Publ. Mat. **53** (2009), 119–140

FAITHFUL LINEAR REPRESENTATIONS OF BANDS

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Abstract

A band is a semigroup consisting of idempotents. It is proved that for any field K and any band S with finitely many components, the semigroup algebra $K[S]$ can be embedded in upper triangular matrices over a commutative K-algebra. The proof of a theorem of Malcev [**4**, Theorem 10] on embeddability of algebras into matrix algebras over a field is corrected and it is proved that if $S = F \cup E$ is a band with two components E, F such that F is an ideal of S and E is finite, then S is a linear semigroup. Certain sufficient conditions for linearity of a band S, expressed in terms of annihilators associated to S, are also obtained.

1. Introduction

Recall that a band is a semigroup consisting of idempotents. This paper is motivated by the problem of embeddability of a band S into the multiplicative semigroup $M_n(K)$ of matrices over a field K, which was first raised in [**3**]. It is known that every band S is a semilattice of rectangular bands. In other words, there exists a semilattice (a commutative band) Γ such that $S = \bigcup_{\alpha\in\Gamma} S_\alpha$, a disjoint union, and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$. Here S_α are called the components of S and they are rectangular bands. This means that every $E = S_\alpha$ satisfies the identity $xyx = x$. Equivalently, $E \cong \mathcal{M}(\{1\}, X, Y; P)$, a completely simple semigroup over the trivial group and with a sandwich matrix $P = (p_{yx})$ where $p_{yx} = 1$ for all $x \in X$, $y \in Y$. Moreover $E = (Ee_0)(e_0E)$ for every $e_0 \in E$ and Ee_0 is a left zero semigroup, that is it satisfies the identity $xy = x$, and e_0E is

2000 *Mathematics Subject Classification.* Primary: 20M25, 16R20, 16S36; Secondary: 20M12, 20M17, 20M30.

Key words. Linear band, semigroup algebra, triangular matrices, annihilator, PI rings, normal band.

Work supported in part by the DGI MEC-Spain and FEDER through grant MTM2005-00934, by Generalitat de Catalunya (Grup de Recerca consolidat 2005SGR00206) and the MNiSW research grant N201 004 32/0088 (Poland).

a right zero semigroup, defined by the identity $xy = y$. For more details we refer to [**7**].

It is well known that every linear semigroup $S \subseteq M_n(K)$ has only finitely many principal factors that contain idempotents (and every such factor is a completely 0-simple or a completely simple semigroup). This follows from a general structure theorem for arbitrary linear semigroups, [**6**, Theorem 3.5]. So, when studying embeddability of bands into matrices, we have to deal with the case where Γ is finite. It is also well known that every band $S \subseteq M_n(K)$ is triangularizable, see [**6**, §4.4], or [**8**].

If $|\Gamma| = 1$, so $S = F$ is a rectangular band, then F has an embedding into $M_3(L)$ for sufficiently big fields L. A matrix embedding of F, given in [**3**], is of the form

$$x \longmapsto \begin{pmatrix} 0 & \mu(xf) & \mu(xf)\eta(fx) \\ 0 & 1 & \eta(fx) \\ 0 & 0 & 0 \end{pmatrix} \in M_3(L)$$

where $f \in F$ is a fixed element and $\mu\colon Ff \longrightarrow L$, $\eta\colon fF \longrightarrow L$ are injective maps. Actually, it is easy to see that, for any field K, choosing the field L big enough and choosing appropriate maps μ, η, we also get an embedding of algebras $K[F] \hookrightarrow M_3(L)$.

The next step is to consider bands with two components. An example constructed in [**3**] shows that there exists a band with two components $S = E \cup F$ such that S is not linear. In this example, both E and F are right zero semigroups. We show that one can construct an even simpler example with E a left zero and F a right zero semigroup.

