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SOME COMMENTS ON THE H_1-INTEGRAL

Abstract

In this note we consider two natural attempts to give a descriptive characterization for H_1-primitives, and discuss why these attempts fail. Meanwhile we get a new descriptive definition of the Henstock integral. Also, we prove that every Henstock integrable function can be written as a sum of a Lebesgue integrable and an H_1-integrable ones.

Let $E \subset \mathbb{R}$. By $|E|$ we denote the Lebesgue outer measure of E. We denote by $\mathcal{I}$ the σ-ideal of sets, the basis of which is the family of all $\mathcal{F}_\sigma$ null sets. If $F\colon E \to \mathbb{R}$ and $A \subset E$ is nonvoid, then $\omega_F(A) = \sup F(A) - \inf F(A)$, i.e., $\omega_F(A)$ is the oscillation of F on A. We say that F is *Baire*1* if for every set $A \subset E$, closed in E, there is a portion $I \cap A \neq \emptyset$ of A such that $F \restriction (I \cap A)$ is continuous. (Recall that for $E = \operatorname{cl} E$, F is Baire*1 iff there exist a sequence $\{E_n\}_n$ of closed sets, such that $\bigcup_{n=1}^{\infty} E_n = E$ and for each n, $F \restriction E_n$ is continuous.)

By a *division* we mean a finite collection of tagged intervals (I, x), $x \in I$, in which intervals I are pairwise nonoverlapping. (In papers [4] and [6] we used the name *partial tagged partition* instead.) If for all (I, x) we have $x \in E$, then we say that the division is *anchored* in a set E. A division is called a *partition* of an interval $\langle a, b\rangle$ if the union of intervals I from this division gives the whole $\langle a, b\rangle$. For two divisions $\mathcal{P}_1$ and $\mathcal{P}_2$ we will write $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ iff for every $(I, x) \in \mathcal{P}_1$ there is a $(J, y) \in \mathcal{P}_2$ with $I \subset J$. Any positive function δ defined on $\mathbb{R}$ we call a *gauge*. We say that a division $\mathcal{P}$ is δ*-fine* if for every $(I, x) \in \mathcal{P}$ we have $I \subset (x - \delta(x), x + \delta(x))$.

Let $F\colon \langle a, b\rangle \to \mathbb{R}$. When $I = \langle c, d\rangle \subset \langle a, b\rangle$, by $\Delta F(I)$ we mean the increment $F(d) - F(c)$. For convenience, the character F will stand for two functions: the point one and the interval-point one, given by formula $(I, x) \mapsto \Delta F(I)$.

For classical notions of AC_*-, ACG_*-, VB_*-, and VBG_*-functions we refer the reader to [5].

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1 On H_1-Primitives

We have proved in [4], *Example 4.2*, that (contrary to the claim presented in the original paper [2]) the class of H_1-primitives is a proper subset of the class of Henstock primitives, i.e., of all ACG_*-functions. Thus, the problem of characterizing H_1-primitives, *Problem 4.3* in [4], emerges. As the space of H_1-integrable functions is not closed under the uniform limit, see [3], it seems to be natural to look for a characterization of the wider class of primitives, namely the class of primitives of uniform limits of H_1-integrable functions. (We may refer to this wider class as to the class of functions H_1*-integrable in the extended sense*; we will use accordingly the name *extended* H_1*-integral*.)

Definition 1.1. We call a function $f\colon \langle a,b\rangle \to \mathbb{R}$, H_1-integrable to $\mathbf{I} \in \mathbb{R}$ if there exists a gauge δ defined on $\langle a,b\rangle$, such that for any $\varepsilon > 0$ one can find a partition π_1 of $\langle a,b\rangle$, such that for every δ-fine partition $\pi \sqsupseteq \pi_1$,

$$\Big| \sum_{(I,x)\in\pi} f(x)|I| - \mathbf{I} \Big| < \varepsilon.$$

The following theorem was proved in [4], *Corollary 3.5* and *Theorem 3.7* there.

