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Inga Jędrzejewska and Bożenna Szkopińska, Institute of Mathematics, Technical University of Łódź, 90-924 Łódź, al. Politechniki 11, Poland.
email: bszkopin@op.pl

# ON GENERALIZATIONS OF FLETT'S THEOREM

**Abstract**

In 1958 T. M. Flett proved a theorem which is a variant of the Lagrange mean value theorem; namely, let $f : [a, b] \to \mathbb{R}$ be a differentiable function in $[a, b]$ and $f'(a) = f'(b)$. Then there exists a number $\eta \in (a, b)$ such that

$$f(\eta) - f(a) = (\eta - a) \cdot f'(\eta).$$

Manav Das, Thomas Riedel and Prasanna K. Sahoo have given generalizations of Flett's theorem for approximately differentiable functions. Here we provide generalizations of these theorems for some local $\mathcal{S}$-systems.

**Definition 1.** [6] By a local system $\mathcal{S}$ we mean a family $\mathcal{S}$ such that at each point $x \in \mathbb{R}$ there is given a nonempty collection of sets $S(x)$ with the following properties:

($i$) $\{x\} \notin S(x)$,

($ii$) if $A \in S(x)$, then $x \in A$,

($iii$) if $A_1 \in S(x)$ and $A_2 \supset A_1$, then $A_2 \in S(x)$,

($iv$) if $A \in S(x)$ and $\delta > 0$, then $A \cap (x - \delta, x + \delta) \in S(x)$.

Now, we will give a few examples of local systems.

**Example 1.** Let $\mathcal{S}_0$ denote the system defined at each point $x$ as

$$S_0(x) = \{A : \exists_{r>0}(x - r, x + r) \subset A\}$$

so that each $S_0(x)$ is precisely the neighborhood filter at the point $x$.

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Let $A^d$ denote the set of all accumulation points of the set $A$.

**Example 2.** A system closely associated to the above system is defined in the following way. $\mathcal{S}_\infty$ is defined at each point $x$ as

$$S_\infty(x) = \{A : x \in A^d\}.$$

**Example 3.** The approximate system $\mathcal{S}_{ap}$ is defined as

$$A \in S_{ap}(x) \text{ if and only if } x \in A \text{ and } \mathrm{d}(A, x) = 1$$

where for the density of a set $A$ at a point $x$ we write

$$\mathrm{d}(A, x) = \lim_{h \to 0} \frac{|A \cap (x, x+h)|}{|h|}$$

where by interval $(x, x+h)$ we mean the interval $(x+h, x)$ whenever $h < 0$.

The system $\mathcal{S}_{I\text{-}ap}$ is defined in a similar manner as the system $\mathcal{S}_{ap}$, considering another type of density points.

**Example 4.** Let $\mathcal{S}_{I\text{-}ap}$ be the system defined for every point $x$ as

$$S_{I\text{-}ap}(x) = \{A : x \in A \wedge d_I(A, x) = 1\},$$

where $\mathrm{d}_I(A, x) = 1$ means that $x$ is an $I$-density point of set $A$ (see [4]).

**Example 5.** Let $\mathcal{N}$ be an ideal or $\sigma$-ideal of sets of real numbers. The system $\mathcal{S}_\mathcal{N}$ is defined as

$$S_\mathcal{N}(x) = \{A : x \in A \wedge \exists_{\delta > 0}(x - \delta, x + \delta) \setminus A \in \mathcal{N}\}.$$

**Definition 2.** [6] Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two local systems. We will write $\mathcal{S}_1 \ll \mathcal{S}_2$ if at every point $x$ $S_1(x) \subset S_2(x)$.

It is obvious that this relation is a partial order in the family of local systems. One can prove the following.