Example 1.1. Let $S = E \cup F$, a disjoint union, where $E = \{e_i \mid i \geq 1\}$ and $F = X \cup X'$ with $X = \{f_i \mid i \geq 1\}$, $X' = \{f'_i \mid i \geq 1\}$. On E and F we define the operation by the rules: $ee' = e$ and $ff' = f'$ for all $e, e' \in E$, $f, f' \in F$. Then we define also $x'e = x'$ for $x' \in X'$, $e \in E$ and $ef = f$ for $e \in E$, $f \in F$. Moreover, $f_j e_i = f'_j$ for $j = 1, \dots, i-1$ and $f_j e_i = f'_i$ for $j \geq i$. It is easy to see that S is a band with components E, F and F is an ideal of S. Notice that the right annihilator $\operatorname{ann}_r(f_j - f'_j) = \{x \in S \mid f_j x = f'_j x\}$ of $f_j - f'_j$ in S contains e_i with $i \geq j$ and does not contain $e_1, \dots, e_{j-1}$. It follows that there is an infinite descending chain of such annihilators. Therefore S is not a linear semigroup.

In view of these examples, one can ask for general conditions under which a band S is linear. This problem is the main motivation for the results of this paper.

We note that the problem of embeddability of rings into matrices over a commutative ring has attracted a lot of attention because of its role in the theory of PI-algebras, see [**9**]. Several papers were also devoted to the embeddability into triangular matrices, in particular see [**1**], [**5**]. In the context of our general aim, it is then natural to look at these problems for the class of bands and also of their semigroup algebras.

In the first part of this paper, we prove that for any field K and any band S with finitely many components, the algebra $K[S]$ can be embedded in upper triangular matrices over a commutative K-algebra. We also give an explicit embedding in the case where S has 2 components.

In the second part, we prove a technical lemma on algebras which allows us to correct the original proof of a theorem of Malcev [**4**, Theorem 10] and to prove that if $S = F \cup E$ is a band with two components E, F such that F is an ideal of S and E is finite, then S is a linear semigroup. We give also some explicit embeddings of special classes of bands satisfying finiteness conditions on annihilators, and of their algebras, into triangular matrices over fields.

2. Triangular embeddings over commutative rings

Let K be a field. For a K-algebra A, denote by $T_n(A)$ the algebra of upper triangular matrices $m = (m_{ij})$ over A with $m_{ii} \in K$ for every $i = 1, \ldots, n$. First, we prove an auxiliary result.

Lemma 2.1. *Assume that a K-algebra R is of the form $R = R_0 + Ke$ for an ideal R_0 of R and for an idempotent $e \in R$. Let A be a commutative K-algebra. If R_0 is embeddable into the matrix ring $M_n(A)$ for some $n \geq 1$ then R embeds in $M_{2n}(A)$. If R_0 is embeddable into $T_n(A)$ for some $n \geq 1$ then R embeds in $T_{2n}(A)$.*

Proof: Suppose that $\phi\colon R_0 \longrightarrow M_n(A)$ is an embedding. We have a decomposition $R = eRe + eR(1-e) + (1-e)Re + (1-e)R(1-e)$ (existence of a unity in R is not assumed). By the hypothesis, $eRe = eR_0e + Ke$ and the remaining three components are contained in R_0. Define the map $\bar{\phi}\colon R \longrightarrow M_{2n}(A)$ by the rules:

$$\bar{\phi}(\lambda e) = \begin{pmatrix} \lambda I & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{for } \lambda \in K,$$

$$\bar{\phi}(x) = \begin{pmatrix} \phi(x) & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for } x \in eR_0e,$$

$$\bar{\phi}(x) = \begin{pmatrix} 0 & \phi(x) \\ 0 & 0 \end{pmatrix}, \quad \text{for } x \in eR_0(1-e),$$

$$\bar{\phi}(x) = \begin{pmatrix} 0 & 0 \\ \phi(x) & 0 \end{pmatrix}, \quad \text{for } x \in (1-e)R_0e,$$

$$\bar{\phi}(x) = \begin{pmatrix} 0 & 0 \\ 0 & \phi(x) \end{pmatrix}, \quad \text{for } x \in (1-e)R_0(1-e),$$

using the block matrix notation with all blocks of size $n \times n$. It is clear that $\bar{\phi}$ is an injective homomorphism.

