Real Analysis Exchange
Vol. 30(1), 2004/2005, pp. 329–338

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A RIEMANN-TYPE INTEGRAL ON A MEASURE SPACE

Abstract

In a compact Hausdorff measure space we define an integral by partitions of the unity and prove that it is nonabsolutely convergent.

1 Introduction.

In a measure space, usually, a Lebesgue-type integral is defined. In [1], Ahmed and Pfeffer defined a Riemann-type integral on a locally compact Hausdorff space, using partitions of sets and proved that it is equivalent to the Lebesgue integral if the space has suitable properties and the measure is complete.

In [7], a Riemann-type integral has been defined in a compact Hausdorff space, using partitions of the unity (PU-integral) and has been proved that a PU-integrable function is $\mu-$integrable and conversely, and that the $\mu-$integral is equivalent to the PU-integral. Now, in this note, we modify the partitions of the unity and we obtain a nonabsolutely convergent integral (PU*-integral). We give also an example of function which is PU$^*-$ integrable but it is not $\mu-$integrable.

2 Preliminaries.

In this paper X denotes a compact Hausdorff space, $\mathcal{M}$ a σ-algebra of subsets of X such that each open set is in $\mathcal{M}$, μ a non-atomic, finite, complete Radon measure on $\mathcal{M}$.

Key Words: Lebesgue measure, partition of the unity, PU*-integral
Mathematical Reviews subject classification: 28A25
Received by the editors December 16, 2003
Communicated by: Peter Bullen
*This work was supported by M.U.R.S.T.

Definition 1. A partition of the unity (PU-partition) of X is, by definition, a finite collection $P = \{(\theta_i, x_i)\}_{i=1}^p$ where $x_i \in X$ and θ_i are non negative, μ-measurable and μ-integrable real functions on X such that $\sum_{i=1}^p \theta_i(x) = 1$ a.e. in X.

The PU-partition is a PU*-partition if $x_i \in S_{\theta_i} = \{x \in X : \theta_i(x) \neq 0\}$.

We observe that for any PU-partition $P = \{(\theta_i, x_i)\}_{i=1}^p$ we can have a PU*-partition $\bar{P} = \{(\bar{\theta}_i, x_i)\}_{i=1}^p$ where for every $x \in X$ we set $\bar{\theta}_i(x) = \theta_i(x)$ if $x_i \in S_{\theta_i}$, and if $x_i \notin S_{\theta_i}$ we set $\bar{\theta}_i(x) = \theta_i(x)$ for $x \neq x_i$ and $\bar{\theta}_i(x_i) = 1$.

Definition 2. gage δ on X is a map which to each $x \in X$ assigns an open neighborhood of x; set $\delta(x) = U(x)$ and denote by $\mathcal{U}(X)$ the family of all gages on X.

Definition 3. If δ is a gage on X, a PU-partition $P = \{(\theta_i, x_i)\}_{i=1}^p$ is said to be δ-fine if $S_{\theta_i} \subset \delta(x_i) \quad (i = 1, 2, .., p)$.

Definition 4. A real function f on X is said to be (PU)-integrable on X if there exists a real number I with the property that, for every given $\epsilon > 0$, there is a gage δ such that $\mid \sum_{i=1}^p f(x_i) \cdot \int_X \theta_i d\mu - I \mid < \epsilon$ for each δ-fine (PU)-partition $P = \{(\theta_i, x_i)\}_{i=1}^p$ of X.

The number I is called the (PU)-integral of f on X and we write $I = (PU) \int_X f$.

For (PU)*-partitions, we have the (PU)*-integral and set $I = (PU)^* \int_X f$.

3 Main Results.

3.1 Properties of the PU^*-Integral.

Proposition 3.1.1. *If δ is a gage on X then there is a δ-fine PU (PU^*)-partition of X.*

PROOF. Given $\delta \in \mathcal{U}(X)$, let $\{U(x_i)\}_{i=1}^n$ be a finite subcover of neighborhoods. Set

$$V_1 = U(x_1), \quad V_i = U(x_i) - \bigcup_{k=1}^{i-1} U(x_k) \quad i = 2, \dots, n$$

and

$$\theta_i(x) = \chi_{V_i}(x),$$

then the family $\{(\theta_i, x_i)\}_{i=1}^n$ verifies the properties of a $\delta-$fine PU-partition of X.

If we consider $\theta_i(x) = \chi_{V_i \cup x_i}(x)$, we have a PU^*-partition. □

Denoting by $\mathcal{PU}^*(A)$ the family of all the PU*-integrable real functions on X, the following Proposition is an immediate consequence of the Definition 4.

