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DILATATIONS OF GRAPHS AND TAYLOR'S FORMULA: SOME RESULTS ABOUT CONVERGENCE

Abstract

The graph of a function f is subjected to non-homogeneous dilatations around the point $(x_0; f(x_0))$, related to the Taylor's expansion of f at x_0. Some questions about convergence are considered. In particular the dilated images of the graph are proved to behave nicely with respect to a certain varifold-like convergence. Further and stronger results are shown to hold in such a context, by suitably reinforcing the assumptions.

1 Introduction

Throughout this paper h, k, n are positive integer numbers, with $h \geq 2$, and f is a map in $C^{h-1}(\mathbb{R}^n, \mathbb{R}^k)$. The graph of f is denoted by G_f. The d-th degree Taylor's polynomial of f at a point x_0 is indicated with $P_d^{x_0} f$, while $f_{d,0}^{x_0}$ is the d-th degree monomial in $P_d^{x_0} f$. For $d = 1, \ldots, h$ we can consider the following families of transformations parametrized by $r > 0$.

$$T_{d,r}^{x_0} : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^k,\ (x; y) \mapsto T_{d,r}^{x_0}(x; y) := \left(\frac{x - x_0}{r}; \frac{y - P_{d-1}^{x_0} f(x)}{r^d} \right).$$

As an easy computation shows, the surface $T_{d,r}^{x_0}(G_f)$ coincides with the graph of

$$f_{d,r}^{x_0}(u) := \frac{f(x_0 + ru) - P_{d-1}^{x_0} f(x_0 + ru)}{r^d},\ u \in \mathbb{R}^n.$$

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Note that $T^{x_0}_{1,r}$ is the homothetic of similitude ratio $1/r$, centered at $(x_0; f(x_0))$, while $f^{x_0}_{1,r}$ just coincides with the incremental ratio

$$f^{x_0}_{1,r}(u) = \frac{f(x_0 + ru) - f(x_0)}{r}, \ u \in \mathbb{R}^n.$$

It follows that blowing up G_f through $T^{x_0}_{1,r}$ produces the tangent space to G_f at $(x_0; f(x_0))$, which coincides with the graph of $f^{x_0}_{1,0}$. More precisely, one has that the maps $f^{x_0}_{1,r}$ converge to $f^{x_0}_{1,0}$, uniformly in the compact sets, as $r \downarrow 0$. Hence the Hausdorff measures associated with the graphs of the $f^{x_0}_{1,r}$, i.e. with $T^{x_0}_{1,r}(G_f)$, converge (in the weak* sense of measures) to the Hausdorff measure associated with the graph of $f^{x_0}_{1,0}$; that is,

$$\mathcal{H}^n \llcorner G_{f^{x_0}_{1,r}} = \mathcal{H}^n \llcorner T^{x_0}_{1,r}(G_f) \to \mathcal{H}^n \llcorner G_{f^{x_0}_{1,0}}$$

as $r \downarrow 0$.

Under our assumptions, by the notation introduced above, such well known facts can easily be generalized to the following statements holding for all $d = 1, \ldots, h-1$ ([2, §3]).

(A_d) The maps $f^{x_0}_{d,r}$ converge to $f^{x_0}_{d,0}$, uniformly in the compact sets, as $r \downarrow 0$.

(B_d) The Hausdorff measures associated with the graphs of the $f^{x_0}_{d,r}$, i.e. with $T^{x_0}_{d,r}(G_f)$, converge (in the weak* sense of measures) to the Hausdorff measure associated with the graph of $f^{x_0}_{d,0}$, namely

$$\mathcal{H}^n \llcorner G_{f^{x_0}_{d,r}} = \mathcal{H}^n \llcorner T^{x_0}_{d,r}(G_f) \to \mathcal{H}^n \llcorner G_{f^{x_0}_{d,0}}$$

as $r \downarrow 0$.

As for statements (A_h) and (B_h), let us observe that they do not make sense in that $f^{x_0}_{h.0}$ does not exist. However a suitable generalization of them has been proposed and studied in [2] from where we do now recap some notation and facts.

Let a family of fields

$$g_1, \ldots, g_k \in C^{h-1}(\mathbb{R}^n; \mathbb{R}^n)$$

be given and set

$$f_i := f \cdot e_{n+i} \ (i = 1, \ldots, k)$$

where $\{e_{n+i}\}$ is the standard orthonormal basis of $\mathbb{R}^k$. Then consider the closed set

$$K := \left\{ x \in \mathbb{R}^n \,\middle|\, \nabla f_i(x) = g_i(x), \text{ for all } i = 1, \ldots, k \right\}.$$

Also define the map

$$\Gamma_h^{x_0} : \mathbb{R}^n \to \mathbb{R}^k,\ \Gamma_h^{x_0}(u) := \frac{1}{h!} \sum_{i=1}^{k} \left(u \cdot \langle D^{h-1} g_i(x_0) | u^{h-1} \rangle \right) e_{n+i}$$
$$= \frac{1}{h!} \sum_{i=1}^{k} \left(u \cdot \sum_{\lambda \in \{1,\dots,n\}^{h-1}} u_\lambda D_\lambda g_i(x_0) \right) e_{n+i}$$

and observe that:

- $\Gamma_h^{x_0}$ generalizes $f_{h,0}^{x_0}$, in the sense that if f is regular enough, then one has $\Gamma_h^{x_0} \equiv f_{h,0}^{x_0}$ (*Remark 3.1*);
- If x_0 is internal to K, then f has to be of class C^h in a neighborhood of x_0. In such a case, obviously, the statements (A_h) and (B_h) make sense and are true.

As a consequence, it becomes natural to pose the following question.

(Q) Let K have density one at x_0. Then, do $f_{h,r}^{x_0}$ (resp. $T_{h,r}^{x_0}(G_{f|K})$) converge in some sense to $\Gamma_h^{x_0}$ (resp. $G_{\Gamma_h^{x_0}}$), as $r \downarrow 0$?

In [2, Proposition 4.1] we proved that the answer to (Q) in general is negative. A simple example in which $\Gamma_h^{x_0} \equiv 0$ while $f_{h,r}^{x_0}$ goes to infinity (as $r \downarrow 0$) is provided in [2, §5]. Related to this point, a mistake occurring in [4] is discussed in [2, §6].

This paper is devoted to present some new developments about the subject surrounding question (Q) and our main achievements are summarized in the remainder of this introduction.

In Theorem 3.1 and Corollary 3.1 we prove that, despite the example we just mentioned, the surfaces $T_{h,r}^{x_0}(G_{f|K})$ behave nicely with respect to a certain varifold-like convergence in which the test functions do not depend on the variable of $\mathbb{R}^k$, where f and $f_{h,r}^{x_0}$ take values. In particular the measures of $T_{h,r}^{x_0}(G_{f|K})$ inside any cylinder $E \times \mathbb{R}^n$, with E bounded subset of $\mathbb{R}^n$, have to converge.

The results in §4 provide some affirmative answers to question (Q), under the assumption that K has "density one of order high enough at x_0". More precisely, we show that if

$$\lim_{r \downarrow 0} \frac{\mathcal{L}^n(B_r(x_0) \setminus K)}{r^{h+n-1}} = 0. \tag{1}$$

then (as $r \downarrow 0$):

- for $n = 1$, $f_{h,r}^{x_0}$ converges to $\Gamma_h^{x_0}$ with respect to pointwise convergence (Theorem 4.1). For $n \geq 2$, in general, such a convergence does not occur (Example following Theorem 4.1).

- $f_{h,r}^{x_0}$ converges to $\Gamma_h^{x_0}$ in L_{loc}^1 (Theorem 4.2).

- The graph measure $\mathcal{H}^n \llcorner T_{h,r}^{x_0}(G_{f|K})$ converges to $\mathcal{H}^n \llcorner G_{\Gamma_h^{x_0}}$ in the weak* sense of measures, on condition that a certain Schwarz-like equation about mixed partials is satisfied (Corollary 4.2).

Originally, we stated the results of §4 under the assumption

$$\lim_{r \downarrow 0} \frac{\int_{B_r(x_0)} \mathcal{H}^1\big([x_0; x] \setminus K\big)\, dx}{r^{h+n}} = 0$$

rather than (1). We are grateful to Pertti Mattila who, on occasion of his recent visit to the Department of Mathematics in Trento, pointed out to us the equivalence between these assumptions. His proof is given in the Appendix §5 (Theorem 5.1).

2 Notation

This section is devoted to introduce the notation used throughout the present paper, included that which has already been introduced in §1 (for the reader's convenience).

$\mathbb{R}^n$, $\mathbb{R}^k$ and $\mathbb{R}^n \times \mathbb{R}^k$ are the euclidean spaces mainly considered throughout this paper. $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_{n+k}\}$ denote the standard orthonormal bases of $\mathbb{R}^n$ and $\mathbb{R}^k$, respectively. The projection mapping $\mathbb{R}^n \times \mathbb{R}^k$ onto $\mathbb{R}^n$ is indicated with π.

Recall that a j-vector ($j = 1, \ldots, n$) in $\mathbb{R}^n$ can be represented by the multi-index notation in the form $\sum_{\alpha \in I(n,j)} a_\alpha e_\alpha$ where

$$I(n,j) := \left\{\alpha = (\alpha_1, \ldots, \alpha_j) \in \mathbf{Z}^j \;\middle|\; 1 \leq \alpha_1 < \cdots < \alpha_j \leq n\right\}$$

and

$$a_\alpha \in \mathbb{R}, \; e_\alpha := e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_j}.$$

The linear space of the j-vectors in $\mathbb{R}^n$ is equipped with the inner product and hence, the norm, naturally induced from $\mathbb{R}^n$. The same notation is obviously adopted for multivectors in $\mathbb{R}^k$ or in $\mathbb{R}^n \times \mathbb{R}^k$.