Next, assume that $\phi\colon R_0 \longrightarrow T_n(A)$. In this case, a slightly more complicated approach is needed to prove the second assertion. Write $\phi(x) = (\phi_{ij}(x))$. Put $\phi(e) = I$, the identity matrix. Then define: $\bar{\phi}\colon R \longrightarrow T_{2n}(A)$ as follows: $\bar{\phi}(x) = (\bar{\phi}_{ij}(x))$, where

$$\bar{\phi}_{2i-1,2j-1}(x) = \phi_{ij}(x) \text{ for } x \in eRe,$$

$$\bar{\phi}_{2i-1,2j}(x) = \phi_{ij}(x) \text{ for } x \in eR(1-e),$$

$$\bar{\phi}_{2i,2j-1}(x) = \phi_{ij}(x) \text{ for } x \in (1-e)Re,$$

$$\bar{\phi}_{2i,2j}(x) = \phi_{ij}(x) \text{ for } x \in (1-e)R(1-e),$$

where i, j run over the set $\{1, \dots, n\}$ with the convention that $\bar{\phi}_{k,l}(x) = 0$ for all unspecified pairs $k, l \in \{1, \dots, n\}$ in each of the four cases listed above. It is clear that $\bar{\phi}(x)$ is an embedding. A direct verification shows that $\bar{\phi}(xy) = \bar{\phi}(x)\bar{\phi}(y)$ for every $x, y \in R$. To illustrate this, we verify the case where $x \in eR(1-e)$ and $y \in (1-e)Re$. Then the nonzero entries of the matrix

$$\bar{\phi}(x)\bar{\phi}(y) = (\bar{\phi}_{ij}(x))(\bar{\phi}_{kl}(y)) = \left(\sum_{q=1}^{2n} \bar{\phi}_{iq}(x)\bar{\phi}_{ql}(y)\right) = \left(\sum_{q=1}^{n} \bar{\phi}_{i,2q}(x)\bar{\phi}_{2q,l}(y)\right)$$

are in positions (i, l) with odd i and l and they are equal to

$$\sum_{q=1}^{n} \phi_{(i+1)/2,q}(x)\phi_{q,(l+1)/2}(y) = \phi_{(i+1)/2,(l+1)/2}(xy) = \bar{\phi}_{il}(xy).$$

So the claim follows.

In order to verify that $\bar{\phi}(x) \in T_{2n}(A)$ for every $x \in R$ we first note that $\bar{\phi}_{i,i}(x)$ is nonzero only if $x \in eRe$ or $(1-e)R(1-e)$ and then $\bar{\phi}_{i,i}(x)$ is equal to $\phi_{k,k}(x)$ for some k, which lies in K by the assumption that $\phi(R_0) \subseteq T_n(A)$. On the other hand, suppose that $\bar{\phi}_{i+1,i}(x)$ is nonzero. Then $x \in (1-e)Re$ and in this case $x^2 = 0$. So, again by the assumption on the image of ϕ, we must get $\bar{\phi}_{i+1,i}(x) = \phi_{k,k}(x) = 0$, for some k, a contradiction. Since other entries below the diagonal in every matrix $\bar{\phi}(x)$ are zero, the result follows. $\square$

Theorem 2.2. *Let S be a band with finitely many components. Then the Jacobson radical $J(K[S])$ of $K[S]$ is nilpotent and $K[S]$ embeds into $T_n(A)$ for a commutative K-algebra A. Moreover, the K-algebra $K[S]/J(K[S])$ has finite dimension.*

Proof: Let $S = \bigcup_{\gamma\in\Gamma} S_\gamma$, a semilattice Γ of completely simple components S_γ, $\gamma \in \Gamma$. We claim that $J(K[S]) = \sum_{\gamma\in\Gamma} \omega(K[S_\gamma])$, where $\omega(K[S_\gamma])$ denotes the augmentation ideal of $K[S_\gamma]$, and the radical is nilpotent. In particular, this implies that the radical has finite codimension. We proceed by induction on $|\Gamma|$. If $|\Gamma| = 1$, then it is well known and easy to check that $K[S]$ is a local algebra with radical $\omega(K[S])$ and $\omega(K[S])^3 = 0$.