**Lemma 3.** [6] *For any local system $\mathcal{S}$ we have*

$$\mathcal{S}_0 \ll \mathcal{S} \ll \mathcal{S}_\infty.$$

**Definition 4.** [6] We will say that $\mathcal{S}$ is filtering at a point $x$ if $A_1 \cap A_2 \in S(x)$ whenever $A_1$ and $A_2$ belong to $S(x)$.

It is clear that if a local system $\mathcal{S}$ is filtering at $x$, then the family of sets $S(x)$ is a filter converging to $x$. Conversely, if at each point $x$ there is given a filter $S(x)$ converging to $x$ and nontrivial at $x$ in the sense that $\{x\}$ does not belong to $S(x)$, then $\mathcal{S}$ is a local system.

**Definition 5.** [6] A system $\mathcal{S}$ is bilateral provided every set $A$ in $S(x)$ contains points on either side of $x$.

**Definition 6.** [6] Let $\mathcal{S}$ be a local system, $f$ be a real function and $x$ any point in $\mathbb{R}$. Then $(\mathcal{S})$-limit of $f$ at $x$ is defined as any extended real number $c$ for which the following condition holds: for every neighborhood $U_c$ of $c$ the set

$$\{t : t = x \vee f(t) \in U_c\}$$

belongs to $S(x)$. This limit is denoted by $(\mathcal{S}) \lim\limits_{y \to x} f(y)$.

The extreme limits relative to a system $\mathcal{S}$ at a point $x$ are defined as

$$(\mathcal{S}) \limsup_{y \to x} f(y) = \inf\{y : \{t : t = x \vee f(t) < y\} \in S(x)\}$$

and

$$(\mathcal{S}) \liminf_{y \to x} f(y) = \sup\{y : \{t : t = x \vee f(t) > y\} \in S(x)\}.$$

**Lemma 7.** *Let $\mathcal{S}$ be a system such that for every $x \in \mathbb{R}$ we have $A_1 \cap A_2 \neq \{x\}$ whenever $A_1 \in S(x)$ and $A_2 \in S(x)$. Then for any function $f$ and any $x \in \mathbb{R}$*

$$(\mathcal{S}) \lim_{y \to x} \inf f(y) \leq (\mathcal{S}) \lim_{y \to x} \sup f(y)$$

*and $(\mathcal{S}) \lim_{y \to x} f(y)$ is unique.*

**Remark 1.** If the system $\mathcal{S}$ is filtering, then $(\mathcal{S}) \lim_{y \to x} f(y)$ exists if and only if

$$(\mathcal{S}) \limsup_{y \to x} f(y) = (\mathcal{S}) \liminf_{y \to x} f(y)$$

and its common value equals to the limit $(\mathcal{S}) \lim_{y \to x} f(y)$.

Now we will prove the following property.

**Lemma 8.** *Let $\mathcal{S}$ be a system filtering at $x$ for $x \in \mathbb{R}$. Let us assume that there exists a finite $(\mathcal{S})$-limits of functions $f$ and $g$. Then*

$$(\mathcal{S}) \lim_{x \to x_0} (f(x) + g(x)) = (\mathcal{S}) \lim_{x \to x_0} f(x) + (\mathcal{S}) \lim_{x \to x_0} g(x); \tag{1}$$

*if moreover one of the functions $f$ or $g$ is bounded, then*

$$(\mathcal{S}) \lim_{x \to x_0} (f(x) \cdot g(x)) = (\mathcal{S}) \lim_{x \to x_0} f(x) \cdot (\mathcal{S}) \lim_{x \to x_0} g(x). \tag{2}$$