Except for the standard euclidean length, which is denoted by $|\cdot|$, every other norm is indicated by $\|\cdot\|$. For example

$$\Big\| \sum_{\alpha\in I(n,j)} a_\alpha e_\alpha \Big\| = \Big(\sum_{\alpha\in I(n,j)} a_\alpha^2 \Big)^{1/2}.$$

Set

$$B_r(x_0) := \big\{ x\in\mathbb{R}^n \big| \, |x-x_0| \le r \big\} \ (x_0\in\mathbb{R}^n, r>0)$$

and

$$\mathbf{S}^{n-1} := \big\{ x\in\mathbb{R}^n \big| \, |x| = 1 \big\} = \partial B_1(0).$$

If $h:\mathbb{R}^n\to\mathbb{R}^k$ is a map of class C^l (with $l\ge 1$) in a neighborhood of a given point x_0, then define

$$\langle D^l h(x_0) \,|\, u^l\rangle := \sum_{\lambda\in\{1,\ldots,n\}^l} u_\lambda D_\lambda h(x_0), \ u\in\mathbb{R}^n$$

where

$$u_\lambda := u_{\lambda_1}\cdots u_{\lambda_l}, \ D_\lambda := \frac{\partial^l}{\partial x_{\lambda_1}\cdots\partial x_{\lambda_l}}.$$

We will deal with functions

$$f\in C^{h-1}(\mathbb{R}^n;\mathbb{R}^k), \ g_1,\ldots,g_k\in C^{h-1}(\mathbb{R}^n;\mathbb{R}^n)$$

where $h\ge 2$ is an integer number. Define

$$f_i := f\cdot e_{n+i}, \ g_{ij} := g_i\cdot e_j, \ g_{*j} := \sum_{i=1}^k g_{ij} e_{n+i}.$$

The graph of f is denoted by G_f, i.e. $G_f := \{(x;f(x)) \,|\, x\in\mathbb{R}^n\}$. Throughout the present paper we will deal with the closed set

$$K := \big\{ x\in\mathbb{R}^n \,\big|\, \nabla f_i(x) = g_i(x), \text{ for all } i=1,\ldots,k \big\}.$$

The operator associating an argument map with its d-th degree Taylor's polynomial at x_0 is indicated with $P_d^{x_0}$, e.g. $P_{h-1}^{x_0}(f)$ is the $(h-1)$-th degree Taylor's polynomial at $x_0\in\mathbb{R}^n$ of f.

In the following formulas we assume $x_0\in\mathbb{R}^n$, $i=1,\ldots,k$ and $j=1,\ldots,n$. Let

$$\rho_{ij}^{x_0} := g_{ij} - P_{h-2}^{x_0}(g_{ij}), \ \varphi_{ij}^{x_0} := \frac{\partial f_i}{\partial x_j} - P_{h-2}^{x_0}\Big(\frac{\partial f_i}{\partial x_j}\Big)$$

and

$$\rho_{i*}^{x_0} := \sum_{j=1}^{n} \rho_{ij}^{x_0} e_j = g_i - P_{h-2}^{x_0}(g_i),\ \rho_{*j}^{x_0} := \sum_{i=1}^{k} \rho_{ij}^{x_0} e_{n+i} = g_{*j} - P_{h-2}^{x_0}(g_{*j})$$

$$\varphi_{i*}^{x_0} := \sum_{j=1}^{n} \varphi_{ij}^{x_0} e_j = \nabla f_i - P_{h-2}^{x_0}(\nabla f_i),\ \varphi_{*j}^{x_0} := \sum_{i=1}^{k} \varphi_{ij}^{x_0} e_{n+i} = \frac{\partial f}{\partial x_j} - P_{h-2}^{x_0}\left(\frac{\partial f}{\partial x_j}\right).$$

By the Taylor's Theorem (e.g. [6, V, §6]), one has

$$\rho_{*j}^{x_0} = G_j^{x_0} + \sigma_j^{x_0} \tag{2}$$

where $G_j^{x_0}$ denotes the maximal degree monomial in $P_{h-1}^{x_0}(g_{*j})$, i.e.

$$\begin{aligned} G_j^{x_0} : \mathbb{R}^n \to \mathbb{R}^k,\ G_j^{x_0}(x) :=& \frac{1}{(h-1)!} \langle D^{h-1} g_{*j}(x_0) \,|\, (x-x_0)^{h-1} \rangle \\ =& \frac{1}{(h-1)!} \sum_{\lambda \in \{1,\dots,n\}^{h-1}} (x-x_0)_\lambda D_\lambda g_{*j}(x_0) \end{aligned}$$

and $\frac{\sigma_j^{x_0}(x)}{|x-x_0|^{h-1}} \to 0$ as $x \to x_0$. Observe that

$$\varepsilon_1(r) := \max_j \max_{x \in B_r(x_0)} \frac{|\sigma_j^{x_0}(x)|}{|x-x_0|^{h-1}} \to 0 \tag{3}$$

as $r \downarrow 0$. Analogously, one has

$$\varepsilon_2(r) := \max_j \max_{x \in B_r(x_0)} \frac{|\varphi_{*j}^{x_0}(x)|}{|x-x_0|^{h-2}} \to 0 \tag{4}$$

as $r \downarrow 0$.

Another map involved in our statements below is

$$\begin{aligned} \Gamma_h^{x_0} : \mathbb{R}^n \to \mathbb{R}^k,\ \Gamma_h^{x_0}(u) :=& \frac{1}{h!} \sum_{i=1}^{k} \left(u \cdot \langle D^{h-1} g_i(x_0) | u^{h-1} \rangle \right) e_{n+i} \\ =& \frac{1}{h!} \sum_{i=1}^{k} \left(u \cdot \sum_{\lambda \in \{1,\dots,n\}^{h-1}} u_\lambda D_\lambda g_i(x_0) \right) e_{n+i}. \end{aligned}$$

Define the following family of transformations, parametrized by $r > 0$.

$$T_{h,r}^{x_0} : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^k, \quad (x;y) \mapsto T_{h,r}^{x_0}(x;y) := \left(\frac{x-x_0}{r} ;\ \frac{y - P_{h-1}^{x_0} f(x)}{r^h} \right)$$

and

$$t_r^{x_0} : \mathbb{R}^n \to \mathbb{R}^n, \ x \mapsto t_r^{x_0}(x) := \frac{x - x_0}{r}.$$

As an easy computation shows, the surface $T_{h,r}^{x_0}(G_f)$ coincides with the graph of

$$f_{h,r}^{x_0}(u) := \frac{f(x_0 + ru) - P_{h-1}^{x_0} f(x_0 + ru)}{r^h}, \ u \in \mathbb{R}^n.$$

Consider the map $\Phi := (\pi | G_f)^{-1}$ namely

$$\Phi : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^k, \ x \mapsto \Phi(x) := (x \, ; f(x))$$

and set $\xi := \Lambda^n d\Phi(e_1 \wedge \cdots \wedge e_n)$. Let $M := \Phi(K)$ and denote by τ_r the unit simple n-vector field tangent to $T_{h,r}^{x_0}(G_f)$ obtained by pushing forward the field ξ through $T_{h,r}^{x_0}$, i.e.

$$\begin{aligned}\tau_r :=& \frac{\Lambda^n dT_{h,r}^{x_0}(\xi)}{\|\Lambda^n dT_{h,r}^{x_0}(\xi)\|} \circ \pi \circ (T_{h,r}^{x_0})^{-1} = \frac{\Lambda^n dT_{h,r}^{x_0}(\xi)}{\|\Lambda^n dT_{h,r}^{x_0}(\xi)\|} \circ (t_r^{x_0})^{-1} \circ \pi \\ =& \frac{\Lambda^n d(I \times f_{h,r}^{x_0})(e_1 \wedge \cdots \wedge e_n)}{\|\Lambda^n d(I \times f_{h,r}^{x_0})(e_1 \wedge \cdots \wedge e_n)\|} \circ \pi.\end{aligned}$$

Moreover, let τ_0 be the unit simple n-vector field tangent to the graph of $\Gamma_h^{x_0}$ having π-projection oriented as $e_1 \wedge \cdots \wedge e_n$, that is

$$\tau_0 := \frac{\Lambda^n d(I \times \Gamma_h^{x_0})(e_1 \wedge \cdots \wedge e_n)}{\|\Lambda^n d(I \times \Gamma_h^{x_0})(e_1 \wedge \cdots \wedge e_n)\|} \circ \pi.$$

The segment joining a couple of point P, Q in $\mathbb{R}^n$ is indicated by $[P; Q]$, i.e.

$$[P; Q] := \{tQ + (1-t)P \,|\, 0 \le t \le 1\}.$$

$\mathcal{L}^d$ and $\mathcal{H}^d$ are the d-dimensional Lebesgue measure and the d-dimensional Hausdorff measure in $\mathbb{R}^n$, respectively. Finally, if E is a Lebesgue measurable set in $\mathbb{R}^n$, let

$$\mathcal{D}(E) := \{x \in \mathbb{R}^n \,|\, x \text{ is a point of density (w.r.t. } \mathcal{L}^n) \text{ of } E\}.$$

3 Varifold-Like Convergence of the Dilated Graphs

Before stating the main result of this section, i.e. Theorem 3.1 below, we'll prove some useful lemmas.