So, assume that $|\Gamma| > 1$. Let β be a maximal element of Γ. Let $\Gamma' = \Gamma \setminus \{\beta\}$ and $S' = \bigcup_{\gamma\in\Gamma'} S_\gamma$. By the induction hypothesis, $J(K[S']) = \sum_{\gamma\in\Gamma'} \omega(K[S_\gamma])$ and it is nilpotent. Note that $\sum_{\gamma\in\Gamma} \omega(K[S_\gamma])$ is an ideal of $K[S]$. Since $(\omega(K[S_\beta]))^3 = 0$, we have that

$$\left(\sum_{\gamma\in\Gamma} \omega(K[S_\gamma])\right)^3 \subseteq J(K[S']).$$

Hence $\sum_{\gamma\in\Gamma} \omega(K[S_\gamma])$ is nilpotent and $\sum_{\gamma\in\Gamma} \omega(K[S_\gamma]) \subseteq J(K[S])$. We define the map $f\colon K[S] \longrightarrow K[S_\beta]$ by

$$f\left(\sum_{s\in S} \alpha(s)s\right) = \sum_{s\in S_\beta} \alpha(s)s,$$

for all $\sum_{s\in S} \alpha(s)s \in K[S]$. Since β is a maximal element of Γ, f is a ring homomorphism which is surjective. It follows that $f(J(K[S])) \subseteq J(K[S_\beta]) = \omega(K[S_\beta])$. Hence $f(J(K[S])) \subseteq J(K[S])$. Since $K[S']$ is an ideal of $K[S]$, we have that $J(K[S]) \cap K[S'] \subseteq J(K[S'])$. Therefore

$$J(K[S]) = f(J(K[S])) + (J(K[S] \cap K[S'])) \subseteq \sum_{\gamma\in\Gamma} \omega(K[S_\gamma]).$$

It follows that $J(K[S]) = \sum_{\gamma\in\Gamma} \omega(K[S_\gamma])$ and it is nilpotent.

From [**1**] we know that there exists an embedding $\phi\colon J(K[S]) \longrightarrow T_n(A)$ (actually into strictly upper triangular matrices) for some $n \geq 1$ and a commutative algebra A. Let J be an ideal of Γ that is not a singleton. Then $J = L \cup \{\delta\}$, a disjoint union, for an ideal L of Γ

and a maximal element $\delta \in J$. Define $R_J = J(K[S]) + \sum_{\gamma \in J} K[S_\gamma]$. From the first part of the proof it follows that R_J is an ideal of $K[S]$ such that $R_J = R_L + Ke_\delta$ for an element $e_\delta \in S_\delta$, and R_L is an ideal of R_J. Therefore the assertion follows by an easy induction based on Lemma 2.1. □

As an immediate consequence of Theorem 2.2 we get the following result.

Corollary 2.3. *The semigroup algebra $K[S]$ of a band S with finitely many components satisfies a standard polynomial identity.*

Our second aim is to construct a concrete embedding of a band S with two components into an algebra over a commutative ring. This is based on a commutative algebra $\bar{R}$ naturally associated to S. Therefore, it also leads to the question whether linearity of S (over a field) can be characterized in terms of this algebra.