Theorem 1.2. *The function $f\colon \langle a,b\rangle \to \mathbb{R}$ is H_1-integrable if and only if it is Henstock integrable and there exists an $E \in \mathcal{I}$, such that $f \restriction (\langle a,b\rangle \setminus E)$ is Baire*1 in its domain.*

The function $f\colon \langle a,b\rangle \to \mathbb{R}$ is H_1-integrable in the extended sense if and only if it is Henstock integrable and there exists an $E \in \mathcal{I}$, such that $f \restriction (\langle a,b\rangle \setminus E)$ is Baire 1 in its domain.

1.1 The Set of Nondifferentiability

Consider the following fact.

Observation 1.3. *Suppose that F is an ACG_*-function differentiable outside an $E \in \mathcal{I}$. Then, F is Henstock primitive of a function f, which is H_1-integrable in the extended sense.*

PROOF. F is a Henstock primitive of

$$f(x) = \begin{cases} F'(x) & \text{if } x \in \langle a,b\rangle \setminus E \\ 0 & \text{if } x \in E. \end{cases}$$

Of course, F' is Baire one in its domain. Thus, by Theorem 1.2, f is H_1-integrable in the extended sense. ☐

The H_1-integral is in some sense close to the Riemann integral. Therefore, the following conjecture seems to be justified. Every H_1-primitive is differentiable outside an $E \in \mathcal{I}$, as all Riemann primitives are. If this were true, it would imply an interesting *characterization* of primitives of functions H_1-integrable in the extended sense. These are the ACG_*-functions differentiable outside an $E \in \mathcal{I}$. Moreover, since H_1-integrability of a function depends on its behavior outside an $E \in \mathcal{I}$, it would give us a descriptive *definition* of the extended H_1-integral. Alas, the claim is false.

It is well known that the set of nondifferentiability points of an absolutely continuous function (even a Lipschitz function) can be generic. (Nevertheless, it is always a null set.) But this is not the case for H_1-primitives. These are differentiable almost everywhere and outside a first category set. (This follows from Theorem 1.2.) However, there are sets of the first category and of measure zero which do not belong to $\mathcal{I}$. The simplest example of such a set is a dense null $\mathcal{G}_\delta$ subset of nowhere dense perfect set of positive measure. The exceptional set we shall indicate below is of this kind. With the usual notation $\overline{d}(A, x)$ and $\underline{d}(A, x)$ for the upper and lower density of a measurable set $A \subset \mathbb{R}$ at a point x, we have the following example.

Example 1.4. *There exists a closed set $D \subset \langle 0, 1\rangle$ and a $\mathcal{G}_\delta$ subset $P \subset D$, dense in D, such that for each $x \in P$ one has $\overline{d}(D, x) = 1$ and $\underline{d}(D, x) = 0$.*

CONSTRUCTION. Take any open set $O_1 \subset \langle 0, 1\rangle$, dense in $\langle 0, 1\rangle$, with $\langle 0, 1\rangle \setminus O_1$ perfect. Let $\{I_i^{(1)}\}_i$ be the family of components of O_1. For each i, let $J_i^{(1)}$ be a closed interval concentric with $I_i^{(1)}$, with $0 < |J_i^{(1)}| < \frac{1}{2}|I_i^{(1)}|$. Let $D_1 = \langle 0, 1\rangle \setminus \bigcup_i (I_i^{(1)} \setminus J_i^{(1)})$.

We proceed by induction. Having defined O_{n-1}, D_{n-1}, $I_i^{(n-1)}$'s, and $J_i^{(n-1)}$'s, we take an open set O_n, dense in D_{n-1}, with $D_{n-1} \setminus O_n$ perfect, satisfying the following three conditions:

(a) $O_n \subset O_{n-1}$.

(b) O_n does not intersect the union $\bigcup_i (I_i^{(n-1)} \setminus J_i^{(n-1)})$.

