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# SOME LOCAL PROPERTIES DEFINING $\mathcal{T}_0$-GROUPS AND RELATED CLASSES OF GROUPS

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**Abstract:** We call $G$ a Hall$_{\mathcal{X}}$-group if there exists a normal nilpotent subgroup $N$ of $G$ for which $G/N'$ is an $\mathcal{X}$-group. We call $G$ a $\mathcal{T}_0$-group provided $G/\Phi(G)$ is a $\mathcal{T}$-group, that is, one in which normality is a transitive relation. We present several new local classes of groups which locally define Hall$_{\mathcal{X}}$-groups and $\mathcal{T}_0$-groups where $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$; the classes $\mathcal{PT}$ and $\mathcal{PST}$ denote, respectively, the classes of groups in which permutability and S-permutability are transitive relations.

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## 1. Introduction

All groups considered will be finite. All unexplained notation or terminology can be found in [**3**] or [**8**].

There is a well-known theorem of Philip Hall which states that if a group $G$ possesses a normal nilpotent subgroup $N$ such that $G/N'$ is nilpotent, then $G$ is nilpotent. It has been the subject of several papers to consider what can be said about the class of groups $G$ possessing a normal nilpotent subgroup $N$ such that $G/N'$ is an $\mathcal{X}$-group for various classes $\mathcal{X}$. In particular, the fourth author in [**6**], for the solvable case, answered this question when $\mathcal{X}$ is any one of the classes $\mathcal{T}$, $\mathcal{PT}$, or $\mathcal{PST}$, that is, respectively, the classes where normality, permutability, and S-permutability are transitive relations. Let us call $G$ a Hall$_{\mathcal{X}}$-group if there exists a normal nilpotent subgroup $N$ of $G$ for which $G/N'$ is an $\mathcal{X}$-group. Since the Fitting subgroup of a group $G$ contains all normal nilpotent subgroups one can simply replace $N$ in the definition just given with $F$, the Fitting subgroup of $G$.

It is a classical problem to determine the structure of a group $G$ such that $G/\Phi(G)$ is an $\mathcal{X}$-group for various classes $\mathcal{X}$. For a class $\mathcal{X}$, let us call $G$ an $\mathcal{X}_0$-group if $G/\Phi(G)$ is an $\mathcal{X}$-group. The fourth author in [**6**] characterized the solvable $\mathcal{X}_0$-groups when $\mathcal{X}$ is one of the classes $\mathcal{T}$, $\mathcal{PT}$, or $\mathcal{PST}$. In particular, it was shown that within the class of solvable groups, the classes $\mathcal{T}_0$, $\mathcal{PT}_0$, and $\mathcal{PST}_0$ coincide.

In this paper, we turn to the concept of a "local" characterization theorem for solvable $\mathcal{T}_0$-groups and $\mathrm{Hall}_{\mathcal{X}}$-groups for $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$. That is, we are interested in characterizations in terms of the different prime divisors of the orders of such groups.

## 2. Preliminaries

The classes $\mathscr{C}_p$, $\mathscr{X}_p$, and $\mathscr{Y}_p$, and the theorems that follow, will be critical in giving the desired local characterization theorems.

**Definition.**

(1) $\mathscr{C}_p$ is the class of groups $G$ for which each subgroup of a Sylow $p$-subgroup $P$ of $G$ is normal in the normalizer, $\mathrm{N}_G(P)$, of $P$.

(2) $\mathscr{X}_p$ is the class of groups $G$ for which each subgroup of a Sylow $p$-subgroup $P$ of $G$ is permutable in the normalizer, $\mathrm{N}_G(P)$, of $P$.

(3) $\mathscr{Y}_p$ is the class of groups $G$ where $H \le S \le P$ with $P$ a Sylow $p$-subgroup of $G$, always implies $H$ is S-permutable in the normalizer, $\mathrm{N}_G(S)$, of $S$.

**Theorem 1.** *Let $G$ be a group.*

(1) (Robinson [**7**], see also [**3**, 2.2.2]) *$G$ is a solvable $\mathcal{T}$-group if and only if $G$ is a $\mathscr{C}_p$-group for all primes $p$.*

(2) (Beidleman, Brewster, and Robinson [**5**], see also [**3**, 2.2.3]) *$G$ is a solvable $\mathcal{PT}$-group if and only if $G$ is an $\mathscr{X}_p$-group for all primes $p$.*

(3) (Ballester-Bolinches and Esteban-Romero [**2**], see also [**3**, 2.2.9]) *$G$ is a solvable $\mathcal{PST}$-group if and only if $G$ is a $\mathscr{Y}_p$-group for all primes $p$.*

We will need some results on the classes $\mathscr{C}_p$, $\mathscr{X}_p$, and $\mathscr{Y}_p$ to aid in our characterization theorems.