Lemma 3.1. *The equality* $(D^l\nabla f_i)|\mathcal{D}(K) = (D^l g_i)|\mathcal{D}(K)$ *holds for all* $i = 1,\dots,k$ *and* $l = 0,1,\dots,h-2$.

Given $x_0 \in \mathcal{D}(K)$, *it follows at once that:*

(i) One has $P^{x_0}_{h-2}\left(\frac{\partial f_i}{\partial x_j}\right) \equiv P^{x_0}_{h-2}(g_{ij})$ *for all* $i = 1,\dots,k$ *and* $j = 1,\dots,n$, *hence* $\rho^{x_0}_{*j}|K \equiv \varphi^{x_0}_{*j}|K$ *for all* $j = 1,\dots,n$;

(ii) If $h \geq 3$, *then*
$$\frac{\partial g_{*j}}{\partial x_m}(x_0) = \frac{\partial g_{*m}}{\partial x_j}(x_0) \tag{5}$$
for all $j, m = 1,\dots,n$.

PROOF. We can assume $h \geq 3$ (for $h = 2$ the statement is obvious, in that $\mathcal{D}(K) \subset K$). Then the result is an immediate consequence of the following fact.

Let C *be a closed subset of* $\mathbb{R}^n$ *and* $\psi \in C^1(\mathbb{R}^n)$ *be such that* $\psi|C \equiv 0$. *Then* $\nabla\psi|\mathcal{D}(C) \equiv 0$.

In order to prove such a statement, note that $\mathcal{D}(C) \subset C$ and $\mathcal{L}^n\big(C\setminus\mathcal{D}(C)\big) = 0$ by the Lebesgue-Besicovitch Differentiation Theorem (e.g. [3, §1.7.1]). Then a standard argument will show that
$$\nabla\psi(x_0) = 0 \tag{6}$$
when $x_0 \in \mathcal{D}(C)$. Suppose to the contrary that there is an $x_0 \in \mathcal{D}(C)$ such that $\nabla\psi(x_0) \neq 0$. Then $\bar{u} \in \mathbf{S}^{n-1}$ and $\varepsilon > 0$ have to exist such that the function $(u,x) \mapsto \nabla\psi(x)\cdot u$ is positive, provided $|x - x_0| \leq \varepsilon$ and $|u - \bar{u}| \leq \varepsilon$. For the wedge shaped set
$$W := \left\{ x \in B_\varepsilon(x_0)\setminus\{x_0\} \,\middle|\, u_x := \frac{x - x_0}{|x - x_0|} \in B_\varepsilon(\bar{u}) \right\}$$
one has
$$\begin{aligned}\psi(x) =& \psi(x) - \psi(x_0) = \psi(x_0 + |x - x_0|u_x) - \psi(x_0)\\ =& \int_0^{|x-x_0|} \frac{d}{dt}\psi(x_0 + tu_x)\,dt = \int_0^{|x-x_0|} \nabla\psi(x_0 + tu_x)\cdot u_x\,dt > 0\end{aligned}$$
for all $x \in W$. In fact the integrand $\nabla\psi(x_0 + tu_x)\cdot u_x$ is positive, in that
$$|x_0 + tu_x - x_0| = t|u_x| \leq |x - x_0| \leq \varepsilon \text{ and } |u_x - \bar{u}| \leq \varepsilon$$
for all $x \in W$. This conclusion contradicts the assumption $x_0 \in \mathcal{D}(C)$. Hence we must admit that (6) holds. $\square$

Remark 3.1. Let x_0 be a point of density of K and assume that $D^h f(x_0)$ exists. Then $\Gamma_h^{x_0}(u)$ coincides with the value of the h-th degree monomial $f_{h,0}^{x_0}$ in the Taylor's polynomial $P_h^{x_0} f$ at $x_0 + u$. Indeed, for $i = 1, \ldots, k$, one has

$$\begin{aligned}\langle D^h f_i(x_0) \,|\, u^h \rangle =& \sum_{\mu \in \{1,\ldots,n\}^h} u_\mu D_\mu f_i(x_0) = \sum_{q=1}^{n} \sum_{\lambda \in \{1,\ldots,n\}^{h-1}} u_\lambda u_q D_\lambda D_q f_i(x_0) \\ =& u \cdot \sum_{\lambda \in \{1,\ldots,n\}^{h-1}} u_\lambda D_\lambda \nabla f_i(x_0) = u \cdot \sum_{\lambda \in \{1,\ldots,n\}^{h-1}} u_\lambda D_\lambda g_i(x_0)\end{aligned}$$

by Lemma 3.1. In particular, if f is of class C^h, then it follows that $f_{h,r}^{x_0}$ converges, uniformly in the compact sets (as $r \downarrow 0$), to $\Gamma_h^{x_0}$ [2, Proposition 3.1].

Remark 3.2. Formula (5), which says that the g_i are irrotational fields, is an immediate consequence of the well known Schwarz theorem about equality of mixed partial derivatives. This is the reason why, in the sequel, such a formula will be referred as the "Schwarz-like equality". As for the case $h = 2$, observe that (5) is in general false. Indeed, any $g \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{curl} g \neq 0$ everywhere has to coincide with the gradient of a certain $f \in C^1(\mathbb{R}^n, \mathbb{R})$ in a set of positive measure (e.g. by [1, Theorem 1]).

Lemma 3.2. *Given $x_0 \in \mathbb{R}^n$, one has*

$$\begin{aligned}\Lambda^n dT_{h,r}^{x_0}(\xi) =& \frac{1}{r^n} \left(e_1; \frac{\varphi_{*1}^{x_0}}{r^{h-1}} \right) \wedge \cdots \wedge \left(e_n; \frac{\varphi_{*n}^{x_0}}{r^{h-1}} \right) \\ =& \frac{1}{r^n} \left(e_1 \wedge \cdots \wedge e_n + \sum_{j=1}^{m} \frac{1}{r^{j(h-1)}} \sum_{\alpha \in I(n,j)} \sigma(\alpha, \overline{\alpha})\, \varphi_{*\alpha}^{x_0} \wedge e_{\overline{\alpha}} \right)\end{aligned}$$

where $m := \min\{n, k\}$. Hence

$$\begin{aligned}\left\| \Lambda^n dT_{h,r}^{x_0}(\xi) \right\| =& \frac{1}{r^n} \left(1 + \sum_{j=1}^{m} \frac{1}{r^{2j(h-1)}} \sum_{\alpha \in I(n,j)} \|\varphi_{*\alpha}^{x_0}\|^2 \right)^{\frac{1}{2}} \\ =& \frac{1}{r^n} \left(1 + \sum_{j=1}^{m} \frac{1}{r^{2j(h-1)}} \sum_{\substack{\alpha \in I(n,j) \\ \beta \in I(k,j)}} \left[\det \varphi_{\beta\alpha}^{x_0} \right]^2 \right)^{\frac{1}{2}}.\end{aligned}$$

Proof. Indeed one has

$$\Lambda^n dT_{h,r}^{x_0}(\xi) = dT_{h,r}^{x_0}\big(d\Phi(e_1)\big) \wedge \cdots \wedge dT_{h,r}^{x_0}\big(d\Phi(e_n)\big)$$

where

$$
\begin{aligned}
dT^{x_0}_{h,r}\big(d\Phi(e_j)\big)(x) =& \frac{d}{dt}\Big|_{t=0} T^{x_0}_{h,r}\big(\Phi(x+te_j)\big) \\
=& \frac{d}{dt}\Big|_{t=0}\left(\frac{x+te_j-x_0}{r}\,;\,\frac{f(x+te_j)-P^{x_0}_{h-1}f(x+te_j)}{r^h}\right) \\
=& \left(\frac{e_j}{r}\,;\,\frac{Df(x)e_j-D(P^{x_0}_{h-1}f)(x)e_j}{r^h}\right) \\
=& \left(\frac{e_j}{r}\,;\,\frac{1}{r^h}\left(\frac{\partial f}{\partial x_j}-P^{x_0}_{h-2}\left(\frac{\partial f}{\partial x_j}\right)\right)(x)\right) = \left(\frac{e_j}{r}\,;\,\frac{\varphi^{x_0}_{*j}(x)}{r^h}\right)
\end{aligned}
$$

for all $x \in \mathbb{R}^n$ and $j = 1, \ldots, n$. □

Now we can prove the following useful estimate.

Lemma 3.3. *Let $L > 0$, $x_0 \in \mathbb{R}^n$, $m := \min\{n, k\}$ and consider the field of simple n-vectors defined by*

$$
\begin{aligned}
\eta(u) :=& \big(e_1; G^{x_0}_1(x_0+u)\big) \wedge \cdots \wedge \big(e_n; G^{x_0}_n(x_0+u)\big) \\
=& e_1 \wedge \cdots \wedge e_n + \sum_{j=1}^{m} \sum_{\alpha \in I(n,j)} \sigma(\alpha,\overline{\alpha}) G^{x_0}_\alpha(x_0+u) \wedge e_{\overline{\alpha}}
\end{aligned}
$$

for all $u \in \mathbb{R}^n$. Then the following estimates hold

(i) $\|r^n \Lambda^n dT^{x_0}_{h,r}(\xi(x_0+ru)) - \eta(u)\| \le c\,\varepsilon_1(rL)$, *for all $u \in B_L(0)$ such that $x_0 + ru \in K$;*

(ii) $\|r^n \Lambda^n dT^{x_0}_{h,r}(\xi(x_0+ru))\| \le 1 + c\varepsilon_2(rL) r^{-m}$, *for all $u \in B_L(0)$;*

provided $r \le 1$, where c is a suitable positive constant which does not depend on r and u.