PROOF. For a real number $x$ and a positive number $r$ let $U(x, r)$ denote the interval $(x-r, x+r)$. Let $(\mathcal{S})\lim_{x\to x_0} f(x) = a$ and $(\mathcal{S})\lim_{x\to x_0} g(x) = b$. Let $r$ be any real number. Let

$$A_1 = \{t : t = x_0 \vee f(t) \in U(a, \frac{r}{2})\}.$$

From the definition of $(\mathcal{S})$-limit of function $f$ at $x_0$ we have that $A_1 \in S(x_0)$. Similarly, for the function $g$ we have that

$$A_2 = \{t : t = x_0 \vee g(t) \in U\Big(b, \frac{r}{2}\Big)\} \in S(x_0).$$

Since $\mathcal{S}$ is filtering at the point $x_0$, then $A_1 \cap A_2 \in S(x_0)$. We put

$$A_3 = \{t : t = x_0 \vee f(t) + g(t) \in U(a + b, r)\}.$$

It follows from the inequality

$$|f(t) + g(t) - a - b| \leq |f(t) - a| + |g(t) - b|$$

that $A_1 \cap A_2 \subset A_3$. From Definition 1 (iii) of the local system $\mathcal{S}$ we have, that $A_3 \in S(x_0)$, which completes the proof of (1).

We assume now, that $f$ is bounded by a real number $M > 0$. Let $r$ be any real number. We see, that the sets

$$A_1 = \{t : t = x_0 \vee f(t) \in U\Big(a, \frac{r}{2 \cdot |b|}\Big)\} \in S(x_0)$$

and

$$A_2 = \{t : t = x_0 \vee g(t) \in U\Big(b, \frac{r}{2M}\Big)\} \in S(x_0).$$

Hence $A_1 \cap A_2 \in S(x_0)$, since $\mathcal{S}$ is filtering at $x_0$. From inequality

$$|f(t) \cdot g(t) - ab| \leq |f(t)||g(t) - b| + |b||f(t) - a| \leq M\frac{r}{2M} + |b|\frac{r}{2|b|} = r$$

we have

$$A_1 \cap A_2 \subset A_3 = \{t : t = x_0 \vee f(t) \cdot g(t) \in U(a \cdot b, r)\}.$$

Thus $A_3 \in S(x_0)$ and we have (2). □

It can easily be seen, that if one of the functions $f$ or $g$ has a finite limit, then Lemma 8 is true for any local system $\mathcal{S}$ and without the additional assumption that $f$ or $g$ is bounded.

**Definition 9.** [6] Let $\mathcal{S}$ be a local system on $X$ and let $f$ be an arbitrary function. We say that $f$ is $(\mathcal{S})$-continuous at $x$ provided

$$\forall_{\varepsilon>0}\,(\{t : |f(t) - f(x)| < \varepsilon\} \in S(x)).$$

**Lemma 10.** [6] *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be local systems such that $\mathcal{S}_1 \ll \mathcal{S}_2$. If function $f$ is $(\mathcal{S}_1)$-continuous at point $x$ it is also $(\mathcal{S}_2)$-continuous there.*

**Definition 11.** Let $x \in \mathbb{R}$ and let $\mathcal{S}$ be a local system defined in a neighborhood $U(x)$ and $f$ be a finite real function defined in $U(x)$. We put

$$C(y,x) = \frac{f(y) - f(x)}{y - x} \text{ if } y \neq x \text{ and } y \in U(x).$$

If $(\mathcal{S})\lim_{y\to x} C(y,x)$ exists, then its value is called the $(\mathcal{S})$-derivative of function $f$ at point $x$.

The number

$$(\mathcal{S})\overline{D}f(x) = (\mathcal{S})\limsup_{y\to x} C(y,x)$$

is called the upper $(\mathcal{S})$-derivative of function $f$ at point $x$.

Similarly we define the lower $(\mathcal{S})$-derivative of function $f$ at $x$ as the lower $(\mathcal{S})$-limit of $C(y,x)$ at $x$ and denote it by $(\mathcal{S})\underline{D}f(x)$.

Immediately, from Remark 1 it follows that if the system $\mathcal{S}$ is filtering, then the $(\mathcal{S})$-derivative of function $f$ at $x$ exists if and only if both the upper and lower $\mathcal{S}$-derivatives exist at $x$ and are equal.