Proposition 3.1.2. *1) $\mathcal{PU}^*(X)$ is a linear space and the map $f \to (PU)^* \int_X f$ is a non negative linear functional on $\mathcal{PU}^*(X)$;*

2) if $k \in \Re$ and $f(x) = k$ for each $x \in X$ then $f \in \mathcal{PU}^*(X)$ and $(PU)^* \int_X f = k\mu(X)$.

3) if $f\ ,\ g\ \in \mathcal{PU}^*(\mathcal{X})$ and $f \leq g$ then $(PU)^* \int_X f \leq (PU)^* \int_X g$.

Proposition 3.1.3. *If A is a compact subset of X and if $f \in \mathcal{PU}^*(X)$, then $f \in \mathcal{PU}^*(A)$.*

Proof. See Proposition 1.3 in [5]. □

If $P = \{(\theta_i, x_i)\}_{i=1}^n$ is a partition of X, set $\sigma(f, P) = \sum_{i=1}^n f(x_i) \int_X \theta_i d\mu$.

Proposition 3.1.4. *If f is a real function on X, then $f \in \mathcal{PU}^*(X)$ if and only if for each $\epsilon > 0$ there is a gage δ on X such that $|\sigma(f, P) - \sigma(f, Q)| < \epsilon$ for every $P = \{(\theta_i, x_i)\}_{i=1}^n$ and $Q = \{(\theta_i', x_i')\}_{i=1}^p$ $\delta-$fine PU*-partitions of X.*

Proof. See proposition 1.4 in [7]. □

3.2 Measurability and Properties of PU^*-Integrable Functions.

Proposition 3.2.1. *If f is $\mu-$measurable and $\mu-$integrable on X, then $f \in \mathcal{PU}^*(X)$ and $(PU)^* \int_X f = \int_X f d\mu$.*

Proof. It follows by the equivalence between the PU-integral and the $\mu-$integral (see [7]) and because a PU*-partition is also a PU-partition. □

Proposition 3.2.2. *A PU*-integrable function is $\mu-$measurable.*

Proof. It is analogue to that used in [7] Propositions 3.1, 3.2 and 3.3. □

Proposition 3.2.3. *If f, g are two real functions on X and $f = g$ a.e. in X then g is (PU)*-integrable if and only if f is (PU)*-integrable and the two integral coincide.*

Proof. If f is (PU)*-integrable then by Proposition 2.2 it is $\mu-$measurable and by completeness of measure also g is $\mu-$measurable, then $f - g = 0$ a.e. in X and it is $\mu-$measurable, $\mu-$integrable and (PU)*-integrable with
$(PU)^* \int_X (f - g) = 0$. So $g = f - (f - g)$ is (PU)*-integrable. □

Lemma 1. *If f is a real $\mu-$integrable function on X, A, $B \in \mathcal{M}$, with $A \subset B$, and if $c \in \Re$ and $\int_A f d\mu \le c \le \int_B f d\mu$ then there exists a $\mu-$measurable set C such that $A \subset C \subset B$ and $\int_C f d\mu = c$.*

PROOF. Consider the $\sigma-$algebra $\mathcal{D} = \{D \in \mathcal{M} : D \subset B - A\}$ and the signed measure $\alpha : D \to \int_D f d\mu$ for $D \in \mathcal{D}$.

By Liapounoff theorem (see [9]), the set $\{\alpha(D) : D \in \mathcal{D}\}$ is a compact interval. So

$$\alpha(\emptyset) = 0 < c - \int_A f d\mu < \int_{B-A} f d\mu$$

and exists $D_1 \in \mathcal{D}$ such that

$$\int_{D_1} f d\mu = c - \int_A f d\mu$$

$$c = \int_{A\cup D_1} f d\mu, \quad A \subset A \cup D_1 \subset B. \qquad \square$$

Proposition 3.2.4. *If f is a PU^*-integrable function on X, then for each $\epsilon > 0$ there is a $\mu-$measurable set E such that $\mu(X - E) < \epsilon$, f is $\mu-$integrable on E and $\int_E f d\mu = (PU)^* \int_X f$.*

PROOF. Suppose that f be not $\mu-$integrable; set

$$E_n = \{x \in X : n - 1 \le f(x) < n\},$$
$$F_n = \{x \in X : -n \le f(x) < -n + 1\} \quad n = 1, 2, 3, \dots,$$

then

$$X = \bigcup_{n=1}^{\infty}(E_n \cup F_n) = \bigcup_{n=1}^{\infty}(\bigcup_{i=1}^{n}(E_i \cup F_i)) = \bigcup_{n=1}^{\infty} H_n,$$

where $H_n = \bigcup_{i=1}^n (E_i \cup F_i)$ is an increasing sequence of measurable sets.