PROOF. Consider $u \in B_L(0)$ such that $x_0 + ru \in K$. For $l = 1, \ldots, n$ let us define

$$
A_l := \left(e_1; \frac{\rho^{x_0}_{*1}(x_0+ru)}{r^{h-1}}\right) \wedge \left(e_2; \frac{\rho^{x_0}_{*2}(x_0+ru)}{r^{h-1}}\right) \wedge \cdots \wedge \left(e_l; \frac{\rho^{x_0}_{*l}(x_0+ru)}{r^{h-1}}\right)
$$

$$
B_l := \Big(e_{n-l+1}; G^{x_0}_{n-l+1}(x_0+u)\Big) \wedge \cdots \wedge \Big(e_{n-1}; G^{x_0}_{n-1}(x_0+u)\Big) \wedge \Big(e_n; G^{x_0}_n(x_0+u)\Big).
$$

Moreover set $A_0 := 1$ and $B_0 := 1$. Then Lemma 3.1 and Lemma 3.2 yield

$$\begin{aligned} \|r^n\,\Lambda^n dT^{x_0}_{h,r}(\xi(x_0+ru)) \,-\, \eta(u)\| =& \|A_n - B_n\| \\ \le& \sum_{l=0}^{n-1} \|A_{n-l}\wedge B_l - A_{n-l-1}\wedge B_{l+1}\| \end{aligned} \tag{7}$$

where

$$\begin{aligned} \|A_{n-l}\wedge B_l - A_{n-l-1}\wedge B_{l+1}\| = & \left\| A_{n-l-1}\wedge \left[\left(e_{n-l}\frac{\rho^{x_0}_{*n-l}(x_0+ru)}{r^{h-1}}\right)\right.\right. \\ & \left.\left. - \left(e_{n-l}; G^{x_0}_{n-l}(x_0+u)\right)\right]\wedge B_l \right\| \\ \le & \|A_{n-l-1}\|\,\|B_l\| \left| \frac{\rho^{x_0}_{*n-l}(x_0+ru)}{r^{h-1}} - G^{x_0}_{n-l}(x_0+u) \right| \end{aligned} \tag{8}$$

for $l = 0, \dots, n-1$.

By recalling (2) and (3), now we obtain that the following estimates

$$|(e_j; G^{x_0}_j(x_0+u))| \le 1 + |G^{x_0}_j(x_0+u)| \le 1 + \frac{\|D^{h-1}g(x_0)\|L^{h-1}}{(h-1)!} \tag{9}$$

$$\begin{aligned} \left|\left(e_j; \frac{\rho^{x_0}_{*j}(x_0+ru)}{r^{h-1}}\right)\right| \le & 1 + \frac{|G^{x_0}_j(x_0+ru)|}{r^{h-1}} + \frac{|\sigma^{x_0}_j(x_0+ru)|}{r^{h-1}} \\ \le & 1 + |G^{x_0}_j(x_0+u)| + L^{h-1}\varepsilon_1(rL) \\ \le & 1 + \frac{\|D^{h-1}g(x_0)\|L^{h-1}}{(h-1)!} + L^{h-1}\varepsilon_1(rL) \end{aligned} \tag{10}$$

and

$$\left| \frac{\rho^{x_0}_{*n-l}(x_0+ru)}{r^{h-1}} - G^{x_0}_{n-l}(x_0+u) \right| = \frac{|\sigma^{x_0}_{n-l}(x_0+ru)|}{r^{h-1}} \le L^{h-1}\varepsilon_1(rL) \tag{11}$$

hold for all $j = 1, \dots, n$.

From the estimates (9) and (10) it follows that there exists a positive constant c_1, not depending on u and r (provided $r \le 1$), such that

$$\|A_{n-l-1}\|\,\|B_l\| \le c_1$$

for all $l = 0, \dots, n-1$. Hence we get (i), by recalling (7), (8) and (11).

In order to prove (ii), consider $u \in B_L(0)$ and recall again Lemma 3.2. We obtain

$$\begin{aligned}
\|r^n \Lambda^n dT^{x_0}_{h,r}(\xi(x_0+ru))\| \leq & 1 + \sum_{j=1}^{m} \frac{1}{r^{j(h-1)}} \sum_{\alpha \in I(n,j)} \|\varphi^{x_0}_{*\alpha}(x_0+ru)\| \\
\leq & 1 + \sum_{j=1}^{m} \binom{n}{j} \frac{1}{r^{j(h-1)}} \left(\varepsilon_2(rL) r^{h-2} L^{h-2}\right)^j \\
\leq & 1 + c_2 \sum_{j=1}^{m} \frac{\varepsilon_2(rL)^j}{r^j} \\
= & 1 + \frac{c_2 \varepsilon_2(rL)}{r^m} \sum_{j=1}^{m} \varepsilon_2(rL)^{j-1} r^{m-j}
\end{aligned}$$

where $m = \min\{n,k\}$ and c_2 is independent from u and r (provided $r \leq 1$). Now the conclusion follows trivially. □

As an easy consequence, we can estimate the measure of $T^{x_0}_{h,r}(G_{f|\mathbb{R}^n \setminus K})$ in the cylinders. Indeed the following result holds.

Lemma 3.4. *Let $L > 0$, $x_0 \in \mathbb{R}^n$ and $m := \min\{n,k\}$. Then one has*

$$\mathcal{H}^n\left(T^{x_0}_{h,r}(G_{f|\mathbb{R}^n \setminus K}) \cap \pi^{-1}(B_L(0))\right) \leq (r^m + c\varepsilon_2(rL)) \frac{\mathcal{L}^n(B_{rL}(x_0) \setminus K)}{r^{n+m}}$$

for all $r > 0$, where c is as il Lemma 3.3. In particular

$$\lim_{r \downarrow 0} \mathcal{H}^n\left(T^{x_0}_{h,r}(G_{f|\mathbb{R}^n \setminus K}) \cap \pi^{-1}(B_L(0))\right) = 0$$

provided

$$\lim_{r \downarrow 0} \frac{\mathcal{L}^n(B_r(x_0) \setminus K)}{r^{n+m}} = 0. \tag{12}$$

PROOF. In fact

$$\begin{aligned}
\mathcal{H}^n\left(T^{x_0}_{h,r}(G_{f|\mathbb{R}^n \setminus K}) \cap \pi^{-1}(B_L(0))\right) = & \mathcal{H}^n\left(T^{x_0}_{h,r}\left(G_{f|\mathbb{R}^n \setminus K} \cap \pi^{-1}(B_{rL}(x_0))\right)\right) \\
= & \mathcal{H}^n\left(T^{x_0}_{h,r}\left(G_{f|B_{rL}(x_0) \setminus K}\right)\right) \\
= & \int_{B_{rL}(x_0) \setminus K} \|\Lambda^n dT^{x_0}_{h,r}(\xi(x))\| \, dx.
\end{aligned}$$

Hence the conclusion follows by Lemma 3.3(ii). □

The next result proves that, under stronger regularity conditions, the field η defined in Lemma 3.3 is tangent to the graph of $\Gamma_h^{x_0}$.

Lemma 3.5. *Let the Schwarz-like equality (5) be satisfied at a point $x_0 \in \mathbb{R}^n$ (not necessarily in $\mathcal{D}(K)$) and for all $j, m = 1, \ldots, n$. Then one has*

$$d\left(\Gamma_h^{x_0}\right)_u e_m = G_m^{x_0}(x_0 + u) \tag{13}$$

for all $m = 1, \ldots, n$ and for all $u \in \mathbb{R}^n$. As a consequence

$$\eta = \Lambda^n d(I \times \Gamma_h^{x_0})\,(e_1 \wedge \cdots \wedge e_n) \tag{14}$$

where $I : \mathbb{R}^n \to \mathbb{R}^n$ denotes the identity map.

In particular, (13) and (14) hold provided $h \geq 3$ and x_0 be a point of density of K.

Proof. Once fixed m and u, by assumption (5), we find

$$\begin{aligned}
\sum_{j=1}^{n} \sum_{\lambda \in \{1,\ldots,n\}^{h-1}} \frac{\partial(u_j u_\lambda)}{\partial u_m} D_\lambda g_{ij}(x_0) = & \sum_{\lambda \in \{1,\ldots,n\}^{h-1}} u_\lambda D_\lambda g_{im}(x_0) \\
& + \sum_{j=1}^{n} u_j \sum_{\lambda \in \{1,\ldots,n\}^{h-1}} \frac{\partial u_\lambda}{\partial u_m} D_\lambda g_{ij}(x_0) \\
= & \langle D^{h-1} g_{im}(x_0) | u^{h-1} \rangle \\
& + (h-1) \sum_{j=1}^{n} u_j \sum_{\mu \in \{1,\ldots,n\}^{h-2}} u_\mu D_\mu \left(\frac{\partial g_{ij}}{\partial x_m} \right)(x_0) \\
= & \langle D^{h-1} g_{im}(x_0) | u^{h-1} \rangle \\
& + (h-1) \sum_{j=1}^{n} \sum_{\mu \in \{1,\ldots,n\}^{h-2}} u_\mu u_j D_\mu \left(\frac{\partial g_{im}}{\partial x_j} \right)(x_0) \\
= & \langle D^{h-1} g_{im}(x_0) | u^{h-1} \rangle \\
& + (h-1) \sum_{\lambda \in \{1,\ldots,n\}^{h-1}} u_\lambda D_\lambda g_{im}(x_0) \\
= & h \langle D^{h-1} g_{im}(x_0) | u^{h-1} \rangle
\end{aligned}$$

for all $i = 1, \dots, k$. Hence

$$d\left(\Gamma_h^{x_0}\right)_u e_m = \frac{\partial \Gamma_h^{x_0}}{\partial u_m}(u) = \frac{1}{h!}\sum_{i=1}^{k}\left(\sum_{j=1}^{n}\sum_{\lambda\in\{1,\dots,n\}^{h-1}}\frac{\partial(u_j u_\lambda)}{\partial u_m} D_\lambda g_{ij}(x_0)\right) e_{n+i}$$

$$= \frac{1}{(h-1)!}\sum_{i=1}^{k}\langle D^{h-1} g_{im}(x_0) | u^{h-1}\rangle\, e_{n+i} = G_m^{x_0}(x_0+u).$$

