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James V. Peters, Department of Mathematics, Long Island University, Brookville, NY 11548. email: jpeters@liu.edu

RADON TRANSFORMS OF TEMPERED DISTRIBUTIONS

Abstract

The Radon transform is not uniquely defined for distributions. Moreover, on even dimensional Euclidean space, the formal integral defining the transform converges only for a subspace of tempered distributions.

1 Introduction.

Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz class of functions $f(x)$ which, together with their partial derivatives of all order, go to 0 faster than $|x|^{-k}$ for all positive integers k as $x \to \infty$. The Radon transform of $f(x) \in \mathcal{S}(\mathbb{R}^n)$ is given by

$$\mathcal{R}f(\theta, t) = \int_{<\theta,\ x>=t} f(x)\,\omega(x). \tag{1.1}$$

The hyperplane $<\theta,\ x>= t$ is parametrized by a unit normal vector θ and its directed distance $t \in \mathbb{R}^1$ from the origin; $\omega(x)$ is the differential form for integration on the hyperplane. Thus $\mathcal{R}f$ is defined on the product space $S^{n-1} \times \mathbb{R}^1$ where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$. Since $\mathcal{R}f(\theta, t) = \mathcal{R}f(-\theta, -t)$, it is convenient to choose $\theta = (\theta_1, \ldots, \theta_n)$ so that the right most non-zero component is positive. This choice is consistent with the application of the Radon transform to reconstructive tomography. For lines in the plane, $\theta_1 = \cos\varphi$ and $\theta_2 = \sin\varphi$ with scans taken over the range of angles $0 \le \varphi < \pi$. Basic properties of the Radon transform along with a wealth of applications can be found in [2].

There is a simple relationship between the Radon transform and the Fourier transform

$$\mathcal{F}f(y) = \int_{\mathbb{R}^n} f(x) e^{-i<x,\ y>}\,dx. \tag{1.2}$$

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Write $y = \alpha\theta$ with $\alpha \in \mathbb{R}^1$ and $\theta \in S^{n-1}$. Integrate (1.2) over the hyperplane $< \theta,\ x >= t$ and then with respect to t. This yields

$$\mathcal{F}f(\alpha\theta) = \int_{-\infty}^{+\infty} \mathcal{R}f(\theta,\ t)e^{-i\alpha t}\,dt, \tag{1.3}$$

which implies that $\mathcal{R}f$ is the one dimensional inverse Fourier transform of $\mathcal{F}f$. The Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ onto itself. In our notation, the range space of the Fourier transform is $\mathcal{S}(S^{n-1} \times \mathbb{R}^1)$. It follows from (1.3) that the Radon transform maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(S^{n-1} \times \mathbb{R}^1)$. The mapping is not onto since, for example, each $f(x) \in \mathcal{S}(\mathbb{R}^n)$ satisfies

$$\int_{-\infty}^{+\infty} \mathcal{R}f(\theta, t)\ dt = C, \tag{1.4}$$

where C is a constant independent of θ. There are additional moment conditions that $\mathcal{R}f$ satisfies but these will not be needed in what follows.

The Fourier transform of a tempered distribution F is uniquely defined as a functional $\mathcal{F}$ on $\mathcal{S}(\mathbb{R}^n)$ satisfying

$$F(f) = (2\pi)^{-n}\mathcal{F}F(\mathcal{F}f) \tag{1.5}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. The analog of (1.5) for the Radon transform [3, p. 12] is

$$F(f) = \frac{1}{2}(2\pi)^{1-n}\int_{-\infty}^{+\infty}\int_{S^{n-1}} \mathcal{R}F(\theta, t)\Psi_f(\theta, t)\,d\theta\,dt = \frac{1}{2}(2\pi)^{1-n}\mathcal{R}F(\Psi_f), \tag{1.6}$$

where $\Psi_f(\theta, t) = D^{n-1}\mathcal{R}f(\theta, t)$. The operator D^{n-1} corresponds to the Fourier multiplier $(\mathtt{i} \mid \alpha \mid)^{n-1}$. Thus,

$$\Psi_f(\theta, t) = (-1)^{(\frac{n-1}{2})}\frac{\partial^{n-1}}{\partial t^{n-1}}\mathcal{R}f$$

if n is odd and the Hilbert transform of $\frac{\partial^{n-1}}{\partial t^{n-1}}\mathcal{R}f$ for n even. Evidently, $\mathcal{R}F$ is not uniquely defined since any polynomial $\sum c_j(\theta)t^j$ of degree $\leq n-2$ annihilates $D^{n-1}\mathcal{R}f$. Beyond non-uniqueness, the convergence of the integral on the right hand side of (1.6) depends on F when n is even. Indeed, $\Psi_f(\theta, t)$ is infinitely differentiable and $\frac{\partial^{n-1}}{\partial t^{n-1}}\mathcal{R}f \in \mathcal{S}(S^{n-1} \times \mathbb{R}^1)$. However, the order estimate $\Psi_f = O(|t|^{-n})$ is best possible for n even due to the Hilbert transform. In section 2 we show that (1.6) converges for distributions that can be identified with functions in $L^p(\mathbb{R}^n)$ for $1 \leq p \leq 2$. The annihilating polynomials are also determined.