**Lemma 1** (Ballester-Bolinches and Esteban-Romero [**2**], see also [**3**, 2.2.13])**.** *A group $G$ is a $\mathscr{Y}_p$-group if and only if $G$ is $p$-nilpotent or $G$ is a $\mathscr{C}_p$-group with abelian Sylow $p$-subgroups.*

**Lemma 2** (Ballester-Bolinches and Esteban-Romero [**2**], see also [**3**, 2.2.4])**.** *A group $G$ is an $\mathscr{X}_p$-group (respectively, $\mathscr{C}_p$-group) if and only if $G$ is a $\mathscr{Y}_p$-group and the Sylow $p$-subgroups of $G$ are Iwasawa (respectively, Dedekind) groups.*

In the previous lemma, Iwasawa groups (respectively, Dedekind groups) are the groups for which every subgroup is permutable (respectively, normal).

**Lemma 3.** *Let $G$ be a group with $N$ a normal $p'$-subgroup of $G$ and $M$ a normal $p$-subgroup of $G$.*

(1) *If $G$ is a $\mathscr{Y}_p$-group, then $G/M$ is a $\mathscr{Y}_p$-group.*
(2) *If $G$ is an $\mathscr{X}_p$-group, then $G/M$ is an $\mathscr{X}_p$-group.*
(3) *If $G$ is a $\mathscr{C}_p$-group, then $G/M$ is a $\mathscr{C}_p$-group.*
(4) *$G$ is a $\mathscr{Y}_p$-group if and only if $G/N$ is a $\mathscr{Y}_p$-group.*
(5) *$G$ is an $\mathscr{X}_p$-group if and only if $G/N$ is an $\mathscr{X}_p$-group.*
(6) *$G$ is a $\mathscr{C}_p$-group if and only if $G/N$ is a $\mathscr{C}_p$-group.*

*Proof:* (1), (2), and (3) follow directly from the definitions of $\mathscr{Y}_p$, $\mathscr{X}_p$, and $\mathscr{C}_p$.

(4) This can be found in [**4**] as Lemma 6.

(5) This can be found in [**1**] as Lemma 4.1.

(6) Let $G$ be a $\mathscr{C}_p$-group. By Lemma 2, this is equivalent to affirming that $G$ is a $\mathscr{Y}_p$-group with Dedekind Sylow $p$-subgroups. Since the Sylow $p$-subgroups of $G$ and $G/N$ are isomorphic, by (4) we have that this is equivalent to stating that $G/N$ is a $\mathscr{Y}_p$-group with Dedekind Sylow $p$-subgroups. By Lemma 2, this is to say that $G/N$ is a $\mathscr{C}_p$-group. This completes the proof. □

*Remark.* In fact, the classes $\mathscr{C}_p$, $\mathscr{X}_p$, and $\mathscr{Y}_p$ are quotient closed in general, however, this is a much more difficult fact to deduce. For our purposes, quotient closure for normal $p'$-subgroups and $p$-subgroups is sufficient.

Lastly, we record some results on $\mathcal{T}_0$-groups which will be needed.

**Theorem 2.**

(1) (van der Waall and Fransman [**9**]) *The class of solvable $\mathcal{T}_0$-groups is a quotient closed class of groups.*
(2) (van der Waall and Fransman [**9**]) *A solvable $\mathcal{T}_0$-group is supersolvable.*
(3) (Ragland [**6**]) *A group $G$ is a solvable* $\mathrm{Hall}_{\mathcal{PST}}$*-group if and only if $G$ is a solvable $\mathcal{T}_0$-group.*
(4) (Ragland [**6**]) *A group $G$ is a solvable $\mathcal{PST}_0$-group if and only if $G$ is a solvable $\mathcal{T}_0$-group.*
(5) (Ragland [**6**]) *A group $G$ is a solvable $\mathcal{T}_0$-group with abelian nilpotent residual if and only if $G$ is a solvable $\mathcal{PST}$-group.*