Finally, the last assertion follows from Lemma 3.1. $\square$

Theorem 3.1. *Let $x_0 \in \mathcal{D}(K)$ and η be the field defined in Lemma 3.3. Consider a bounded measurable set $E \subset \mathbb{R}^n$ and a continuous function*

$$F : \mathbb{R}^n \times \Sigma_1 \to \mathbb{R}.$$

Then one has

$$\lim_{r\downarrow 0}\int_{T_{h,r}^{x_0}(G_{f|K})\cap\pi^{-1}(E)} F(u;\tau_r(u,v))\, d\mathcal{H}^n(u,v) = \int_E F\left(u;\frac{\eta(u)}{\|\eta(u)\|}\right)\|\eta(u)\|\, du.$$

In particular ($F \equiv 1$) the following equality holds

$$\lim_{r\downarrow 0}\mathcal{H}^n\left(T_{h,r}^{x_0}(G_{f|K})\cap\pi^{-1}(E)\right) = \int_E \|\eta(u)\|\, du$$

$$= \int_E\left(1 + \sum_{j=1}^{m}\sum_{\alpha\in I(n,j)}\|G_\alpha^{x_0}(x_0+u)\|^2\right)^{\frac{1}{2}} du$$

$$= \int_E\left(1 + \sum_{j=1}^{m}\sum_{\substack{\alpha\in I(n,j)\\ \beta\in I(k,j)}}\left[\det G_{\beta\alpha}^{x_0}(x_0+u)\right]^2\right)^{\frac{1}{2}} du$$

where $m := \min\{n,k\}$.

PROOF. First of all, consider a positive real number L such that $E \subset B_L(0)$. Then one has

$$\begin{aligned}\mathcal{L}^n(E\setminus t_r^{x_0}(K)) \le& \mathcal{L}^n(B_L(0)\setminus t_r^{x_0}(K)) = \mathcal{L}^n(t_r^{x_0}(B_{rL}(x_0)\setminus K))\\ =& \frac{\mathcal{L}^n(B_{rL}(x_0)\setminus K)}{(rL)^n}L^n \to 0\end{aligned} \tag{15}$$

as $r \downarrow 0$. We get

$$\int_E F\left(u;\frac{\eta(u)}{\|\eta(u)\|}\right)\|\eta(u)\|\, du = \lim_{r\downarrow 0}\int_{E\cap t_r^{x_0}(K)} F\left(u;\frac{\eta(u)}{\|\eta(u)\|}\right)\|\eta(u)\|\, du.$$

Hence it follows that it will be enough to prove that

$$\begin{aligned}\Delta(r) :=& \int_{T^{x_0}_{h,r}(G_{f|K})\cap\pi^{-1}(E)} F(u;\tau_r(u,v))\,d\mathcal{H}^n(u,v)\\ &- \int_{E\cap t^{x_0}_r(K)} F\left(u;\frac{\eta(u)}{\|\eta(u)\|}\right)\|\eta(u)\|\,du \to 0\end{aligned} \tag{16}$$

as $r \downarrow 0$.

For $r > 0$, let us define

$$\begin{aligned}\Delta_1(r) &:= \int_{E\cap t^{x_0}_r(K)} \delta_1(r,u)\,r^n\|\Lambda^n dT^{x_0}_{h,r}(\xi(x_0+ru))\|\,du\\ \Delta_2(r) &:= \int_{E\cap t^{x_0}_r(K)} \delta_2(r,u)\,F\left(u;\frac{\eta(u)}{\|\eta(u)\|}\right)du\end{aligned}$$

where

$$\begin{aligned}\delta_1(r,u) :=& F\left(u;\frac{\Lambda^n dT^{x_0}_{h,r}(\xi(x_0+ru))}{\|\Lambda^n dT^{x_0}_{h,r}(\xi(x_0+ru))\|}\right) - F\left(u;\frac{\eta(u)}{\|\eta(u)\|}\right)\\ \delta_2(r,u) :=& r^n\|\Lambda^n dT^{x_0}_{h,r}(\xi(x_0+ru))\| - \|\eta(u)\|.\end{aligned}$$

Now observe that

- $\Delta_1(r) \to 0$, as $r \downarrow 0$. Indeed one has $\Delta_1(r) = \int_E \psi_r(u)\,du$ where

$$\psi_r(u) := \begin{cases}\delta_1(r,u)\,r^n\|\Lambda^n dT^{x_0}_{h,r}(\xi(x_0+ru))\| & \text{if } u \in t^{x_0}_r(K)\\ 0 & \text{if } u \notin t^{x_0}_r(K).\end{cases}$$

  Hence one concludes by recalling Lemma 3.3 and the dominated convergence theorem;

- $\Delta_2(r) \to 0$, as $r \downarrow 0$. Indeed F is continuous and Lemma 3.3 holds.

Finally (16) follows at once from the identity $\Delta \equiv \Delta_1 + \Delta_2$ which can be easily proved by recalling the definitions of ξ and τ_r, given in §2. $\square$

By recalling Lemma 3.5, we obtain at once the following result.

Corollary 3.1. *Let $x_0 \in \mathcal{D}(K)$ and the Schwarz-like equality (5) be satisfied at x_0, e.g. assume $h \geq 3$ (recall Lemma 3.1). Consider a bounded measurable* *set $E \subset \mathbb{R}^n$ and a continuous function $F : \mathbb{R}^n \times \Sigma_1 \to \mathbb{R}$. Then one has*

$$\lim_{r \downarrow 0} \int_{T^{x_0}_{h,r}(G_{f|K}) \cap \pi^{-1}(E)} F(u; \tau_r(u,v))\, d\mathcal{H}^n(u,v)$$
$$= \int_{G_{\Gamma^{x_0}_h} \cap \pi^{-1}(E)} F(u; \tau_0(u,v))\, d\mathcal{H}^n(u,v).$$

In particular ($F \equiv 1$) the following equality holds.

$$\lim_{r \downarrow 0} \mathcal{H}^n \left(T^{x_0}_{h,r}(G_{f|K}) \cap \pi^{-1}(E) \right) = \mathcal{H}^n \left(G_{\Gamma^{x_0}_h} \cap \pi^{-1}(E) \right).$$

Remark 3.3. The nice behavior of the surfaces $T^{x_0}_{h,r}(G_{f|K})$, with respect to convergence, stated in Theorem 3.1 and in Corollary 3.1, is due to the strong relation existing in K between g and ∇f. In fact they coincide! Since outside K the fields g and ∇f are (in general) unrelated, we cannot expect the mentioned results to hold with f in place of $f|K$, unless some further assumption is considered. For example, one can prescribe condition (12) and then apply Lemma 3.4.

4 Some Further Convergence Results under Reinforced Assumptions

Let us consider the following generic question.

> How to strengthen the assumption that K has density one at x_0, in order to get the convergence of $f^{x_0}_{h,r}$ (resp. $T^{x_0}_{h,r}(G_{f|K})$) to $\Gamma^{x_0}_h$ (resp. $G_{\Gamma^{x_0}_h}$), as $r \downarrow 0$?

In this section we will provide some answers with respect to pointwise, mean and graph measures convergence. In short, it turns out that all of them (except for the pointwise convergence in the case $n \geq 2$) occur as soon as K is assumed to have "density one of order $h + n - 1$ at x_0"; namely

$$\lim_{r \downarrow 0} \frac{\mathcal{L}^n(B_r(x_0) \setminus K)}{r^{h+n-1}} = 0.$$

4.1 Pointwise Convergence

First of all, we will consider the case $n = 1$. In such a particular setting, the pointwise convergence actually occurs by assuming that K has density one of order h at x_0. Indeed, one has the following result.

Theorem 4.1 (n=1). *If*

$$\lim_{r\downarrow 0} \frac{\mathcal{L}^1((x_0-r, x_0+r)\setminus K)}{r^h} = 0 \tag{17}$$

then $\lim_{r\downarrow 0} f_{h,r}^{x_0}(u) = \Gamma_h^{x_0}(u) = \frac{u^h}{h!} D^{h-1}g(x_0)$ *for all* $u \in \mathbb{R}$.

PROOF. We have to verify that $\Delta(r) := f_{h,r}^{x_0}(u) - \frac{u^h}{h!} D^{h-1}g(x_0) \to 0$ as $r \downarrow 0$, for all $u \in \mathbb{R}$.