Note that the corollary below follows from Lemma 8.

**Corollary 12.** *Let $\mathcal{S}$ be any filtering system. If $f$ has a finite $(\mathcal{S})$-derivative at $x$, then it is $(\mathcal{S})$-continuous at that point.*

R. O'Malley in [3] introduced the concept of selection.

By selection $p$ we mean a function $p(x,y)$ of two variables such that

(i) $p(x,y) = p(y,x)$;

(ii) if $x < y$, then $x < p(x,y) < y$.

**Example 6.** For a selection $p$, define the selective system $\mathcal{S}_p$ at $x$ by

$$S_p(x) = \{A : x \in A \wedge \exists_{\delta>0}(A \supset \{p(x,y) : 0 < |x-y| < \delta\})\}.$$

This system we shall denote by $\mathcal{S}_p$.

**Definition 13.** A system $\mathcal{S}$ has intersection condition means for every choice of sets $\{S_x : x \in \mathbb{R} \wedge S_x \in S(x)\}$ there is a positive function $\delta : X \to \mathbb{R}$ such that

$$S_x \cap S_y \cap (x, y) \neq \emptyset \tag{3}$$

whenever $0 < y - x < \min\{\delta(x), \delta(y)\}$.

Let us assume that the system $\mathcal{S}$ is bilateral and fulfills the intersection condition (3). Then there exists a selection $p$ for which $\mathcal{S} \ll \mathcal{S}_p$.

**Corollary 14.** *Let $\mathcal{S}$ be bilateral and satisfy the intersection condition* (3) *and let $p$ be a selection for which $\mathcal{S} \ll \mathcal{S}_p$. If $c$ is the derivative of $f$ with respect to the system $\mathcal{S}$ (i.e., it is the $(\mathcal{S})$-derivative), then it is the derivative with respect to the system $\mathcal{S}_p$ as well.*

From properties of $(\mathcal{S})$-continuous functions (see [6]) the next theorem follows.

**Theorem 15.** *Let $\mathcal{S}$ be a bilateral system, fulfilling the intersection condition. If a function $f$ is $(\mathcal{S})$-continuous, then $f$ has the Darboux property.*

In view of properties of selective derivatives (see [3]) and Corollary 12 we have the following theorems.

**Theorem 16.** *Let a bilateral and filtering system $\mathcal{S}$ have the intersection property. If $f\colon [0,1] \to \mathbb{R}$ has a finite $(\mathcal{S})$-derivative for all $x$ in $[0,1]$, then the $(\mathcal{S})$-derivative of $f$ has the Darboux property.*

**Theorem 17.** *Let $\mathcal{S}$ be a filtering system that fulfills the intersection condition. If $f$ is $(\mathcal{S})$-differentiable at every point $x \in [a,b]$ and $(\mathcal{S})f'(x) \geq 0$ for every $x \in [a,b]$, then $f$ is non-decreasing in $[a,b]$.*

**Definition 18.** ([2], [6]) We say that a system $\mathcal{S}$ satisfies condition $(J_3)$ at a point $x$ if every set $E$ such that $x \in E$ and $(x-\delta, x+\delta) \cap E$ contains a nonempty open interval for each positive number $\delta$ belongs to $\mathcal{S}(x)$.

We say that a system $\mathcal{S}$ satisfies condition $(J_3)$ in a subset $X$ of the set of real numbers if it satisfies that condition at every point $x$ from the set $X$.

**Theorem 19.** *Let $\mathcal{S}$ be a bilateral filtering system which fulfills condition $(J_3)$ and $f : [a,b] \to \mathbb{R}$ be a non-decreasing function. If $x_0 \in [a,b]$ and $f$ is $(\mathcal{S})$-differentiable at $x_0$, then it is also differentiable at $x_0$.*

Proof. Note that if $f$ is differentiable at $x_0$, then it is also $(\mathcal{S})$-differentiable at $x_0$ and $(\mathcal{S})f'(x_0) = f'(x_0)$.