By a property of the measure, it results $\lim_{n\to\infty} \mu(H_n) = \mu(X)$ and for each $\epsilon > 0$ there is $\bar{n} \in N$ such that for $n_0 > \bar{n}$ it is

$$\mu(X) - \mu(H_{n_0}) = \mu(X - H_{n_0}) < \epsilon \qquad (*)$$

f is bounded on H_{n_0} so it is $\mu-$integrable on H_{n_0}.

Suppose that $\int_{H_{n_0}} f d\mu < (PU^*) \int_X f$; since f is not $\mu-$integrable, then the series $\sum_n \int_{E_n} f d\mu$ and $\sum_n \int_{F_n} f d\mu$ are divergent to $+\infty$ and to $-\infty$ respectively. In fact, if $\sum_n \int_{E_n} f d\mu = +\infty$ and $\sum_n \int_{F_n} f d\mu > -\infty$, consider the functions

$$f_1(x) = f(x) \quad \text{if} \quad x \in \bigcup_n E_n \quad \text{and} \quad f_1(x) = 0 \quad \text{elsewhere},$$

$$f_2(x) = f(x) \quad \text{if} \quad x \in \bigcup_n F_n \quad \text{and} \quad f_2(x) = 0 \quad \text{elsewhere},$$

then $f_2(x)$ is $\mu-$integrable and hence (PU)*-integrable and $f_1(x) = f(x) - f_2(x)$ is (PU)*-integrable, but it is also $\mu-$integrable with integral $+\infty$ and this is impossible. So for $\epsilon > 0$ there exists $K > n_0$ such that

$$\int_{H_{n_0}} f d\mu + \int_{E_{n_0+1}} f d\mu + \cdots + \int_{E_{n_0+k}} f d\mu > (PU)^* \int_X f$$

and set $H = H_{n_0} \cup E_{n_0+1} \cup \cdots \cup E_{n_0+k}$, it results

$$\int_{H_{n_0}} f d\mu < (PU)^* \int_X f < \int_H f d\mu.$$

By Lemma 1 there exists a $\mu-$measurable set E with $H_{n_0} \subset E \subset H$ such that $\int_E f d\mu = (PU)^* \int_X f$ and by relation (*) we have

$$\mu(X - E) \leq \mu(X - H_{n_0}) < \epsilon. \qquad \square$$

Lemma 2. *If f is $\mu-$measurable and there exists finite $\int_X f d\mu$, given $\epsilon > 0$ there is a gage δ on X such that*

$$\sum_i |(f(x_i) \int_X \theta_i d\mu - \int_X f\theta_i d\mu)| < \epsilon$$

for each $\delta-$fine (PU)*-partition $P = \{(\theta_i, x_i)\}$ in X.

Proof. It is a consequence of Vitali-Caratheodory theorem. See Proposition 3.1 in [5]. $\square$

Proposition 3.2.5. *A $\mu-$measurable function f is (PU)*-integrable on X if and only if given $\epsilon > 0$ there is a gage δ on X and a $\mu-$measurable set E such that $\mu(E^C) < \epsilon$, f is $\mu-$integrable on E and $|\sum_i f\chi_{E^C}(x_i) \int_X \theta_i d\mu| < \epsilon$ for each $\delta-$fine (PU)*-partition $P = \{(\theta_i, x_i)\}$. Moreover $\int_E f d\mu = (PU)^* \int_X f$. We have set $E^C = X - E$.*

Proof. If f is (PU)*-integrable, by previous Proposition, let $\epsilon > 0$ there is $E \in \mathcal{M}$ such that $\mu(E^C) < \epsilon$, f is $\mu-$integrable on E and $\int_E f d\mu = (PU)^* \int_X f$; so $f\chi_E$ is $\mu-$integrable and hence (PU)*-integrable and

$$(PU)^* \int_X f\chi_E = \int_X f\chi_E d\mu = \int_E f d\mu = (PU)^* \int_X f.$$