Let us define the functions

$$\Delta_1(r) := \frac{1}{r^h} \int_{(x_0, x_0+ru)\setminus K} (f'(x) - g(x))\, dx$$

$$\Delta_2(r) := \frac{1}{r^h} \int_{x_0}^{x_0+ru} \left(g(x) - \sum_{j=0}^{h-1} \frac{(x-x_0)^j}{j!} D^j g(x_0) \right) dx$$

and observe that, by Lemma 3.1, we get

$$\begin{aligned}
\frac{f(x_0+ru) - P_{h-1}^{x_0} f(x_0+ru)}{r^h} - \frac{u^h}{h!} D^{h-1}g(x_0)& \\
&= \frac{1}{r^h}\left(f(x_0+ru) - f(x_0) - \sum_{j=1}^{h} \frac{(ru)^j}{j!} D^{j-1} g(x_0) \right) \\
&= \frac{1}{r^h}\left(\int_{x_0}^{x_0+ru} f'(x)\, dx - \sum_{j=0}^{h-1} \frac{(ru)^{j+1}}{(j+1)!} D^j g(x_0) \right) \\
&= \frac{1}{r^h}\left(\int_{x_0}^{x_0+ru} g(x)\, dx + \int_{(x_0,x_0+ru)\setminus K} (f'(x) - g(x))\, dx + \right.\\
&\qquad \left. - \sum_{j=0}^{h-1} \frac{D^j g(x_0)}{(j+1)!} \int_{x_0}^{x_0+ru} D(x-x_0)^{j+1} dx \right);
\end{aligned}$$

namely $\Delta(r) = \Delta_1(r) + \Delta_2(r)$. The conclusion immediately follows, in that:

- $\Delta_1(r) \to 0$, as $r \downarrow 0$, by assumption (17);
- a standard estimate of the remainder in the Taylor's formula (e.g. see [6, V, §6]) yields

$$\begin{aligned}
|\Delta_2(r)| &\le \frac{1}{r^h (h-1)!} \left| \int_{x_0}^{x_0+ru} (x-x_0)^{h-1} dx \right| \sup_{(x_0,x_0+ru)} \|D^{h-1}g - D^{h-1}g(x_0)\| \\
&= \frac{|u|^h}{h!} \sup_{(x_0,x_0+ru)} \|D^{h-1}g - D^{h-1}g(x_0)\| \to 0
\end{aligned}$$

as $r \downarrow 0$. □

As for the case $n \geq 2$, here there is an example showing that pointwise convergence does not occur, in general, irrespective of the order of density one (of K at x_0) which one is assuming.

Example ($n \geq 2$). We will assume $n = 2$, $k = 1$ and $h = 2$, but our argument is completely general and can be easily arranged in order to produce similar examples in any different situation (provided $n \geq 2$). Let

$$C := \mathbb{R} \setminus \cup_{j=1}^{\infty} I_j, \ I_j := \left(\frac{1}{2^j}, \frac{1}{2^j} + \frac{1}{j 2^j} \right)$$

and

$$K_1 := \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, |y| \geq e^{-1/x^2} \right\}.$$

Then consider the function studied in [2, Second example], which will be denoted by φ. For the convenience of the reader, recall from [2] that φ' is piecewise linear, $\varphi' | C \equiv 0$ and $\varphi' | I_j$ is a tent-like function attaining its maximum value at the middle point m_j of I_j, with $\varphi'(m_j) = 2^{-j/2}$. An easy computation [2, Proposition 5.3] shows that $\lim_{r \downarrow 0} \varphi^0_{2,r}(t) = +\infty$ for all $t > 0$.

Now we have to define $f \in C^1(\mathbb{R}^2)$ and $g_1 \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. Set $g_1 := (0, 0)$ while f can be any function such that $f | K_1 \equiv 0$ and $f(t, 0) = \varphi(t)$ for all $t \in \mathbb{R}$. Then

$$K = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, \nabla f(x, y) = g_1(x, y) = 0 \right\}$$

includes the set K_1. Hence

$$\frac{\mathcal{L}^2(B_r(0,0) \setminus K)}{r^l} \leq \frac{\mathcal{L}^2(B_r(0,0) \setminus K_1)}{r^l} \to 0$$

as $r \downarrow 0$, for every fixed integer number l.

Despite such a very strong condition on the density of K at $(0, 0)$, the function $f^{(0,0)}_{2,r}$ does not converge everywhere to $\Gamma^{(0,0)}_2 \equiv 0$, as $r \downarrow 0$. Indeed, for instance, one has $\lim_{r \downarrow 0} f^{(0,0)}_{2,r}(t, 0) = \lim_{r \downarrow 0} \varphi^0_{2,r}(t) = +\infty$ as $r \downarrow 0$, for all $t > 0$.

4.2 Convergence L^1_{loc}

One has the following result.

Theorem 4.2. *The equalities*

$$\lim_{r \downarrow 0} \frac{\int_{B_r(x_0)} \mathcal{H}^1\big([x_0; x] \setminus K\big)\, dx}{r^{h+n}} = 0 \tag{18}$$

and

$$\lim_{r\downarrow 0} \frac{\mathcal{L}^n(B_r(x_0)\setminus K)}{r^{h+n-1}} = 0. \tag{19}$$

are equivalent. If they are satisfied, then $f_{h,r}^{x_0}$ *converges to* $\Gamma_h^{x_0}$ *in* L^1_{loc}*, as* $r \downarrow 0$*.*

PROOF. The part of the statement concerning the equivalence of the two equalities follows immediately from Theorem 5.1.

Then let us assume the two equalities are true (Our argument below is based on the first one.) and observe that, as a consequence, K has density one at x_0. Let R be any fixed positive real number and, for $r > 0$, let

$$\Delta_i(r) := \int_{B_R(0)} |f_{h,r}^{x_0}(u)\cdot e_{n+i} - \Gamma_h^{x_0}(u)\cdot e_{n+i}|\, du \ (i = 1,\dots,k).$$

Since the inequality

$$\int_{B_R(0)} |f_{h,r}^{x_0}(u) - \Gamma_h^{x_0}(u)|\, du \le \sum_{i=1}^{k} \Delta_i(r)$$

holds for all $r > 0$, it will be enough to prove that

$$\lim_{r\downarrow 0} \Delta_i(r) = 0 \tag{20}$$

for all $i = 1,\dots,k$.

First of all, by the change of variables formula for integrals (with $x = x_0 + ru$) and Lemma 3.1, it follows that

$$\begin{aligned}
\Delta_i(r) &= \int_{B_R(0)} \left| \frac{f_i(x_0+ru) - P_{h-1}^{x_0} f_i(x_0+ru)}{r^h} - \frac{\langle D^{h-1}g_i(x_0)|u^{h-1}\rangle\cdot u}{h!}\right| du \\
&= \frac{1}{r^{n+h}} \int_{B_{rR}(x_0)} \Bigg| f_i(x) - P_{h-1}^{x_0} f_i(x) \\
&\qquad\qquad - \frac{1}{h!}\langle D^{h-1}g_i(x_0)|(x-x_0)^{h-1}\rangle\cdot(x-x_0)\Bigg|\, dx \\
&= \frac{1}{r^{n+h}} \int_{B_{rR}(x_0)} \Bigg| \int_0^1 \nabla f_i(x_0 + t(x-x_0))\cdot(x-x_0)\, dt \\
&\qquad\qquad - \sum_{j=0}^{h-2} \frac{\langle D^j g_i(x_0)|(x-x_0)^j\rangle\cdot(x-x_0)}{(j+1)!} \\
&\qquad\qquad - \frac{\langle D^{h-1}g_i(x_0)|(x-x_0)^{h-1}\rangle\cdot(x-x_0)}{h!}\Bigg|\, dx.
\end{aligned}$$

Hence, by setting $K_x := \{t \in \mathbb{R} \,|\, x_0 + t(x - x_0) \in K\}$ and

$$\begin{aligned}\Delta_i^{(1)}(r) :=& \frac{1}{r^{n+h}} \int_{B_{rR}(x_0)} \left(\int_{[0,1]\setminus K_x} |\nabla f_i(x_0 + t(x - x_0)) \right.\\ &\left. - g_i(x_0 + t(x - x_0))| \, dt \right) |x - x_0| \, dx\end{aligned}$$

$$\begin{aligned}\Delta_i^{(2)}(r) :=& \frac{1}{r^{n+h}} \int_{B_{rR}(x_0)} \left| \int_0^1 g_i(x_0 + t(x - x_0)) \, dt \right.\\ &\left. - \sum_{j=0}^{h-1} \frac{\langle D^j g_i(x_0) | (x - x_0)^j \rangle}{(j+1)!} \right| |x - x_0| \, dx\end{aligned}$$

we obtain

$$\begin{aligned}\Delta_i(r) =& \frac{1}{r^{n+h}} \int_{B_{rR}(x_0)} \left| \int_0^1 g_i(x_0 + t(x - x_0)) \cdot (x - x_0) \, dt \right.\\ & - \int_{[0,1]\setminus K_x} g_i(x_0 + t(x - x_0)) \cdot (x - x_0) \, dt \\ & + \int_{[0,1]\setminus K_x} \nabla f_i(x_0 + t(x - x_0)) \cdot (x - x_0) \, dt \\ & \left. - \sum_{j=0}^{h-1} \frac{\langle D^j g_i(x_0) | (x - x_0)^j \rangle \cdot (x - x_0)}{(j+1)!} \right| dx \le \Delta_i^{(1)}(r) + \Delta_i^{(2)}(r).\end{aligned}$$

Then (20) follows, in that

- $\Delta_i^{(1)}(r) \to 0$, as $r \downarrow 0$. Indeed, if assume $rR \le 1$ and set $c_i := \sup_{B_1(x_0)} |\nabla f_i - g_i|$, then one has

$$\begin{aligned}\Delta_i^{(1)}(r) \le& \frac{c_i}{r^{n+h}} \int_{B_{rR}(x_0)} \left(\int_{[0,1]\setminus K_x} dt \right) |x - x_0| \, dx \\ =& \frac{c_i}{r^{n+h}} \int_{B_{rR}(x_0)} \mathcal{H}^1([x_0; x] \setminus K) \, dx \to 0\end{aligned}$$

  as $r \downarrow 0$, by the hypothesis (18).