(c) For each i, letting $\langle x, y\rangle = J_i^{(n-1)}$, for each $z \in J_i^{(n-1)}$

$$|O_n \cap \langle x, z\rangle| \le \frac{1}{2^n}(z - x), \quad |O_n \cap \langle z, y\rangle| \le \frac{1}{2^n}(y - z).$$

Let $\{I_i^{(n)}\}_i$ be the family of components of O_n. For each i, let $J_i^{(n)}$ be a closed interval concentric with $I_i^{(n)}$, with

$$0 < |J_i^{(n)}| < \frac{1}{2^n}|I_i^{(n)}|. \tag{1}$$

Put $D_n = D_{n-1} \setminus \bigcup_i (I_i^{(n)} \setminus J_i^{(n)})$.

We put $D = \bigcap_{n=1}^{\infty} D_n$ and $P = \bigcap_{n=1}^{\infty} \bigcup_i J_i^{(n)}$. It is clear that $P \subset D$, P is dense in D, and it is of $\mathcal{G}_\delta$ type. Fix an $x \in P$ and take arbitrary $h > 0$ and n. There exist an $m \geq n$ and an i such that $(x-h, x+h) \supset I_i^{(m)}$, $x \in \operatorname{int} J_i^{(m)}$. Denote by y and z those endpoints of $I_i^{(m)}$ and $J_i^{(m)}$ respectively, which are closer to x. In view of (c) we get

$$|\langle z, 2x-z\rangle \cap D| \geq |\langle z, 2x-z\rangle \setminus O_{m+1}| \geq 2\left(1 - \frac{1}{2^{m+1}}\right)(x-z) \text{ if } x > z,$$
$$|\langle 2x-z, z\rangle \cap D| \geq |\langle 2x-z, z\rangle \setminus O_{m+1}| \geq 2\left(1 - \frac{1}{2^{m+1}}\right)(z-x) \text{ if } x < z.$$

Hence $\overline{d}(D, x) = 1$. On the other hand, by (1),

$$|\langle y, 2x-y\rangle \cap D| \leq |J_i^{(m)}| < \frac{1}{2^m}|I_i^{(m)}| \text{ if } x > y,$$
$$|\langle 2x-y, y\rangle \cap D| \leq |J_i^{(m)}| < \frac{1}{2^m}|I_i^{(m)}| \text{ if } x < y.$$

Since $(1 - \frac{1}{2^m})|I_i^{(m)}| < |I_i^{(m)}| - |J_i^{(m)}| < 2|x-y|$, we have

$$\frac{1}{2^m}|I_i^{(m)}| < \frac{2}{2^m - 1}|x-y|.$$

That means $\underline{d}(D, x) = 0$. We are done. □

Corollary 1.5. *There exists an H_1-integrable function f whose primitive is symmetrically nondifferentiable on a set which does not belong to $\mathcal{I}$.*

PROOF. Take the set D constructed in Example 1.4 and put $f = \chi_D$. Since f is Baire*1, it is H_1-integrable (Theorem 1.2). The primitive of f is symmetrically differentiable exactly at points at which the density of D exists, so we apply the fact that $P \notin \mathcal{I}$. □

In a descriptive definition of an integral, a derivative of a primitive must be integrable for an arbitrary extension to the set where it does not exist. But, by Theorem 1.2 the H_1-integrability of a function depends on its values outside an $E \in \mathcal{I}$. Hence, Corollary 1.5 shows that for the H_1-integral the exceptional set can be too large, if we consider the ordinary derivative (even the symmetric one). Since the primitive of $f = \chi_D$ is monotone, it seems natural to suppose there is no generalized derivative for which the exceptional set of nondifferentiability would always belong to $\mathcal{I}$. Having this in mind we can make the following assertion.

Statement 1.6. *Descriptive definitions of the H_1-integral and of the extended H_1-integral are unavailable.*

1.2 A Variational Equivalence

In the second attempt we take into consideration the variational equivalence which is used to define an integral in the H_1 sense.

In the sequel let θ and γ be interval-point functions; i.e., functions defined on family of tagged subintervals of $\langle a, b\rangle$.