Now suppose, that $f$ doesn't have a derivative at $x_0$ but it has an $(\mathcal{S})$-derivative at $x_0$. Assume that $(\mathcal{S})f'(x_0) = \alpha$. Then one of the extreme derivates of $f$ at $x_0$ is different from $\alpha$; i.e.,

$$\underline{D}f(x_0) < (\mathcal{S})f'(x_0) \text{ or } \overline{D}f(x_0) > (\mathcal{S})f'(x_0).$$

Assume that

$$\underline{D}_+ f(x_0) < (\mathcal{S})f'(x_0) = \alpha.$$

Since $f$ is a nondecreasing function, $\underline{D}_+ f(x_0) \geq 0$ and $\alpha > 0$, of course.

Choose $\varepsilon_0$ such that $0 < \varepsilon_0 < \frac{\alpha}{2}$ and $\underline{D}_+ f(x_0) < \alpha - 2\varepsilon_0$. So there exists a sequence $(h_n)$ such that $h_n \searrow 0$ and

$$\frac{f(x_0 + h_n) - f(x_0)}{h_n} < \alpha - 2\varepsilon_0$$

for every $n \in \mathbb{N}$. Note that for every $n \in N$ and for any

$$x \in [x_0 + (1 - \frac{\varepsilon_0}{\alpha - \varepsilon_0}) \cdot h_n, x_0 + h_n]$$

we have $\frac{f(x) - f(x_0)}{x - x_0} < \alpha - \varepsilon_0$.

Let

$$E = \{x : \frac{f(x) - f(x_0)}{x - x_0} < \alpha - \varepsilon_0\}$$

In view of condition $(J_3)$ the set $E$ belongs to $S(x_0)$ since

$$E \cap (x_0 - \delta, x_0 + \delta) \supset [x_0 + (1 - \frac{\varepsilon_0}{\alpha - \varepsilon_0}) \cdot h_n, x_0 + h_n]$$

for some $n$. Since

$$A = \{x \in [a, b] : \frac{f(x) - f(x_0)}{x - x_0} \geq \alpha - \varepsilon_0\} \in S(x_0),$$

$A \cap E \neq \emptyset$, a contradiction.

The proofs in other cases are similar. □

One can obtain the same result if condition $(J_3)$ is replaced by the Khintchine condition.

**Definition 20.** [5] We say that a system $\mathcal{S}$ fulfills the Khintchine condition if for each sequences $(x_n)$ and $(\delta_n)$ such that $\delta_n \searrow 0$, $x_n \to x_0$ and $\liminf_{n \to \infty} \frac{\delta_n}{|x_n - x_0|} > 0$, the set $\bigcup_{n=1}^{\infty} (x_n - \delta_n, x_n + \delta_n)$ belongs to $S(x_0)$.

**Theorem 21.** [5] *Let $\mathcal{S}$ be a bilateral filtering system which fulfills the Khintchine condition and let $f : [a,b] \to \mathbb{R}$ be a non-decreasing function. If $x_0 \in [a,b]$ and $f$ is $(\mathcal{S})$-differentiable at $x_0$, then it is also differentiable at $x_0$.*

**Corollary 22.** *Let $\mathcal{S}$ be a bilateral and filtering system which fulfills the intersection condition and condition $(J_3)$. If a function $f : [a,b] \to \mathbb{R}$ is $(\mathcal{S})$-differentiable at every point $x \in [a,b]$ and $g : [a,b] \to \mathbb{R}$ is differentiable at every point $x \in [a,b]$ and $(\mathcal{S})f'(x) \leq g'(x)$ for every $x \in [a,b]$ or if $(\mathcal{S})f'(x) \geq g'(x)$ for every $x \in [a,b]$, then $f$ is differentiable at every point $x \in [a,b]$.*

