x|$.

The following theorem is an extension of a well known result about the median.

Theorem 1. *For every r.v. X with finite expected value $E(|X|)$ the function $E(F_s(X-t))$ of the real variable t attains its minimum at $t\in m_r(X)$ where $r=\frac{1}{s+1}$.*

Proof. Replacing X by $X-m$, $m\in m_r(X)$ we may assume that $0\in m_r(X)$. For $t>0$ we want to show that $E(F_s(X\pm t))\ -E(F_s(X))=E(F_s(X\pm t)-F_s(X))=E(\Delta_{\pm t}F_s(X))\ge 0$. Observe that $\Delta_{-t}F_s(x)$ equals respectively to st if $x<0$, to $s(t-x)-x\ge -t$ if $0\le x\le t$, and to $-t$ if $x\ge t$. To translate

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this into the desired conclusion, denote by $\mathbf{1}_S$ the indicator function of an event S, and write

$$\Delta_{-t}F_s(X) = st\mathbf{1}_{\{X\le 0\}} + \mathbf{1}_{\{0<X\le t\}}(s(t-X)-X) + \mathbf{1}_{\{X>t\}}(-t) \ge st\mathbf{1}_{\{X\le 0\}} - t\mathbf{1}_{\{X>0\}}$$

and

$$E(\Delta_{-t}F_s(X)) \ge t(\frac{s}{s+1} - \frac{s}{s+1}) = 0.$$

The inequality $E(\Delta_t F_s(X)) \ge 0$ is obtained in the same manner by writing $\mathbf{1} = \mathbf{1}_{\{X\le -t\}} + \mathbf{1}_{\{-t<X<0\}} + \mathbf{1}_{\{X\ge 0\}}$ giving rise to the inequality

$$E(\Delta_t F_s(X)) \ge -stP(X<0) + tP(X\ge 0) \ge 0$$

. □

Similarly as in the case of the median, see [1], there is the following converse to Theorem 1.

Theorem 2. *If f is a function such that for every two-valued r.v. X, $E(f(X-t))$ attains its minimum at every $t \in m_r(X)$, where $r = \frac{1}{s+1}$, then f is of the form $f(x) = \alpha F_s(x) + \beta$, where α, β are constants, $\alpha \ge 0$.*

PROOF. We carry the argument under the assumption that f is continuous and then similarly as in [1] observe that this assumption is a consequence of the hypothesis of the theorem.

Replacing $f(x)$ by $f(x) - f(0)$ we may assume that $f(0) = 0$. Let X be a two-valued r.v., $P(X = a) = p$, $P(X = b) = q = 1 - p$, and $a < b$. It is easily checked that the r-percentile of X, $m_r(X) = a$ if $p > r$, $m_r(X) = b$ if $p < r$ and $m_r(X) = [a, b]$ when $p = r$ (in this case m_r is multivalued). For this choice of p, that is, $p = \frac{1}{s+1}$ and $q = \frac{s}{s+1}$, the hypothesis on f can now be stated as follows:

For all real t, a, b, τ, $a < b$, $0 \le \tau \le 1$, we have

$$\frac{1}{s+1}f(a-t) + \frac{s}{s+1}f(b-t) \ge \frac{1}{1+s}f(a-(\tau a+(1-\tau)b)) + \frac{s}{s+1}f(b-(\tau a+(1-\tau)b)))$$

$$= \frac{1}{s+1}f((1-\tau)(a-b)) + \frac{s}{s+1}f(\tau(b-a)).$$

With $t = a$ and $\tau = 0$ we get $\frac{s}{s+1}f(b-a) \ge \frac{1}{s+1}f(a-b)$, i.e., $f(-x) \le sf(x)$ for $x \ge 0$. Similarly, with $t = b$ and $\tau = 1$ we get the reverse inequality, to conclude that $f(-x) = sf(x)$ for $x \ge 0$. Now we complete the argument by showing that for $x > 0$ $f(x) = \alpha x$. To this effect observe that $f(a-t) + sf(b-t)$ is minimized by every $t = \tau a + (1-\tau)b$, $0 \le \tau \le 1$. It follows that as a function of $\tau \in [0,1]$, $f((1-\tau)(a-b))+sf(\tau(b-a)) = s(f((1-\tau)(b-a))+sf(\tau(b-a))$ is constant and equals $sf(b-a)$. Letting $t=0$ and $a=b$ we get $f(b) \geq f(0) = 0$ for all b. Also, letting $\tau(b-a) = x$, $(1-\tau)(b-a) = y$ this implies that $f(x+y) = f(x)+f(y)$ for $0 \leq x, y \leq b-a$. Equivalently, $f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$ which using continuity implies that f is positive homogenous (note that this is the only place where the continuity of f is used). Letting $b-a \to \infty$ yields the conclusion.

We now finish the proof by showing that the hypotheses of the theorem imply that f is continuous at any $x > 0$. This is done in a similar way as in the proof of Lemma 4 in [1], where instead of being additive f is subadditive. First we notice that for $0 \leq x \leq M$, $M > 0$, we have $0 \leq f(x) = f(M)+f(M-x) \leq f(M)$ and f is bounded on any finite interval in $[0,\infty)$. Let $x > 0$. Then both $\limsup_{y\to x} f(y) = L$ and $\liminf y \to xf(y) = l$ exist at x and are finite. For $\epsilon > 0$ we write $f(x) = f(\frac{x+\epsilon+x-\epsilon}{2}) = \frac{f(x+\epsilon)+f(x-\epsilon)}{2}$ implying by taking the $\limsup_{\epsilon\to 0}$ that $f(x) \leq L$. With $x_n \to x$ such that $f(x_n) \to l$ we have $l \leq \liminf f(\frac{x+x_n}{2}) = \frac{f(x)+l}{2}$, hence $f(x) \geq l$. Next we choose $x_n \to x$, $y_n \to x$ such that $f(\frac{x_n+y_n}{2}) \to L$ and $f(x_n) \to l$ to conclude that $L \leq \frac{l+L}{2}$ so that $L \leq l$. Hence $l = f(x) = L$ and f is continuous at x. The proof is complete. □

References

[1] Jan Mycielski, Pawel Szeptycki, *Minimizing moments*, Real Anal. Exchange, this issue.