By the (PU)*-integrability of f and $f\chi_E$, at corrispondence of $\epsilon > 0$ there is a δ on X such that for each $\delta-$fine (PU)*-partition $\{(\theta_i, x_i)\}$, it results

$$|\sum_i f(x_i) \int_X \theta_i d\mu - (PU)^* \int_X f| < \frac{\epsilon}{2}$$

and

$$|\sum_i f(x_i)\chi_E \int_X \theta_i d\mu - (PU)^* \int_X f| < \frac{\epsilon}{2}.$$

So we have

$$|\sum_i f(x_i)\chi_{E^C} \int_X \theta_i d\mu| = |\sum_i f(x_i) \int_X \theta_i d\mu - \sum_i f(x_i)\chi_E \int_X \theta_i d\mu| \leq$$

$$\leq |\sum_i f(x_i) \int_X \theta_i d\mu - (PU)^* \int_X f| + |\sum_i f\chi_E(x_i) \int_X \theta_i d\mu - (PU)^* \int_X f| < \epsilon.$$

Conversely, for $\epsilon > 0$ let E be a $\mu-$measurable and $\mu-$integrable set with $\mu(E^C) < \epsilon$ and let δ be a gage on X such that $|\sum_i f\chi_E^C(x_i) \int_X \theta_i d\mu| < \frac{\epsilon}{2}$ for each $\delta-$fine (PU)*-partition P of X.

By the $\mu-$integrability of f on E , then also the function $f\chi_E$ is $\mu-$integrable and, by lemma 2, there is a gage δ_1 on X such that

$$|\sum_i f\chi_E(x_i) \int_X \theta_i d\mu - \int_X f\chi_E d\mu| < \frac{\epsilon}{2}.$$

If $\bar{\delta}(x) = \delta(x) \bigcap \delta_1(x)$ for each $x \in X$, then for each $\bar{\delta}-$fine (PU)*-partition P consider:

$$|\sum_i f(x_i) \int_X \theta_i d\mu - \int_E f d\mu| \leq |\sum_i f\chi_E(x_i) \int_X \theta_i d\mu - \int_E f d\mu| +$$

$$+ |\sum_i f\chi_E^C(x_i) \int_X \theta_i d\mu| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So f is (PU)*-integrable and $(PU)^* \int_X f = \int_E f d\mu$. □

3.3 Convergence Theorems and Nonabsolutely Convergence of the PU$^*-$Integral.

Proposition 3.3.1. *If f and $|f|$ are (PU)*-integrable then f is $\mu-$integrable.*

Proof. If f and $|f|$ are (PU)*-integrable, consider the bounded sequence $f_n = |f| \wedge n$ for each $n \in N$ it converges increasing to $|f|$ and it is $\mu-$integrable and

$$\int_X |f| d\mu = \lim_n \int_X f_n d\mu = \lim_n (PU)^* \int_X f_n \leq (PU)^* \int_X |f| < +\infty.$$

So $|f|$ and f are $\mu-$integrable. □

Proposition 3.3.2. *If $(f_n)_n$ is an increasing sequence of (PU)*-integrable functions converging to f pointwisely and $\lim_n (PU)^* \int_X f_n < \infty$ then f is (PU)*-integrable and $(PU)^* \int_X f = \lim_n (PU)^* \int_X f_n$.*

Proof. Consider the increasing sequence $(f_n - f_1)_n$ converging to $f - f_1$; since the functions $(f_n - f_1)_n$ are non negative, then by Proposition 3.3.1, they are $\mu-$integrable and

$$\lim_n \int_X (f_n - f_1) d\mu = \lim_n (PU)^* \int_X (f_n - f_1) =$$

$$= \lim_n (PU)^* \int_X f_n - (PU)^* \int_X f_1 < +\infty.$$

So by the monotone theorem for the $\mu-$integrable functions, the function $(f - f_1)$ is $\mu-$integrable and hence (PU)*-integrable. Therefore $f = (f - f_1) + f_1$ is (PU)*-integrable. □

Proposition 3.3.3. *If $(f_n)_n$ is a sequence of (PU)* integrable functions converging pointwisely to f and such that there are two functions h and g (PU)*-integrable with $h \leq f_n \leq g$ for each $n \in N$ then f is (PU)*-integrable and $(PU)^* \int_X f = \lim_n (PU)^* \int_X f_n$.*

Proof. Consider the sequence $(f_n - h)_n$; it is non negative and (PU)*-integrable, so it is $\mu-$integrable and results:

$$0 \leq (f_n - h) \leq (g - h).$$

Since the function $g - h$ is non negative and (PU)*-integrable, it is $\mu-$integrable and by the dominate convergent theorem, the sequence of functions $(f_n - h)$ converges to $f - h$ which is a $\mu-$integrable function and hence (PU)*-integrable. By the equality $f = (f - h) + h$ it follows the (PU)*-integrability of f. □