Definition 1.7. We say that θ and γ are strongly equivalent on a set $E \subset \langle a, b\rangle$ if there exists a gauge δ on E such that for each $\varepsilon > 0$ one can find a partition π_ε of $\langle a, b\rangle$, such that for all δ-fine divisions $\mathcal{P} \sqsupseteq \pi_\varepsilon$, anchored in E, we have $\sum_{(I,x)\in\mathcal{P}} |\theta(I, x) - \gamma(I, x)| < \varepsilon$.

We say that $F : \langle a, b\rangle \to \mathbb{R}$ is $\mathcal{I}$-null if F is strongly equivalent to zero on each $A \in \mathcal{I}$.

Note that f is H_1-integrable to primitive F iff the functions $(I, x) \mapsto f(x)|I|$ and F are strongly equivalent on $\langle a, b\rangle$. This follows from the Saks-Henstock lemma for the H_1-integral.

$\mathcal{I}$-nullity is a substitute for the condition of absolute continuity of variational measure, which is considered in the theory of the Henstock integral.

Definition 1.8. Let δ be a gauge on $E \subset \langle a, b\rangle$. By δ-variation of θ on E, denoted by $V_\delta^\theta(E)$, we mean the supremum of values

$$|\Delta|\theta(\mathcal{P}) = \sum_{(I,x)\in\mathcal{P}} |\theta(I, x)|, \tag{2}$$

taken over all δ-fine divisions $\mathcal{P}$ anchored in E. The infimum of $V_\delta^\theta(E)$, taken over all gauges δ, we name the variational measure of E induced by θ. We denote it by $\mu_\theta(E)$. We say that μ_θ is absolutely continuous if $\mu_\theta(N) = 0$ for all null sets $N \subset \langle a, b\rangle$.

The following theorem was proved in [1], *Theorem 1.*

Theorem 1.9. *Let $F : \langle a, b\rangle \to \mathbb{R}$. Then, μ_F is absolutely continuous iff F is an ACG_*-function.*

Let θ be additive; i.e., $\theta(I \cup J, x) = \theta(I, x) + \theta(J, x)$ for any two nonoverlapping intervals I and J, with a common endpoint x. Then, comparing two conditions: the $\mathcal{I}$-nullity of θ and the absolute continuity of μ_θ, we see that in the first one the equivalence to zero is understood in a strengthened sense, but on a smaller class of sets.

We will show that for $F\colon \langle a,b\rangle \to \mathbb{R}$, the $\mathcal{I}$-nullity is equivalent to the ACG_* property and so it cannot characterize H_1-primitives.

Lemma 1.10. *Let F be strongly equivalent to zero on sets $E_1, E_2, E_3, \dots$. Then, it is strongly equivalent to zero on $\bigcup_{n=1}^{\infty} E_n$.*

PROOF. Let F be strongly equivalent to zero on $E_1, E_2, E_3, \dots$ using gauges $\delta_1, \delta_2, \delta_3, \dots$ respectively. Since $\mu_F(E_n) = 0$, we can assume that $V^F_{\delta_n}(E_n) < \frac{1}{2^n}$ and that E_n's are pairwise disjoint. We define $\delta(x) = \delta_n(x)$ when $x \in E_n$. Consider arbitrary $\varepsilon > 0$. For each n, there are partitions π_n of $\langle a,b\rangle$ such that for all δ_n-fine divisions $\mathcal{P} \sqsupseteq \pi_n$, anchored in E_n, we have $|\Delta|F(\mathcal{P}) < \frac{\varepsilon}{2^{n+1}}$. We may assume that $\dots \sqsupseteq \pi_3 \sqsupseteq \pi_2 \sqsupseteq \pi_1$. There is an N so that $\frac{1}{2^{N-1}} < \varepsilon$. Take a δ-fine division $\mathcal{P} \sqsupseteq \pi_N$, anchored in $\bigcup_{n=1}^{\infty} E_n$. Let $\mathcal{P}_n = \{(I,x) \in \mathcal{P} : x \in E_n\}$. Since $\mathcal{P}_n$ is δ_n-fine and $\mathcal{P}_n \sqsupseteq \pi_N \sqsupseteq \pi_n$, $n = 1,2,\dots,N$, we have