Proof. Assume that $(\mathcal{S})f'(x) \leq g'(x)$ for $x \in [a,b]$. Hence by Lemma 8, $(\mathcal{S})(g-f)' \geq 0$ for every $x \in [a,b]$. Therefore, from Theorem 17 it follows that function $g-f$ is non-decreasing in $[a,b]$. So from Theorem 19 we infer that $g-f$ is differentiable at every point $x \in [a,b]$. Hence, $f = g-(g-f)$ is also differentiable at every point $x \in [a,b]$. □

The same proof can be used for the next corollary.

**Corollary 23.** *Let $\mathcal{S}$ be a bilateral filtering system that fulfills the Khintchine condition. If a function $f : [a,b] \to \mathbb{R}$ is $(\mathcal{S})$-differentiable at every point $x \in [a,b]$ and $g : [a,b] \to \mathbb{R}$ is differentiable at every point $x \in [a,b]$ and $(\mathcal{S})f'(x) \leq g'(x)$ for every $x \in [a,b]$ or if $(\mathcal{S})f'(x) \geq g'(x)$ for every $x \in [a,b]$, then $f$ is differentiable at every point $x \in [a,b]$.*

**Theorem 24.** [Mean Value Theorem] *Let $\mathcal{S}$ be a bilateral and filtering system which fulfills the intersection condition and condition $(J_3)$. If a function $f : [a,b] \to \mathbb{R}$ is $(\mathcal{S})$-differentiable at every point $x \in [a,b]$, then there exists $c \in (a,b)$ such that*

$$(\mathcal{S})f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Proof. Let $L = \frac{f(b)-f(a)}{b-a}$. If $(\mathcal{S})f'$ is bounded from above or from below, then it follows from Corollary 22 that $f$ is differentiable in $[a,b]$ and we can use Lagrange's theorem on mean values for the ordinary derivative. Otherwise there are numbers $u$, $v$ in $(a,b)$ such that $(\mathcal{S})f'(u) < L$ and $(\mathcal{S})f'(v) > L$. Since $(\mathcal{S})f'$ has Darboux property by Theorem 16, there exists a number $c$ between $u$ and $v$ such that $(\mathcal{S})f'(c) = L$, which completes the proof. □

The same proof can be shown for the next theorem.

**Theorem 25.** [Mean Value Theorem] *Let $\mathcal{S}$ be a bilateral and filtering system which fulfills the Khintchine condition. If a function $f : [a,b] \to \mathbb{R}$ is $(\mathcal{S})$-differentiable at every point $x \in [a,b]$ then there exists $c \in (a,b)$ such that*

$$(\mathcal{S})f'(c) = \frac{f(b)-f(a)}{b-a}.$$

## 1 Main Results.

From now on we consider bilateral and filtering systems which fulfil the intersection condition and one of the following conditions: $(J_3)$ or the Khintchine condition.

**Corollary 26.** *Let $f : [a, b] \to \mathbb{R}$ be $(\mathcal{S})$-differentiable in $[a, b]$. If $f(b) > f(a)$ and $(\mathcal{S})f'(b) < 0$, then there exists a number $\eta \in (a, b)$ such that $(\mathcal{S})f'(\eta) = 0$.*

Proof. Since $(\mathcal{S})f'(b) < 0$, it follows from the definition of $(\mathcal{S})$-derivative that there exists $x_0 < b$, for which $f(x_0) - f(b) > 0$. Hence $f(a) < f(b) < f(x_0)$. It follows from theorem 15 that there exists a number $c \in (a, x_0)$ for which $f(c) = f(b)$. If we apply theorem 24 to interval $[c, b]$, we obtain the conclusion of the theorem. □

**Corollary 27.** *Let $f : [a, b] \to \mathbb{R}$ be $(\mathcal{S})$-continuous in $[a, b]$ and $(\mathcal{S})$-differentiable in $(a, b]$. If*

$$[f(b) - f(a)] \cdot (\mathcal{S})f'(b) \leq 0.$$

*then there exists a number $\eta \in (a, b]$ such that $(\mathcal{S})f'(\eta) = 0$.*

Proof. Consider the following possibilities:

(1) Let $(\mathcal{S})f'(b) = 0$. Then it is sufficient to put $\eta = b$.