Definition 5. We say that a real function f has finite $\int_X f d\mu$ but $\int_X |f| d\mu$ is infinite if

i) or exists a sequence $A_n \in \mathcal{M}$ with $A_n \subset A_{n+1}$, $\bigcup A_n = X$, f is $\mu-$integrable on A_n for each n and exists finite $\lim_n \int_{A_n} f d\mu$ while $\int_X |f| d\mu = +\infty$. Then we set

$$\int_X f d\mu = \lim_n \int_{A_n} f d\mu;$$

ii) or if $f = \sum_{n=1}^{+\infty} a_n \chi_{A_n}$, $A_n \in \mathcal{M}$, $\bigcup A_n = X$, $A_i \bigcap A_j = \emptyset$ and $\sum_{n=1}^{+\infty} a_n \mu(A_n)$ is finite while $\sum_{n=1}^{+\infty} |a_n| \mu(A_n) = +\infty$, then we set

$$\sum_{n=1}^{+\infty} a_n \mu(A_n) = \int_X f d\mu.$$

Proposition 3.3.4. *If f is $\mu-$measurable and exists finite $\int_X f d\mu$ but $\int_X |f| d\mu = +\infty$ then f is (PU)*-integrable and $\int_X f d\mu = (PU)^* \int_X f$.*

PROOF. If $\epsilon > 0$, by lemma 2, there is a gage δ on X such that if $P = \{(\theta_i, x_i)\}$ is a (PU)*-partition of X, then we have:

$$\epsilon > |\sum_i (f(x_i) \int_X \theta_i d\mu - \int_X f \theta_i d\mu)| = |\sum_i f(x_i) \int_X \theta_i d\mu - \sum_i f \theta_i d\mu| =$$

$$= |\sum_i (f(x_i) \int_X \theta_i d\mu - \int_X f d\mu)|.$$

□

An example of a function which is PU*-integrable but it is not $\mu-$integrable.

Consider the space $X = \{0,1\}^{\mathbb{N}}$. Let $\bar{\alpha} = \alpha_1 \alpha_2 \ldots \alpha_k$ be a finite string of 0 and 1; consider the set $A_{\bar{\alpha}} = [\bar{\alpha}]_k = \{\gamma \in X : \gamma = \bar{\alpha}\beta$, for some $\beta \in X\}$, it is a clopen set (i.e. an open and closed set) with respect to the topology induced by the metric ρ so defined:

if $\alpha, \beta \in X$ $\rho(\alpha, \beta) = \frac{1}{2^n}$ if $\alpha \neq \beta$ and $\alpha_1 = \beta_1, \ldots \alpha_n = \beta_n, \alpha_{n+1} \neq \beta_{n+1}$
$\rho(\alpha, \alpha) = 0$.

With respect to this metric ρ, $X = \{0,1\}^{\mathbb{N}}$ is a complete, separable and compact metric space (see [3]). Define on the family $\{A_{\bar{\alpha}}\}$ the following set function m:

$$m(A_{\bar{\alpha}}) = \frac{1}{2^k}$$

and let m^* be the outer measure induced by m on the family of all the subsets of X. If $\mathcal{M}$ is the $\sigma-$algebra of all the subsets of X m^*-measurable in the

Caratheodory sense, then the open sets are in $\mathcal{M}$ and m^* is a complete measure on $\mathcal{M}$.

Define on X the following real function

$$f(\alpha) = \begin{cases} a_1 & \text{if } \alpha_1 = 0 \\ a_2 & \text{if } \alpha_1 = 1 \text{ and } \alpha_2 = 0 \\ a_n & \text{if } \alpha_1, \alpha_2, \dots \alpha_{n-1} = 1, \alpha_n = 0 \\ \dots \end{cases}$$

$$f(1111\dots111\dots) = 0$$

where $\alpha = (\alpha_1, \alpha_2, \dots) \in \{0,1\}^{\mathbb{N}}$ and $a_n = (-1)^n \frac{2^n}{n}$ for every $n \in \mathbb{N}$. Then, by Proposition 3.3.4, we have:

$$\int_X f dm = \sum_{n=1}^{\infty} a_n \frac{1}{2^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} = (PU)^* \int_X f,$$

so f is PU*-integrable but $|f|$ is not μ-integrable.

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