$$|\Delta|F(\mathcal{P}) < \sum_{n=1}^{N} |\Delta|F(\mathcal{P}_n) + \sum_{n=N+1}^{\infty} |\Delta|F(\mathcal{P}_n) <$$

$$< \sum_{n=1}^{N} \frac{\varepsilon}{2^{n+1}} + \sum_{n=N+1}^{\infty} V^F_{\delta_n}(E_n) < \frac{\varepsilon}{2} + \frac{1}{2^N} < \varepsilon. \qquad \square$$

Theorem 1.11. *Suppose that $F\colon \langle a,b\rangle \to \mathbb{R}$ is an ACG_*-function. Then, it is $\mathcal{I}$-null.*

PROOF. Take an $A \in \mathcal{I}$ and let $A \subset \bigcup_{n=1}^{\infty} A_n$ where all $A_n = \operatorname{cl} A_n$ are null sets and F is AC_* on each A_n. Fix an n. We will show that F is strongly equivalent to zero on A_n, using any gauge δ. We may assume that $a, b \in A_n$. Let (a_i, b_i), $i = 1,2,3,\dots$, be intervals contiguous to A_n in $\langle a,b\rangle$. Take an $\varepsilon > 0$. There are intervals $\langle c_i, d_i\rangle \subset (a_i, b_i)$ such that

$$\sum_{i=1}^{\infty} \big(\omega_F(\langle a_i, c_i\rangle) + \omega_F(\langle d_i, b_i\rangle)\big) < \varepsilon. \tag{3}$$

Also, there is an $\eta > 0$ such that

$$\sum_k |I_k| < \eta \quad \Rightarrow \quad \sum_k \omega_F(I_k) < \varepsilon \tag{4}$$

for each family $\{I_k\}_k$ of nonoverlapping intervals with endpoints in A_n. One can find an N so that

$$\left| \langle a,b\rangle \setminus \bigcup_{i=1}^{N} (a_i, b_i) \right| < \eta. \tag{5}$$

Set $\mathcal{R} = \{(\langle a_i, c_i\rangle, a_i), (\langle d_i, b_i\rangle, b_i)\}_{i=1}^{N}$. Complete $\mathcal{R}$ to any partition π_ε of $\langle a, b\rangle$. Consider a δ-fine division $\mathcal{P} \sqsupseteq \pi_\varepsilon$, anchored in A_n. Let

$$\mathcal{P}' = \{(I, x) \in \mathcal{P} :\ I \subset \langle a_i, c_i\rangle \cup \langle d_i, b_i\rangle,\ i = 1, 2, \dots, N\}.$$

By (3) $|\Delta|F(\mathcal{P}') < \varepsilon$, by (4) and (5) $|\Delta|F(\mathcal{P} \setminus \mathcal{P}') < \varepsilon$. Thus $|\Delta|F(\mathcal{P}) < 2\varepsilon$. By Lemma 1.10, F is strongly equivalent to zero on A. □

Theorem 1.12. *Suppose that $F\colon \langle a, b\rangle \to \mathbb{R}$ is $\mathcal{I}$-null. Then, it is an ACG_*-function.*

PROOF. Suppose first that F is not a VBG_*-function. Then, by the proof of *Theorem 1* in [1], we get that there exists a closed null set $N \subset \langle a, b\rangle$ with $\mu_F(N) \geq 1$. Hence F is not strongly equivalent to zero on N, a contradiction.