(2) Let us assume that $f(a) = f(b)$.

If $f$ is $(\mathcal{S})$-differentiable at $a$, then the conclusion of the corollary follows immediately from theorem 24.

If $f$ is not $(\mathcal{S})$-differentiable at $a$, then we can find a point $x_0 \in (a, b]$ and we can apply theorem 24 or corollary 26 to the interval $[x_0, b]$ and $f$ or $-f$, obtaining our desired conclusion.

(3) Assume, that $[f(b) - f(a)] \cdot (\mathcal{S})f'(b) < 0$. Then either

(i) $(\mathcal{S})f'(b) < 0$ and $f(b) > f(a)$,
or

(ii) $(\mathcal{S})$-$f'(b) > 0$ and $f(b) < f(a)$.

In the case (i) we apply theorem 15 and we can find a point $x_0 \in (a, b)$ such that $f(x_0) = f(b)$. Now applying theorem 24 we get the conclusion.

In the case (ii) we proceed similarly considering the function $-f$. □

And now we will prove a generalization of Flett's theorem for $(\mathcal{S})$-differentiable functions.

**Theorem 28.** *Let $f : [a,b] \to \mathbb{R}$ be $(\mathcal{S})$-differentiable in $[a,b]$. If*

$$[(\mathcal{S})f'(b) - \frac{f(b)-f(a)}{b-a}] \cdot [(\mathcal{S})f'(a) - \frac{f(b)-f(a)}{b-a}] \geq 0,$$

*then there exists $\eta \in (a,b]$ such that*

$$f(\eta) - f(a) = (\eta - a) \cdot (\mathcal{S})f'(\eta).$$

Proof. Let us look at the function $g : [a,b] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} \dfrac{f(x)-f(a)}{x-a} & \text{if } x \in (a,b], \\ (\mathcal{S})f'(a) & \text{if } x = a. \end{cases} \tag{4}$$

Note that $g$ is $(\mathcal{S})$-continuous in $[a,b]$ and that when we use lemma 8, then for every $c \in (a,b]$

$$\begin{aligned} (\mathcal{S})g'(c) &= (\mathcal{S}) \lim_{x \to c} \frac{g(x)-g(c)}{x-c} \\ &= (\mathcal{S}) \lim_{x \to c} \Big[\frac{1}{x-a} \cdot \frac{f(x)-f(c)}{x-c} - \frac{1}{x-a} \cdot \frac{f(c)-f(a)}{c-a}\Big] \\ &= \frac{1}{c-a} \cdot (\mathcal{S})f'(c) - \frac{1}{c-a} \cdot \frac{f(c)-f(a)}{c-a} \\ &= \frac{(\mathcal{S})f'(c) - g(c)}{c-a} \end{aligned} \tag{5}$$

which means that $g$ is $(\mathcal{S})$-differentiable in $(a,b]$.

Moreover, note that it follows from (5) that

$$\begin{aligned} &[g(b) - g(a)] \cdot (\mathcal{S})g'(b) \\ &= \frac{-1}{b-a} \cdot \Big[(\mathcal{S})f'(b) - \frac{f(b)-f(a)}{b-a}\Big] \cdot \Big[(\mathcal{S})f'(a) - \frac{f(b)-f(a)}{b-a}\Big]. \end{aligned}$$

Hence it follows from the assumption that $[g(b) - g(a)] \cdot (\mathcal{S})g'(b) \leq 0$. From corollary 27 it follows that $(\mathcal{S})g'(\eta) = 0$ for some $\eta \in (a,b]$. If we take the above, the definition of $g$ and (5), we have the conclusion of the theorem. □