## 3. Main results

In this section we provide local characterization theorems for the solvable $\mathcal{T}_0$-groups, $\mathrm{Hall}_{\mathcal{T}}$-groups, $\mathrm{Hall}_{\mathcal{PT}}$-groups, and $\mathrm{Hall}_{\mathcal{PST}}$-groups. We will need the following definitions:

**Definition.** Let $p$ be a prime. For a group $G$, let $\Phi(G)_p$ denote the Sylow $p$-subgroup of the Frattini subgroup of $G$. Let $F_p'$ denote the derived subgroup of the Sylow $p$-subgroup of the Fitting subgroup, $F$, of $G$. Let $L_p'$ denote the derived subgroup of a Sylow $p$-subgroup of the nilpotent residual, $L$, of $G$. If $p$ does not divide the order of $\Phi(G)$ then let us denote by $\Phi(G)_p$ the identity subgroup. Likewise, $F_p' = 1$ if $p$ does not divide the order of $F'$ and $L_p' = 1$ if $p$ does not divide the order of $L'$. Let $\mathscr{X}$ be one of the classes $\mathscr{C}_p$, $\mathscr{X}_p$, or $\mathscr{Y}_p$.

(1) Define $\Phi_{\mathscr{X}}$ to be the class of groups $G$ for which $G/\Phi(G)_p$ is a $\mathscr{X}$-group.
(2) Define $\mathscr{F}_{\mathscr{X}}$ to be the class of groups $G$ for which $G/F_p'$ is a $\mathscr{X}$-group.
(3) Define $\mathscr{L}_{\mathscr{X}}$ to be the class of groups $G$ for which $L$ is $p'$-nilpotent and $G/L_p'$ is a $\mathscr{X}$-group.

Since the Frattini factor group of a solvable $\mathcal{T}_0$-group is a $\mathcal{T}$-group, and $\mathcal{T}$-groups can be locally characterized using the class $\mathscr{C}_p$, it stands to reason that $\mathcal{T}_0$-groups can be locally characterized using the class $\mathscr{C}_p$ with certain conditions involving the Frattini subgroup. This is indeed the case, however, recall that $\mathcal{T}_0 = \mathcal{PT}_0 = \mathcal{PST}_0$ in the class of solvable groups. So it also stands to reason that one could replace $\mathscr{C}_p$ with $\mathscr{X}_p$ or $\mathscr{Y}_p$. In view of Theorem 2, part (3), one should be able to characterize the solvable $\mathcal{T}_0$-groups using $\mathscr{Y}_p$ with certain conditions placed upon the Fitting subgroup. The details can be found in the following theorem:

**Theorem 3.** *The following are equivalent for a group $G$.*

(1) *$G$ is an $\mathscr{L}_{\mathscr{Y}_p}$-group for all primes $p$.*
(2) *$G$ is an $\mathscr{F}_{\mathscr{Y}_p}$-group for all primes $p$.*
(3) *$G$ is a solvable $\mathrm{Hall}_{\mathcal{PST}}$-group.*
(4) *$G$ is a $\Phi_{\mathscr{C}_p}$-group for all primes $p$.*
(5) *$G$ is a $\Phi_{\mathscr{X}_p}$-group for all primes $p$.*
(6) *$G$ is a $\Phi_{\mathscr{Y}_p}$-group for all primes $p$.*
(7) *$G$ is a solvable $\mathcal{T}_0$-group.*

*Proof:* Throughout the proof let us denote the nilpotent residual of $G$ by $L$ and the Fitting subgroup of $G$ by $F$. Also, for any group $H$, $H_p$ will denote a Sylow $p$-subgroup of $H$.

(1) $\Rightarrow$ (2) If $G$ is an $\mathscr{L}_{\mathscr{Y}_p}$-group for all primes $p$ then $L$ is $p'$-nilpotent for all primes $p$ and hence nilpotent. Thus $L \leq F$ so that for any prime $p$ we have $L_p' \leq F_p'$. Now since $G/L_p'$ is a $\mathscr{Y}_p$-group for all primes $p$, we have, by Lemma 3, part (1), that $(G/L_p')/(F_p'/L_p') \simeq G/F_p'$ is a $\mathscr{Y}_p$-group for all primes $p$.