Suppose that S is a band with two components F and E such that F is an ideal of S. We may assume that $F = \{f_{i,j} \mid i \in I \text{ and } j \in J\}$, $E = \{e_{a,b} \mid a \in A \text{ and } b \in B\}$, and

$$f_{i,j} f_{k,l} = f_{i,l} \quad \text{and} \quad e_{a,b} e_{c,d} = e_{a,d},$$

for all $i, k \in I$, $j, l \in J$, $a, c \in A$ and $b, d \in B$. Let $\alpha\colon A \times B \times I \to I$ be the map defined by the rule

$$e_{a,b} f_{i,j} = f_{\alpha(a,b,i),j}.$$

Let $\beta\colon J \times A \times B \to J$ be the map defined by the rule

$$f_{i,j} e_{a,b} = f_{i,\beta(j,a,b)}.$$

Since

$$f_{\alpha(a,d,i),j} = e_{a,d} f_{i,j} = e_{a,b} e_{c,d} f_{i,j} = e_{a,b} f_{\alpha(c,d,i),j} = f_{\alpha(a,b,\alpha(c,d,i)),j},$$

we have that

$$\alpha(a, b, \alpha(c, d, i)) = \alpha(a, d, i), \tag{1}$$

for all $a, c \in A$, $b, d \in B$ and $i \in I$. Similarly one can see that

$$\beta(\beta(j, a, b), c, d) = \beta(j, a, d), \tag{2}$$

for all $a, c \in A$, $b, d \in B$ and $j \in J$.

For any field K, we will construct a commutative K-algebra $\bar{R}$ such that the semigroup algebra $K[S]$ embeds into the ring $T_7(\bar{R})$ of 7×7 upper triangular matrices over $\bar{R}$ with diagonal in K. Actually, $\bar{R}$ is a semigroup algebra $K[C]$ of a commutative semigroup C.

Let $X = \{x_i \mid i \in I\}$, $Y = \{y_j \mid j \in J\}$, $Z = \{z_a \mid a \in A\}$, $Z' = \{z'_a \mid a \in A\}$, $T = \{t_b \mid b \in B\}$ and $T' = \{t'_b \mid b \in B\}$ be pairwise disjoint sets of commuting indeterminates over K. Let $R = K[X \cup Y \cup Z \cup Z' \cup T \cup T']$ be the polynomial ring on these indeterminates with coefficients in K. Let M be the ideal of R generated by

(i) $t_b x_i - t_d x_k$, for all $i, k \in I$ and all $b, d \in B$ such that there exists $a \in A$ satisfying $\alpha(a, b, i) = \alpha(a, d, k)$,

(ii) $z_a t_b x_i - z_c t_d x_k$, for all $i, k \in I$, $a, c \in A$ and $b, d \in B$ such that $\alpha(a, b, i) = \alpha(c, d, k)$,

(iii) $y_j z'_a - y_l z'_c$, for all $j, l \in J$ and all $a, c \in A$ such that there exists $b \in B$ satisfying $\beta(j, a, b) = \beta(l, c, b)$,

(iv) $y_j z'_a t'_b - y_l z'_c t'_d$, for all $j, l \in J$, $a, c \in A$ and $b, d \in B$ such that $\beta(j, a, b) = \beta(l, c, d)$.

Let $\bar{R} = R/M$. Let $\rho\colon S \to T_7(\bar{R})$ be the map defined by

$$\rho(e_{a,b}) = \begin{pmatrix} 0 & \overline{z_a} & \overline{z_a t_b} & 0 & 0 & 0 & 0 \\ 0 & 1 & \overline{t_b} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{z'_a} & \overline{z'_a t'_b} \\ 0 & 0 & 0 & 0 & 0 & 1 & \overline{t'_b} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for all $a \in A$ and $b \in B$,

$$\rho(f_{i,j}) = \begin{pmatrix} 0 & 0 & 0 & \overline{z_a t_b x_k} & 0 & \overline{z_a t_b x_k y_l z'_c} & \overline{z_a t_b x_k y_l z'_c t'_d} \\ 0 & 0 & 0 & \overline{t_b x_k} & 0 & \overline{t_b x_k y_l z'_c} & \overline{t_b x_k y_l z'_c t'_d} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \overline{y_l z'_c} & \overline{y_l z'_c t'_d} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for all $f_{i,j} \in EFE$ such that $\alpha(a, b, k) = i$ and $\beta(l, c, d) = j$,