- $\Delta_i^{(2)}(r) \to 0$, as $r \downarrow 0$. Indeed, by the estimate of the remainder for the Taylor's formula already invoked in the proof of Theorem 4.1, we obtain

$$\Delta_i^{(2)}(r) = \frac{1}{r^{n+h}} \int_{B_{rR}(x_0)} \left| \int_0^1 g_i(x_0 + t(x - x_0)) \, dt \right.$$

$$
\begin{aligned}
&\quad - \sum_{j=0}^{h-1} \frac{\langle D^j g_i(x_0) | (x-x_0)^j \rangle}{(j+1)!} \int_0^1 (t^{j+1})' \, dt \Bigg| \, |x-x_0| \, dx \\
&\leq \frac{1}{r^{n+h}} \int_{B_{rR}(x_0)} \Bigg(\int_0^1 \Bigg| g_i(x_0 + t(x-x_0)) \\
&\qquad\qquad - \sum_{j=0}^{h-1} \frac{\langle D^j g_i(x_0) | [t(x-x_0)]^j \rangle}{j!} \Bigg| \, dt \Bigg) |x-x_0| \, dx \\
&\leq \frac{1}{r^{n+h}(h-1)!} \int_{B_{rR}(x_0)} |x-x_0| \Bigg(\sup_{[x_0;x]} \| D^{h-1} g_i - D^{h-1} g_i(x_0) \| \Bigg) \\
&\qquad\qquad \Bigg(\int_0^1 |x-x_0|^{h-1} t^{h-1} dt \Bigg) dx \\
&\leq \frac{(rR)^h \mathcal{L}^n(B_{rR}(x_0))}{r^{n+h} h!} \sup_{B_{rR}(x_0)} \| D^{h-1} g_i - D^{h-1} g_i(x_0) \| \\
&= \frac{R^{n+h} \mathcal{L}^n(B_1(0))}{h!} \sup_{B_{rR}(x_0)} \| D^{h-1} g_i - D^{h-1} g_i(x_0) \| \to 0
\end{aligned}
$$

as $r \downarrow 0$. □

4.3 Convergence of the Graph Measures

Let us prove that the local convergence in measure implies the convergence of the corresponding graph measures.

Theorem 4.3. *If the Schwarz-like equality (5) is satisfied at a point x_0 of density of K, the following statements hold:*

(i) Let E be a bounded open subset of $\mathbb{R}^n$ such that

$$\lim_{r \downarrow 0} \mathcal{L}^n(\mathcal{E}_r^\delta) = 0, \ \mathcal{E}_r^\delta := \{ u \in E \cap t_r^{x_0}(K) \mid |f_{h,r}^{x_0}(u) - \Gamma_h^{x_0}(u)| \geq \delta \}$$

for all $\delta > 0$. Then

$$\lim_{r \downarrow 0} \int_{T_{h,r}^{x_0}(G_{f|K})} \varphi \, d\mathcal{H}^n = \int_{G_{\Gamma_h^{x_0}}} \varphi \, d\mathcal{H}^n \tag{21}$$

for every function $\varphi : \mathbb{R}^{n+k} \to \mathbb{R}$ which is supported, bounded and uniformly continuous in $\overline{E} \times \mathbb{R}^k$.

(ii) If $f_{h,r}^{x_0} \to \Gamma_h^{x_0}$, locally in measure as $r \downarrow 0$, then

$$\mathcal{H}^n \llcorner T_{h,r}^{x_0}(G_{f|K}) \to \mathcal{H}^n \llcorner G_{\Gamma_h^{x_0}}$$

as $r \downarrow 0$, in the weak* sense of measures.

In particular, the statements (i) and (ii) are true provided $h \geq 3$ and x_0 be a point of density of K.

PROOF. Observe that (ii) is an immediate consequence of (i), while the ending assertion trivially follows from Lemma 3.1.

In order to prove (i), consider a function φ satisfying the hypotheses listed in the statement. Recalling the change of variables formula for integrals (with $x = x_0 + ru$), we obtain

$$\begin{aligned}\int_{T_{h,r}^{x_0}(G_{f|K})} \varphi\, d\mathcal{H}^n &= \int_K \varphi(T_{h,r}^{x_0}(x; f(x)))\, \|\Lambda^n dT_{h,r}^{x_0}(\xi(x))\|\, dx \\ &= \int_K \varphi\left(\frac{x-x_0}{r}; \frac{f(x) - P_{h-1}^{x_0} f(x)}{r^h}\right) \|\Lambda^n dT_{h,r}^{x_0}(\xi(x))\|\, dx \\ &= \int_{t_r^{x_0}(K)} \varphi(u; f_{h,r}^{x_0}(u))\, \|r^n \Lambda^n dT_{h,r}^{x_0}(\xi(x_0+ru))\|\, du.\end{aligned}$$

Then, by Lemma 3.5, it follows that

$$\begin{aligned}&\int_{T_{h,r}^{x_0}(G_{f|K})} \varphi\, d\mathcal{H}^n - \int_{G_{\Gamma_h^{x_0}}} \varphi\, d\mathcal{H}^n \\ &= \int_{t_r^{x_0}(K)} \varphi(u; f_{h,r}^{x_0}(u))\, \|r^n \Lambda^n dT_{h,r}^{x_0}(\xi(x_0+ru))\|\, du \\ &\quad - \int_{\mathbb{R}^n} \varphi(u; \Gamma_h^{x_0}(u)) \|\eta(u)\|\, du \\ &= \int_{t_r^{x_0}(K)} \varphi(u; f_{h,r}^{x_0}(u)) \big(\|r^n \Lambda^n dT_{h,r}^{x_0}(\xi(x_0+ru))\| - \|\eta(u)\|\big) du \\ &\quad + \int_{t_r^{x_0}(K)} \Big(\varphi(u; f_{h,r}^{x_0}(u)) - \varphi(u; \Gamma_h^{x_0}(u))\Big)\, \|\eta(u)\|\, du \\ &\quad - \int_{\mathbb{R}^n \setminus t_r^{x_0}(K)} \varphi(u; \Gamma_h^{x_0}(u)) \|\eta(u)\|\, du.\end{aligned}$$

Hence we find

$$\left| \int_{T_{h,r}^{x_0}(G_{f|K})} \varphi\, d\mathcal{H}^n - \int_{G_{\Gamma_h^{x_0}}} \varphi\, d\mathcal{H}^n \right| \leq c_1 \Delta_1(r) + c_2 \Delta_2(r) + c_1 c_2 \mathcal{L}^n(E \setminus t_r^{x_0}(K))$$

where $c_1 := \sup_{E\times R^k} |\varphi| < +\infty,\ c_2 := \sup_E \|\eta\| < +\infty$ and

$$\Delta_1(r) := \int_{E\cap t_r^{x_0}(K)} \|r^n\Lambda^n dT^{x_0}_{h,r}(\xi(x_0+ru)) - \eta(u)\|\, du$$
$$\Delta_2(r) := \int_{E\cap t_r^{x_0}(K)} \left|\varphi(u; f^{x_0}_{h,r}(u)) - \varphi(u;\Gamma^{x_0}_h(u))\right| du.$$

Now the equality (21) follows, observing that

- $\Delta_1(r) \to 0$, as $r \downarrow 0$, by Lemma 3.3;
- $\Delta_2(r) \to 0$, as $r \downarrow 0$. Indeed, for all $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

  $$|\varphi(P) - \varphi(Q)| \le \varepsilon$$

  provided $P, Q \in \overline{E} \times \mathbb{R}^k$ satisfy $|P - Q| \le \delta_\varepsilon$. In particular, fixed $\varepsilon > 0$ arbitrarily. One has $\left|\varphi(u; f^{x_0}_{h,r}(u)) - \varphi(u;\Gamma^{x_0}_h(u))\right| \le \varepsilon$ for all $u \in (E \cap t_r^{x_0}(K)) \setminus \mathcal{E}_r^{\delta_\varepsilon}$ and for all $r > 0$. Thus

  $$\begin{aligned}\Delta_2(r) = &\int_{(E\cap t_r^{x_0}(K))\setminus\mathcal{E}_r^{\delta_\varepsilon}} \left|\varphi(u; f^{x_0}_{h,r}(u)) - \varphi(u;\Gamma^{x_0}_h(u))\right| du\\ &+ \int_{\mathcal{E}_r^{\delta_\varepsilon}} \left|\varphi(u; f^{x_0}_{h,r}(u)) - \varphi(u;\Gamma^{x_0}_h(u))\right| du\\ \le&\varepsilon\mathcal{L}^n(E) + 2c_1\mathcal{L}^n(\mathcal{E}_r^{\delta_\varepsilon}) \to \varepsilon\mathcal{L}^n(E)\end{aligned}$$

  as $r \downarrow 0$. The conclusion follows from the arbitrariness of ε.
- $\mathcal{L}^n(E \setminus t_r^{x_0}(K)) \to 0$, as $r \downarrow 0$, in that K has density one at x_0 (compare (15)). □