(2) $\Rightarrow$ (3) If $G$ is an $\mathscr{F}_{\mathscr{Y}_p}$-group for all primes $p$, then $G/F_p'$ is a $\mathscr{Y}_p$-group for all primes $p$. Using Lemma 3, part (4), we have $(G/F_p')/(F'/F_p') \simeq G/F'$ is a $\mathscr{Y}_p$-group for all primes $p$. Hence by Theorem 1, part (3), we have that $G/F'$ is a solvable $\mathcal{PST}$-group so that $G$ is a solvable $\mathrm{Hall}_{\mathcal{PST}}$-group.

(3) $\Rightarrow$ (4) If $G$ is a solvable $\mathrm{Hall}_{\mathcal{PST}}$-group, then, by Theorem 2, part (3), we have that $G$ is a solvable $\mathcal{T}_0$-group. Let $\bar{G} = G/\Phi(G)_p$. Then $\bar{G}$ is a solvable $\mathcal{T}_0$-group by Theorem 2, part (1). By induction, if $\Phi(G)_p \neq 1$, we have $\bar{G}/\Phi(\bar{G})_p$ is a $\mathscr{C}_p$-group. But $\Phi(\bar{G})_p = 1$ and so $\bar{G}$ is a $\mathscr{C}_p$-group. If $\Phi(G)_p = 1$ then one only needs $G$ itself to be a $\mathscr{C}_p$ group. Now $G/\Phi(G)$ is a $\mathcal{T}$-group and hence a $\mathscr{C}_p$-group by Theorem 1, part (1). Lemma 3, part (6), gives us that $G$ is a $\mathscr{C}_p$-group since $\Phi(G)$ is a $p'$-group.

Both (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (6) follow directly from the definitions of $\mathscr{C}_p$, $\mathscr{X}_p$, and $\mathscr{Y}_p$.

(6) $\Rightarrow$ (7) If $G$ is a $\Phi_{\mathscr{Y}_p}$-group for all primes $p$, then $G/\Phi(G)_p$ is a $\mathscr{Y}_p$-group for all primes $p$. Since $\Phi(G)/\Phi(G)_p$ is a $p'$-group for all primes $p$, we have, by Lemma 3, part (4), that $\big(G/\Phi(G)_p\big)/\big(\Phi(G)/\Phi(G)_p\big) \simeq G/\Phi(G)$ is a $\mathscr{Y}_p$-group for all primes $p$. Thus $G/\Phi(G)$ is a $\mathcal{PST}$-group by Theorem 1, part (3). Hence $G$ is a solvable $\mathcal{PST}_0$-group and thus a solvable $\mathcal{T}_0$-group by Theorem 2, part (4).

(7) $\Rightarrow$ (1) If $G$ is a solvable $\mathcal{T}_0$-group, then $G$ is supersolvable by Theorem 2, part (2). Thus $G'$ is nilpotent in which case so is $L$. Hence $L$ is $p'$-nilpotent for all primes $p$. Note that if $L$ is abelian then, by Theorem 2, part (5), $G$ is a $\mathcal{PST}$-group and hence a $\mathscr{Y}_p$-group for all primes $p$ by Theorem 1, part (3). From Lemma 3, part (1), it would follow that $G$ is a $\mathscr{L}_{\mathscr{Y}_p}$-group for all primes $p$ if $L$ is abelian. So let us assume $L$ is not abelian.

Let $\bar{G} = G/L_p'$ and let $\bar{L}$ be the nilpotent residual of $\bar{G}$. Now $\bar{G}$ is a solvable $\mathcal{T}_0$-group by Theorem 2, part (1). By induction, if $L_p' \neq 1$, we have $\bar{G}/\bar{L}_p'$ is a $\mathscr{Y}_p$-group. But $\bar{L}_p' = 1$ and so $\bar{G}$ is a $\mathscr{Y}_p$-group.

Assume that $L'_p = 1$. Since $L$ is not abelian there exists a prime $q \neq p$ for which $L'_q \neq 1$. Now let $\bar{G}$ denote $G/L'_q$ with $\bar{L}$ the nilpotent residual of $\bar{G}$ and note $\bar{G}$ is a $\mathcal{T}_0$-group by Theorem 2, part (1). So, by induction, $\bar{G}/\bar{L}'_p$ is a $\mathscr{Y}_p$-group. Thus $\bar{G}$ is a $\mathscr{Y}_p$-group and hence $G$ is a $\mathscr{Y}_p$-group by Lemma 3, part (4). It now follows that $G$ is a $\mathscr{L}_{\mathscr{Y}_p}$-group for all primes $p$ completing the proof. □

The classes of solvable $\mathrm{Hall}_{\mathcal{T}}$-groups and $\mathrm{Hall}_{\mathcal{PT}}$-groups admit similar local characterizations which we provide now.