Now, we will prove that μ_F is absolutely continuous. Take any null set $D \subset \langle a, b\rangle$. We may assume that D is Borel. Let $\bigcup_n E_n = \langle a, b\rangle$, where for each n the set E_n is closed and F is VB_* on E_n. Fix an n and consider the value $\mu_F(D \cap E_n)$. Let G be the piecewise linear extension of $F \restriction E_n$, G is of bounded variation. The variational measure μ_F defined on subsets of E_n is the regular Borel measure defined by variation of G, $|.|_G$. So, the value $\mu_F(D \cap E_n) = |D \cap E_n|_G$ can be approximated by values $|P|_G$, where the P's are closed subsets of $D \cap E_n$. Since F is $\mathcal{I}$-null, all these imply that $|P|_G = \mu_F(P) = 0$. Thus, there must be $\mu_F(D \cap E_n) = |D \cap E_n|_G = 0$. So, $\mu_F(D) = 0$. By Theorem 1.9, F is an ACG_*-function. □

In fact, we proved above that for the absolute continuity of μ_F it is enough to assume that μ_F is zero only on $\mathcal{F}_\sigma$ null sets. But then, this value may be approximated by sums of the kind (2), in a strengthened way.

The equivalence of $\mathcal{I}$-nullity and the property ACG_* allows another descriptive definition of the Henstock integral.

Corollary 1.13. *An $f\colon \langle a, b\rangle \to \mathbb{R}$ is Henstock integrable iff there exists an $\mathcal{I}$-null function F such that $F'(x) = f(x)$ for almost all $x \in \langle a, b\rangle$.*

1.3 Riemann Primitives

We end this section with an observation related to the still open problem of characterizing the class of Riemann primitives.

Observation 1.14. *There exists a bounded H_1-integrable function f, which is a derivative, but whose primitive is a primitive of no Riemann integrable function.*

PROOF. One can easily construct a bounded Baire*1 approximately continuous function f, discontinuous exactly at points of some nowhere dense perfect set of positive measure. Such a function f is H_1-integrable and it differs from every Riemann integrand on a set of positive measure. □

Every Riemann primitive is Lipschitz and differentiable outside an $E \in \mathcal{I}$. However, these two properties do not characterize Riemann primitives.

2 Main Result: Henstock=Lebesgue+H_1

In view of *Example 4.2* in [4], another problem arises. Can every Henstock integrable function be written as the sum of a Lebesgue integrable function and an H_1-integrable one (*Problem 4.5* in [4])? We will answer this question in affirmative. We say that f is integrable on a set E in the *improper sense* if $f\chi_E$ is integrable on every subinterval $\langle c,d\rangle \subset (\inf E, \sup E)$ and if a finite double limit of $\int_c^d f\chi_E$ exists when $c \to \inf E$, $d \to \sup E$.

Lemma 2.1. *Let a set $E \subset \langle a,b\rangle$ be closed, $I \subset \langle a,b\rangle$ be an open interval. Suppose that $f\colon \langle a,b\rangle \to \mathbb{R}$ is Lebesgue integrable in the improper sense on the set $E \cap I$. Then, $f = f_1 + f_2$ on $E \cap I$, where f_1 is Lebesgue integrable on $E \cap I$ and f_2 is Baire*1 on $E \cap I$. Moreover, for each $\eta > 0$ the function f_1 can be chosen so that $\int_{E\cap I} |f_1| < \eta$.*

PROOF. We may assume that $a = \inf E$, $b = \sup E$, $I = (a,b)$, and that $f = 0$ outside of E. By assumption, f is Lebesgue integrable on every interval $\langle c,d\rangle \subset (a,b)$. Pick two monotone sequences in (a,b): $a_n \to a$, $b_n \to b$, $a_1 = b_1$. Since the integral of f is absolutely continuous on each interval $\langle a_{n+1}, a_n\rangle$, by Lusin's theorem on $\mathcal{C}$-property one can find a closed subset $P_n \subset E \cap (a_{n+1}, a_n)$ such that

(a) $f \restriction P_n$ is continuous,

(b) $\int_{\langle a_{n+1},a_n\rangle \setminus P_n} |f| < \frac{\eta}{2^{n+1}}$.