Corollary 4.1. *Let $x_0 \in \mathcal{D}(K)$ and the Schwarz-like equality (5) be satisfied at x_0, e.g. assume $h \ge 3$ (recall Lemma 3.1). Then one has*

$$\mathcal{H}^n \llcorner T^{x_0}_{h,r}(G_{f|K}) \to \mathcal{H}^n \llcorner G_{\Gamma^{x_0}_h}$$

as $r \downarrow 0$, in the weak sense of measures, provided $f^{x_0}_{h,r}$ converges in L^1_{loc} to $\Gamma^{x_0}_h$, as $r \downarrow 0$.*

Proof. It's enough to recall the well known result according to which the convergence in L^1_{loc} of a sequence of functions implies the convergence locally in measure of the same sequence to the same limit function, e.g. [5, §25, Theorem A]. □

Corollary 4.2. *Let one of the two equivalent equalities (18) and (19) be satisfied. Moreover assume the Schwarz-like equality (5), e.g. let $h \geq 3$ (recall Lemma 3.1). Then one has $\mathcal{H}^n \llcorner T^{x_0}_{h,r}(G_{f|K}) \to \mathcal{H}^n \llcorner G_{\Gamma^{x_0}_h}$ as $r \downarrow 0$, in the weak* sense of measures. Under the additional condition $h - 1 \geq \min\{n, k\}$, even the weak* convergence of the whole graph measures occurs, i.e. $\mathcal{H}^n \llcorner T^{x_0}_{h,r}(G_f) \to \mathcal{H}^n \llcorner G_{\Gamma^{x_0}_h}$ as $r \downarrow 0$.*

PROOF. The first statement is a consequence of Theorem 4.2 and Corollary 4.1. The second one follows from Lemma 3.4 by observing that (19) implies (12). □

5 Appendix

This appendix is devoted to stating Theorem 5.1 which provides a useful characterization of condition (18).

Theorem 5.1 (P. Mattila)**.** *Given a Lebesgue measurable subset E of $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$, the following hold:*

(i) If there exists a couple of constants $a > 0$ and $m > n - 1$ such that

$$\mathcal{L}^n(E \cap B_r(x_0)) \leq ar^m$$

for all r small enough, then one has

$$\int_{B_r(x_0)} \mathcal{H}^1(E \cap [x_0; x])dx \leq abr^{m+1}$$

for all r small enough, where b is positive and depending only on n, m.

(ii) If a and m are positive constants such that

$$\int_{B_r(x_0)} \mathcal{H}^1(E \cap [x_0; x])dx \leq ar^{m+1}$$

for all r small enough, then there exists b, positive and depending only on n, such that

$$\mathcal{L}^n(E \cap B_r(x_0)) \leq abr^m$$

for all r small enough.

As a consequence, for $m > n - 1$, it follows that

$$\lim_{r\downarrow 0} \frac{\int_{B_r(x_0)} \mathcal{H}^1(E \cap [x_0; x])\, dx}{r^{m+1}} = 0$$

if and only if $\lim_{r\downarrow 0} \frac{\mathcal{L}^n(E\cap B_r(x_0))}{r^m} = 0$.

PROOF. Without affecting the generality of our argument, we can assume $x_0 = 0$. Moreover we will denote $B_r(x_0)$ simply by B_r.

By using a Fubini type argument, one can easily verify that a constant $c = c(n)$ has to exist such that

$$\begin{aligned}\frac{1}{c}\mathcal{L}^n(F \cap B_1 \setminus B_{1/2}) &\leq \int_{\mathbf{S}^{n-1}} \mathcal{H}^1\left(F \cap [0;y] \cap B_1 \setminus B_{1/2}\right) d\mathcal{H}^{n-1}(y)\\ &\leq c\mathcal{L}^n(F \cap B_1 \setminus B_{1/2})\end{aligned}$$

for every Lebesgue measurable subset F of $\mathbb{R}^n$. Since

$$\begin{aligned}&\int_{\mathbf{S}^{n-1}} \mathcal{H}^1\left(E \cap [0;ry] \cap B_{2^{1-j}r} \setminus B_{2^{-j}r}\right) d\mathcal{H}^{n-1}(y)\\ =&2^{1-j}r \int_{\mathbf{S}^{n-1}} \mathcal{H}^1\left(\frac{2^{j-1}}{r}(E \cap [0;ry]) \cap B_1 \setminus B_{1/2}\right) d\mathcal{H}^{n-1}(y)\\ =&2^{1-j}r \int_{\mathbf{S}^{n-1}} \mathcal{H}^1\left(\frac{2^{j-1}}{r}E \cap [0;y] \cap B_1 \setminus B_{1/2}\right) d\mathcal{H}^{n-1}(y)\end{aligned}$$

the next inequalities readily follow.

$$\begin{aligned}&\frac{2^{(j-1)(n-1)}}{cr^{n-1}}\mathcal{L}^n(E \cap B_{2^{1-j}r} \setminus B_{2^{-j}r})\\ \leq&\int_{\mathbf{S}^{n-1}} \mathcal{H}^1(E \cap [0;ry] \cap B_{2^{1-j}r} \setminus B_{2^{-j}r})d\mathcal{H}^{n-1}(y)\\ \leq&\frac{c2^{(j-1)(n-1)}}{r^{n-1}}\mathcal{L}^n(E \cap B_{2^{1-j}r} \setminus B_{2^{-j}r}).\end{aligned} \tag{22}$$

Also observe that

$$\begin{aligned}&\int_{B_r} \mathcal{H}^1(E \cap [0;x])dx = \sum_{j=1}^{\infty}\int_{B_r} \mathcal{H}^1\left(E \cap [0;x] \cap B_{2^{1-j}r} \setminus B_{2^{-j}r}\right) dx\\ =&\sum_{j=1}^{\infty}\int_{\mathbf{S}^{n-1}}\left(\int_{2^{-j}r}^{r} \mathcal{H}^1(E \cap [0;ty] \cap B_{2^{1-j}r} \setminus B_{2^{-j}r})t^{n-1}dt\right) d\mathcal{H}^{n-1}(y).\end{aligned} \tag{23}$$

Now, let us prove the first statement. By (23) and the last inequality in (22), we get

$$\begin{aligned}&\int_{B_r} \mathcal{H}^1(E \cap [0;x])\, dx\\ \leq&\sum_{j=1}^{\infty}\int_{\mathbf{S}^{n-1}}\left(\int_{2^{-j}r}^{r} \mathcal{H}^1(E \cap [0;ry] \cap B_{2^{1-j}r} \setminus B_{2^{-j}r})t^{n-1}dt\right) d\mathcal{H}^{n-1}(y)\end{aligned}$$

$$
\begin{aligned}
&<r^n \sum_{j=1}^{\infty} \int_{\mathbf{S}^{n-1}} \mathcal{H}^1(E \cap [0; ry] \cap B_{2^{1-j}r} \setminus B_{2^{-j}r}) d\mathcal{H}^{n-1}(y) \\
&\leq cr \sum_{j=1}^{\infty} 2^{(n-1)(j-1)} \mathcal{L}^n(E \cap B_{2^{1-j}r} \setminus B_{2^{-j}r}) \leq acr \sum_{j=1}^{\infty} 2^{(n-1)(j-1)} \left(2^{1-j}r\right)^m \\
&= acr^{m+1} \sum_{j=1}^{\infty} 2^{(n-1-m)(j-1)} = \frac{ac}{1-2^{n-1-m}}\, r^{m+1}
\end{aligned}
$$

which concludes the proof of (i).

It remains to prove the second statement. Recalling (23) and the first inequality in (22), we find

$$
\begin{aligned}
&\int_{B_r} \mathcal{H}^1(E \cap [0; x])\, dx \\
&\geq \sum_{j=1}^{\infty} \int_{\mathbf{S}^{n-1}} \left(\int_{2^{1-j}r}^{r} \mathcal{H}^1\left(E \cap [0; ty] \cap B_{2^{1-j}r} \setminus B_{2^{-j}r}\right) t^{n-1} dt \right) d\mathcal{H}^{n-1}(y) \\
&= \sum_{j=1}^{\infty} \int_{\mathbf{S}^{n-1}} \left(\int_{2^{1-j}r}^{r} \mathcal{H}^1\left(E \cap [0; ry] \cap B_{2^{1-j}r} \setminus B_{2^{-j}r}\right) t^{n-1} dt \right) d\mathcal{H}^{n-1}(y) \\
&= \frac{r^n}{n} \sum_{j=1}^{\infty} \left(1 - 2^{(1-j)n}\right) \int_{\mathbf{S}^{n-1}} \mathcal{H}^1\left(E \cap [0; ry] \cap B_{2^{1-j}r} \setminus B_{2^{-j}r}\right) d\mathcal{H}^{n-1}(y) \\
&\geq \frac{r}{nc} \sum_{j=1}^{\infty} \left(1 - 2^{(1-j)n}\right) 2^{(j-1)(n-1)} \mathcal{L}^n\left(E \cap B_{2^{1-j}r} \setminus B_{2^{-j}r}\right) \\
&= \frac{r}{nc} \sum_{j=2}^{\infty} \left(2^{(j-1)(n-1)} - 2^{1-j}\right) \mathcal{L}^n\left(E \cap B_{2^{1-j}r} \setminus B_{2^{-j}r}\right) \\
&\geq \frac{r}{nc} \sum_{j=2}^{\infty} \mathcal{L}^n\left(E \cap B_{2^{1-j}r} \setminus B_{2^{-j}r}\right) = \frac{r}{nc} \mathcal{L}^n(E \cap B_{r/2}).
\end{aligned}
$$

Hence the conclusion immediately follows. □

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