**Theorem 4.** *The following are equivalent for a group $G$.*

(1) *$G$ is a $\mathscr{F}_{\mathscr{X}_p}$-group for all primes $p$.*
(2) *$G$ is a solvable* $\mathrm{Hall}_{\mathcal{PT}}$*-group.*

*Proof:* (1) $\Rightarrow$ (2) The proof is similar to the proof of (2) $\Rightarrow$ (3) from Theorem 3. One only needs to use invoke the appropriate results corresponding to $\mathscr{X}_p$ from Lemma 3 and Theorem 1.

(2) $\Rightarrow$ (1) If $G$ is a solvable $\mathrm{Hall}_{\mathcal{PT}}$-group, then it is a solvable $\mathrm{Hall}_{\mathcal{PST}}$-group and hence by Theorem 3 we have $G/F'_p$ is a $\mathscr{Y}_p$-group for all primes $p$.

Now $G/F'$ a solvable $\mathcal{PT}$-group and thus an $\mathscr{X}_p$-group for all primes $p$ by Theorem 1, part (2). Hence the Sylow subgroups of $G/F'$ are Iwasawa. Let $P$ denote a Sylow $p$-subgroup of $G$. Then $PF'/F' \simeq P/(P \cap F') = P/F'_p$ and thus $P/F'_p$ is an Iwasawa group.

Now one can conclude from Theorem 2 that $G/F'_p$ is an $\mathscr{X}_p$-group for all primes $p$ so that $G$ is a $\mathscr{F}_{\mathscr{X}_p}$-group for all primes $p$. □

**Theorem 5.** *The following are equivalent for a group $G$.*

(1) *$G$ is a $\mathscr{F}_{\mathscr{C}_p}$-group for all primes $p$.*
(2) *$G$ is a solvable* $\mathrm{Hall}_{\mathcal{T}}$*-group.*

*Proof:* Using the appropriate theorems and lemmas, the proof is quite similar to that of Theorem 4 and so we omit the proof. □

The next theorem shows that the difference between solvable $\mathcal{T}_0$-groups and $\mathrm{Hall}_{\mathcal{PT}}$-groups ($\mathrm{Hall}_{\mathcal{T}}$-groups) amounts to the Sylow structure of $G/F'_p$ for all primes $p$.

**Theorem 6.** *Let $G$ be a solvable group with $F = \mathrm{Fit}(G)$. Then $G$ is a $\mathcal{T}_0$-group where the Sylow $p$-subgroups of $G/F'_p$ are Iwasawa (Dedekind) groups for all primes $p$ if and only if $G$ is a* $\mathrm{Hall}_{\mathcal{PT}}$*-group (*$\mathrm{Hall}_{\mathcal{T}}$*-group).*

*Proof:* Suppose $G$ is a $\mathcal{T}_0$-group where the Sylow $p$-subgroups of $G/F'_p$ are Iwasawa (Dedekind) groups. By Theorem 3, we have $G/F'_p$ is a

$\mathscr{Y}_p$-group for all primes $p$. By Lemma 2, we must have $G/F_p'$ is an $\mathscr{X}_p$-group ($\mathscr{C}_p$-group) for all primes $p$. We can now deduce from Theorem 4, part (2), that $G$ is a $\mathrm{Hall}_{\mathcal{PT}}$-group ($\mathrm{Hall}_{\mathcal{T}}$-group).

Suppose $G$ is a $\mathrm{Hall}_{\mathcal{PT}}$-group ($\mathrm{Hall}_{\mathcal{T}}$-group). Then Theorem 4, part (5), says that $G/F_p'$ is an $\mathscr{X}_p$-group ($\mathscr{C}_p$-group) for all primes $p$. Using Lemma 2, we can deduce that $G/F_p'$ is a $\mathscr{Y}_p$-group for all primes $p$ and that the Sylow $p$-subgroups of $G/F_p'$ are Iwasawa (Dedekind) groups. That $G$ is a $\mathcal{T}_0$-group follows from Theorem 3. □

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