**Theorem 29.** *Let $f : [a,b] \to \mathbb{R}$ be $(\mathcal{S})$-differentiable in $[a,b]$. If $(\mathcal{S})f'(a) = (\mathcal{S})f'(b)$, then there exists $\eta \in (a,b)$ such that*

$$f(\eta) - f(a) = (\eta - a) \cdot (\mathcal{S})f'(\eta). \tag{6}$$

Proof. Let $g : [a, b] \to \mathbb{R}$ be defined by (4). First, let us assume that

$$f(b) - f(a) = (b - a) \cdot (\mathcal{S})f'(b).$$

Then it follows from the definition of function $g$ and the assumption that $g(b) = g(a)$. If we apply Theorem 24 we obtain

$$(\mathcal{S})g'(\eta) = 0 \text{ for some } \eta \in (a, b).$$

Hence by (5)

$$f(\eta) - f(a) = (\eta - a) \cdot (\mathcal{S})f'(\eta).$$

But if

$$f(b) - f(a) \neq (b - a) \cdot (\mathcal{S})f'(b),$$

then either

$$\left[(\mathcal{S})f'(b) - \frac{f(b) - f(a)}{b - a}\right] > 0 \text{ or } \left[(\mathcal{S})f'(b) - \frac{f(b) - f(a)}{b - a}\right] < 0.$$

Since $(\mathcal{S})f'(b) = (\mathcal{S})f'(a)$,

$$\left[(\mathcal{S})f'(b) - \frac{f(b) - f(a)}{b - a}\right] \cdot \left[(\mathcal{S})f'(a) - \frac{f(b) - f(a)}{b - a}\right] > 0 \tag{7}$$

and if we apply Theorem 28 we obtain equality (6) for some $\eta \in (a, b]$. Since the inequality (7) is sharp, it is obvious that $\eta \neq b$. □

**Corollary 30.** *If $f : [a, b] \to \mathbb{R}$ is $(\mathcal{S})$-differentiable in $[a, b]$, then there exists $\eta \in (a, b)$ such that*

$$f(\eta) - f(a) = (\eta - a) \cdot (\mathcal{S})f'(\eta) - \frac{1}{2} \cdot \frac{(\mathcal{S})f'(b) - (\mathcal{S})f'(a)}{(b - a)} \cdot (\eta - a)^2.$$

Proof. Let us define function $\psi : [a, b] \to \mathbb{R}$ as

$$\psi(x) = f(x) - \frac{1}{2} \cdot \frac{(\mathcal{S})f'(b) - (\mathcal{S})f'(a)}{(b - a)} \cdot (x - a)^2.$$

It follows from Lemma 8 that $f$ is $(\mathcal{S})$-differentiable in $[a, b]$ and that

$$(\mathcal{S})\psi'(x) = (\mathcal{S})f'(x) - \frac{(\mathcal{S})f'(b) - (\mathcal{S})f'(a)}{b - a} \cdot (x - a).$$

Moreover, we see that $(\mathcal{S})\psi'(a) = (\mathcal{S})f'(a) = (\mathcal{S})\psi'(b)$. If we apply Theorem 29 to $\psi$, then

$$\psi(\eta) - \psi(a) = (\eta - a) \cdot (\mathcal{S})\psi'(\eta) \text{ for some } \eta \in (a, b).$$

If we take the above and the definition of function $\psi$, we have the conclusion. □

Since each of the systems $\mathcal{S}_0$, $\mathcal{S}_{ap}$, $\mathcal{S}_{I-ap}$, $\mathcal{S}_{\mathcal{N}}$ is bilateral, filtering and fulfills the intersection condition and one of the following conditions: $(J_3)$ or the Khintchine condition, we can conclude that all the results are true for each of those systems.

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