2 L^p Spaces.

If $F \in L^1(\mathbb{R}^n)$, then F is integrable over hyperplanes. In fact, integration of $|F|$ over $<\theta,\ x>= t$ and then with respect to t yields

$$\int_{\mathbb{R}^n} |F(x)|\ dx \geq \int_{-\infty}^{+\infty} |\mathcal{R}F(\theta,t)|\ dt. \tag{2.1}$$

Without the absolute values, (2.1) becomes an equality so $F \in L^1(\mathbb{R}^n)$ satisfies (1.4). A weaker condition, which also implies integrability over almost every hyperplane, is

$$\int_{\mathbb{R}^n} |F(x)| \cdot (1+|x|)^{-1}\ dx < \infty. \tag{2.2}$$

By Hölder's inequality, (2.2) is satisfied if $F \in L^p(\mathbb{R}^n)$ for some p with $1 \leq p < \frac{n}{n-1}$.

Example 2.3. Let $F(x) = (|x|^{n-1} \ln |x|)^{-1}$ for $|x| \geq 2$ and 0 otherwise. This function is not integrable over hyperplanes but $F \in L^p(\mathbb{R}^n)$ for all $p \geq \frac{n}{n-1}$. Thus, condition (2.2) is best possible for L^p spaces.

For p such that $\frac{n}{n-1} \leq p \leq 2$ we have recourse to the Fourier transform. By the Hausdorff-Young theorem, $|\alpha|^{(\frac{n-1}{q})}\mathcal{F}F(\alpha\theta) \in L^q(\mathbb{R}^1)$ for each q where $q = \frac{p}{p-1}$. But $|\alpha|^{(\frac{n-1}{q})}c_0(\theta)\delta(\alpha) = 0$ where $\delta(\alpha)$ denotes the Dirac mass. As such, the Radon transform is not uniquely defined.

Theorem 2.4. *If $F(x) \in L^p(\mathbb{R}^n)$ for some p where $\frac{n}{n-1} \leq p \leq 2$, then the integral in (1.6) converges and $\mathcal{R}F(\theta,t)$ is defined (almost everywhere) on $S^{n-1} \times \mathbb{R}^1$ up to an annihilating polynomial $\sum c_j(\theta)t^j$ of degree $< (n-1)(1-\frac{1}{p})$.*

Proof. In the Fourier domain, (1.6) is equivalent to the convergence of

$$\int_{-\infty}^{+\infty} \int_{S^{n-1}} \mathcal{F}F(\alpha\theta)|\alpha|^{n-1}\mathcal{F}f(\alpha\theta)\ d\theta\ d\alpha.$$

Write the integrand as the product of $|\alpha|^{(n-1)/q}\mathcal{F}F(\alpha\theta)$ times $|\alpha|^{(n-1)/p}\mathcal{F}f(\alpha\theta)$. The first function is in $L^q(S^{n-1} \times \mathbb{R}^1)$, while the second is in $L^p(S^{n-1} \times \mathbb{R}^1)$ since $\mathcal{F}f(\alpha\theta)$ is a Schwartz class function. Convergence of the integral follows from Hölder's inequality.

To determine the extent of non-uniqueness of $\mathcal{R}F$, suppose that the support of the Schwartz class function $\mathcal{F}f$ is disjoint from $\alpha = 0$. Then dividing $\mathcal{F}f$ by any power of $|\alpha|$ yields another Schwartz class function. It follows that any distribution that annihilates every $|\alpha|^{(\frac{n-1}{q})}\mathcal{F}f(\alpha\theta)$ must have point support at

$\alpha = 0$. These are distributions of the form $\sum c_j(\theta)\delta^{(j)}(\alpha)$ for $k < \frac{n-1}{q}$ where $\delta^{(j)}$ denotes the j-th derivative of the Dirac mass. Replacing $\frac{1}{q}$ by $1 - \frac{1}{p}$ and computing the one dimensional inverse Fourier transform yields a polynomial $\sum c_j(\theta)t^j$ of degree $< (n-1)(1-\frac{1}{p})$. □

Remark 2.5. The isometry between $F \in L^2(\mathbb{R}^n)$ and $D^{(\frac{n-1}{2})}\mathcal{R}F \in L^2(S^{n-1} \times \mathbb{R}^1)$ is a well known property of the Radon transform [5, p. 29]. In particular, if $F \in L^2(\mathbb{R}^3)$ then $\frac{\partial}{\partial t}\mathcal{R}F$ is uniquely defined. Antidifferentiation gives $\mathcal{R}F$ up to an additive term $c_0(\theta)$. This applies also to the function in Example 2.3 since $(n-1)(1-\frac{1}{p}) = \frac{n-1}{n}$.

While $F(x) \in L^p(\mathbb{R}^n)$ for $p > 2$ does not imply $\mathcal{F}F(\alpha\theta) \in L^q(\mathbb{R}^n)$, the proof of the Theorem 2.4 remains valid for Fourier transforms in the range $1 \leq q \leq 2$. In particular, suppose that $F(x) \in \mathcal{A}(\mathbb{R}^n)$, the algebra of continuous functions with absolutely summable Fourier transforms. Then the integral (1.6) converges and $\mathcal{R}F$ is defined (almost everywhere) up to an annihilating polynomial of degree $n-2$.

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