$$\rho(f_{i,j}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{x_i} & 0 & \overline{x_i y_l z'_c} & \overline{x_i y_l z'_c t'_d} \\ 0 & 0 & 0 & 1 & 0 & \overline{y_l z'_c} & \overline{y_l z'_c t'_d} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for all $f_{i,j} \in FE \setminus EFE$ such that $\beta(l,c,d) = j$,

$$\rho(f_{i,j}) = \begin{pmatrix} 0 & 0 & 0 & \overline{z_a t_b x_k} & \overline{z_a t_b x_k y_j} & 0 & 0 \\ 0 & 0 & 0 & \overline{t_b x_k} & \overline{t_b x_k y_j} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \overline{y_j} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for all $f_{i,j} \in EF \setminus EFE$ such that $\alpha(a,b,k) = i$, and

$$\rho(f_{i,j}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{x_i} & \overline{x_i y_j} & 0 & 0 \\ 0 & 0 & 0 & 1 & \overline{y_j} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for all $f_{i,j} \in F \setminus (EF \cup FE)$.

To prove that ρ is well-defined we need to check that $\overline{z_a t_b x_k} = \overline{z_{a'} t_{b'} x_{k'}}$, $\overline{t_b x_k} = \overline{t_{b'} x_{k'}}$, $\overline{y_l z'_c t'_d} = \overline{y_{l'} z'_{c'} t'_{d'}}$, $\overline{y_l z'_c} = \overline{y_{l'} z'_{c'}}$, for all $a, a', c, c' \in A$, $b, b', d, d' \in B$, $k, k' \in I$ and $l, l' \in J$ such that $\alpha(a,b,k) = \alpha(a',b',k')$ and $\beta(l,c,d) = \beta(l',c',d')$. Suppose that $\alpha(a,b,k) = \alpha(a',b',k')$ and $\beta(l,c,d) = \beta(l',c',d')$. By the definition of M, we have that $\overline{z_a t_b x_k} = \overline{z_{a'} t_{b'} x_{k'}}$ and $\overline{y_l z'_c t'_d} = \overline{y_{l'} z'_{c'} t'_{d'}}$. By (1),

$$\alpha(a,b,k) = \alpha(a,b,\alpha(a,b,k)) = \alpha(a,b,\alpha(a',b',k')) = \alpha(a,b',k').$$

Hence $\overline{t_b x_k} = \overline{t_{b'} x_{k'}}$. By (2),

$$\beta(l,c,d) = \beta(\beta(l,c,d),c,d) = \beta(\beta(l',c',d'),c,d) = \beta(l',c',d).$$

Hence $\overline{y_l z'_c} = \overline{y_{l'} z'_{c'}}$. Therefore ρ is well-defined.

We shall see that ρ is injective. Since M is generated by homogeneous polynomials of degree 2 and 3, it is clear that the restrictions of ρ to E and to $F \setminus (EF \cup FE)$ are injective. It is also clear that we only need to check that each of the restrictions of ρ to the disjoint sets EFE, $FE \setminus EFE$ and $EF \setminus EFE$ is injective. It is sufficient to prove that for $a, c \in A$, $b, d \in B$, $i, k \in I$ and $j, l \in J$,

$$\overline{z_a t_b x_i} = \overline{z_c t_d x_k} \Longrightarrow \alpha(a,b,i) = \alpha(c,d,k) \tag{3}$$

and

$$\overline{y_j z'_a t'_b} = \overline{y_l z'_c t'_d} \Longrightarrow \beta(j,a,b) = \beta(l,c,d). \tag{4}$$