In the same way we find closed subsets $R_n \subset E \cap (b_n, b_{n+1})$. Put $f_2 = f$ on $\bigcup_{n=1}^{\infty}(P_n \cup R_n)$, 0 otherwise, $f_1 = f - f_2$. By (a), it is clear that the function f_2 is Baire*1 on $E \cap I$. By (b), we have $\int_{E\cap I} |f_1| < 2\sum_{n=1}^{\infty} \frac{\eta}{2^{n+1}} = \eta$. □

Theorem 2.2. *Every Henstock integrable function $f\colon \langle a,b\rangle \to \mathbb{R}$ can be written as a sum $f = f_1 + f_2$, where f_1 is Lebesgue integrable, f_2 is H_1-integrable. Moreover, for each $\varepsilon > 0$ the function f_1 can be chosen so that $\int_a^b |f_1| < \varepsilon$.*

PROOF. We use transfinite induction. We define transfinite sequences:

1. $\{U_\alpha\}_{\alpha<\Omega}$ of open sets in $\langle a,b\rangle$ (ascending),
2. $\{\mathcal{J}_\alpha\}_{\alpha<\Omega}$ of families of open subintervals of $\langle a,b\rangle$.

Put $U_1=\emptyset$, denote $E_1=\langle a,b\rangle\setminus U_1=\langle a,b\rangle$.

Let $\mathcal{J}_\alpha$ be the family of all open intervals I such that f is Lebesgue integrable on $E_\alpha\cap I$ in the improper sense, $E_\alpha=\langle a,b\rangle\setminus U_\alpha$. Put $U_{\alpha+1}=U_\alpha\cup(E_\alpha\cap\bigcup\mathcal{J}_\alpha)$. For a limit ordinal β put $U_\beta=\bigcup_{\alpha<\beta}U_\alpha$, $\mathcal{J}_\beta=\emptyset$. For all α's let $\mathcal{I}_\alpha$ denote the family of all compound intervals of $\bigcup\mathcal{J}_\alpha$. (Then $\mathcal{I}_\alpha$ is countable and $\bigcup\mathcal{I}_\alpha=\bigcup\mathcal{J}_\alpha$.) Because every closed set contains a portion on which f is Lebesgue integrable, if $U_\alpha\neq\langle a,b\rangle$, then $U_\alpha\subsetneq U_{\alpha+1}$. Since $\{U_\alpha\}_\alpha$ is ascending, by the Cantor–Baire principle there exists an $\alpha<\Omega$ such that $U_\alpha=\langle a,b\rangle$. Denote $\{P_n\}_{n=1}^\infty=\{I\cap E_\beta\}_{I\in\mathcal{I}_\beta,\beta\leq\alpha}$. For each n, apply Lemma 2.1 for $E=E_\beta$ and $\eta=\frac{\varepsilon}{2^n}$ to find an appropriate sum $f=f_1^{(n)}+f_2^{(n)}$ on $P_n=E_\beta\cap I$. Note that the P_n's are pairwise disjoint and $\bigcup_{n=1}^\infty P_n=\langle a,b\rangle$. Put $f_i(x)=f_i^{(n)}(x)$ if $x\in P_n$, $i=1,2$. One has

$$\int_a^b|f_1|<\sum_{n=1}^\infty\int_{P_n}|f_1^{(n)}|<\sum_{n=1}^\infty\frac{\varepsilon}{2^n}=\varepsilon,$$

so f_1 is Lebesgue integrable. As $f_2=f-f_1$, f_2 is Henstock integrable. Each function $f_2^{(n)}$ is Baire*1 on an $\mathcal{F}_\sigma$ set P_n. Hence, the so-defined function $f_2\colon\langle a,b\rangle\to\mathbb{R}$ is Baire*1. In view of Theorem 1.2, f_2 is H_1-integrable. □

Let us conclude this note with the following query.

Question 2.3. *Does there exist a function, H_1-integrable in the extended sense, which cannot be written as the sum of an H_1-integrable one and a derivative?*

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