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A NEW PROOF OF THE ERGODIC MAXIMAL EQUALITY

Abstract

A new simple proof of the Ergodic Maximal Equality is proposed.

Theorem (Ergodic Maximal Equality). *Let $(X, \mathbb{S}, \mu)$ be a probability space, $\mu(X) = 1$, and $(T_t)_{t \geq 0}$ be an ergodic semigroup of measure-preserving transformations on X. If $f \in L(X)$ and $\lambda > \int_X f d\mu$, then*

$$\mu\{f^* > \lambda\} = \frac{1}{\lambda} \int_{\{f^* > \lambda\}} f d\mu, \tag{1}$$

where $f^*(x) = \sup_{a>0} \frac{1}{a} \int_0^a f(T_t x) dt,\ x \in X$.

This theorem was first proved in [4] (see also [6]). The generalization for infinite measure spaces was given in [7] applying different technic. Using the continuous version of the filling scheme method (see [5],[6]) the different proof was proposed in [1] involving the both finite and infinite measure cases. In this note we give a new very simple proof of the theorem utilizing fully the main idea of ergodicity that single trajectories carry almost all information about the system. A referee has informed us that a similar proof with the same idea was in unpublished paper of D. Engel.

We shall use well-known facts about the one-sided maximal function,

$$Mh(t) = \sup_{a>0} \frac{1}{a} \int_t^{t+a} h dm$$

(m denotes the Lebesgue measure on the line $\mathbb{R}$):

(*) If (α, β) is a connected component of the open set $\{Mh > \lambda\}$, $\lambda \in \mathbb{R}$, then

$$\int_\alpha^\beta h dm = \lambda(\beta - \alpha).$$

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(**) If $(\alpha, \infty) \subset \{Mh > \lambda\}$, then

$$\limsup_{a\to\infty} \frac{1}{a} \int_t^{t+a} hdm \geq \lambda$$

for each $t \in \mathbb{R}$ (see, e.g., [2], p. 58).

PROOF. Let $\{f^* > \lambda\} = E$. Fix any $x \in X$ for which the equalities

$$\lim_{a\to\infty} \frac{1}{a} \int_0^a f(T_t x)dt = \int_X f d\mu,$$

$$\lim_{a\to\infty} \frac{1}{a} \int_0^a \mathbb{1}_E(T_t x)dt = \mu(E), \quad \lim_{a\to\infty} \frac{1}{a} \int_0^a \mathbb{1}_E f(T_t x)dt = \int_E f d\mu \qquad (2)$$

hold (by the Ergodic Theorem almost all $x \in X$ is such). Let

$$h(t) = \begin{cases} f(T_t x) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}$$

Obviously $Mh(t) = f^*(T_t x)$, $t \geq 0$, and hence $0 \leq t \in \{Mh > \lambda\} \Leftrightarrow T_t x \in E$.

Since

$$\lim_{a\to\infty} \frac{1}{a} \int_0^a h(t)dt = \lim_{a\to\infty} \frac{1}{a} \int_0^a f(T_t x)dt = \int_X f d\mu < \lambda,$$

by virtue of property (**) we have $(\alpha, \infty) \not\subset \{Mh > \lambda\}$ for each $\alpha \in \mathbb{R}$. Hence there exists an increasing sequence of positive numbers $a_i \notin \{Mh > \lambda\}$, $i = 1, 2, \ldots$, such that $a_i \to \infty$.

Since $\Delta_i \equiv \{Mh > \lambda\} \cap (-\infty, a_i)$ is the union of some connected components of $\{Mh > \lambda\}$, by virtue of (*) we have

$$\lambda m(\Delta_i) = \int_{\Delta_i} hdm.$$

Let $\Delta_i^+ = \Delta_i \cap (0, \infty)$. Then

$$\int_{\Delta_i} hdm = \int_{\Delta_i^+} hdm = \int_0^{a_i} \mathbb{1}_{\{Mh>\lambda\}} h\, dm = \int_0^{a_i} \mathbb{1}_E f(T_t x)dt.$$

Suppose that (α_0, β_0) is the connected component of $\{Mh > \lambda\}$ which contains 0. (Note that (α_0, β_0) is not infinite.) We assume that $(\alpha_0, \beta_0) =$

$(0,0)=\emptyset$ if $0\notin\{Mh>\lambda\}$. Then

$$\lambda m(\Delta_i)=\lambda(|\alpha_0|+m(\Delta_i^+))=\lambda(|\alpha_0|+\int_0^{a_i}\mathbb{I}_{\{Mh>\lambda\}}dm)$$
$$=\lambda(|\alpha_0|+\int_0^{a_i}\mathbb{I}_E(T_tx)dt).$$

Thus,

$$\lambda(|\alpha_0|+\int_0^{a_i}\mathbb{I}_E(T_tx)dt)=\int_0^{a_i}\mathbb{I}_Ef(T_tx)dt.$$

Dividing this equality by a_i, letting i to tend to the infinity and taking into account (2), we get (1). □

By the same way we can prove the Ergodic Maximal Equality for infinite measure spaces. We have to use then the continuous version of the Chacon-Ornstein theorem (see [3]) instead of Individual Ergodic Theorem to get the corresponding equations in (2).

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