Suppose that $\overline{z_a t_b x_i} = \overline{z_c t_d x_k}$ and that $\alpha(a,b,i) \neq \alpha(c,d,k)$. Since M is homogeneous with respect to the degrees in X, Y, Z, Z', T and in T',

$$z_a t_b x_i - z_c t_d x_k = \sum_{n=1}^{m} \lambda_n (z_{a_n} t_{b_n} x_{i_n} - z_{c_n} t_{d_n} x_{k_n}),$$

for some $\lambda_n \in K$ and some $a_n, c_n \in A$, $b_n, d_n \in B$ and $i_n, k_n \in I$ such that $\alpha(a_n, b_n, i_n) = \alpha(c_n, d_n, k_n)$, for all $n = 1, \dots, m$. Let $N_1 = \{n \mid 1 \leq n \leq m \text{ and } \alpha(a_n, b_n, i_n) = \alpha(a,b,i)\}$ and $N_2 = \{n \mid 1 \leq n \leq m \text{ and } \alpha(a_n, b_n, i_n) = \alpha(c,d,k)\}$. Then, clearly,

$$z_a t_b x_i = \sum_{n \in N_1} \lambda_n (z_{a_n} t_{b_n} x_{i_n} - z_{c_n} t_{d_n} x_{k_n})$$

and

$$-z_c t_d x_k = \sum_{n \in N_2} \lambda_n (z_{a_n} t_{b_n} x_{i_n} - z_{c_n} t_{d_n} x_{k_n}).$$

But this is impossible. Hence $\alpha(a,b,i) = \alpha(c,d,k)$ and (3) follows. Similarly we prove (4). Hence ρ is injective. It is easy to see that $\rho(e_{a,b} e_{c,d}) = \rho(e_{a,b})\rho(e_{c,d})$ and $\rho(f_{i,j} f_{k,l}) = \rho(f_{i,j})\rho(f_{k,l})$. Let $f_{i,j} = e_{a,b} f_{i,j} \in EF$ and $e_{c,d} \in E$. Then

$$\rho(f_{i,j}) = \begin{pmatrix} 0 & 0 & 0 & \overline{z_a t_b x_i} & \overline{z_a t_b x_i}\, r \\ 0 & 0 & 0 & \overline{t_b x_i} & \overline{t_b x_i}\, r \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for some $r \in M_{1\times 3}(\bar{R})$, and

$$\rho(e_{c,d})\rho(f_{i,j}) = \begin{pmatrix} 0 & 0 & 0 & \overline{z_c t_b x_i} & \overline{z_c t_b x_i}\, r \\ 0 & 0 & 0 & \overline{t_b x_i} & \overline{t_b x_i}\, r \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \rho(f_{\alpha(c,b,i),j}) = \rho(e_{c,b} f_{i,j}) = \rho(e_{c,d} e_{a,b} f_{i,j}) = \rho(e_{c,d} f_{i,j}).$$

Let $f_{i,j} \in F \setminus EF$ and $e_{c,d} \in E$. Then

$$\rho(f_{i,j}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{x_i} & \overline{x_i}\, r \\ 0 & 0 & 0 & 1 & r \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for some $r \in M_{1\times 3}(\bar{R})$, and

$$\rho(e_{c,d})\rho(f_{i,j}) = \begin{pmatrix} 0 & 0 & 0 & \overline{z_c t_d x_i} & \overline{z_c t_d x_i}\, r \\ 0 & 0 & 0 & \overline{t_d x_i} & \overline{t_d x_i}\, r \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \rho(f_{\alpha(c,d,i),j}) = \rho(e_{c,d} f_{i,j}).$$

Similarly, it is easy to see that we always have $\rho(f_{i,j} e_{a,b}) = \rho(f_{i,j})\rho(e_{a,b})$. Since $\rho(S)$ is K-linearly independent, ρ extends to a monomorphism of K-algebras $K[S] \to T_7(\bar{R})$. It is clear that $\bar{R} \cong K[C]$, where C is the monoid defined by the same presentation as $\bar{R}$.

3. Embeddings over a field

This section is devoted to the embeddability of bands, and of their semigroup algebras, into the matrix algebras over a field. We start with a discussion of a theorem formulated in [**4**, Theorem 10]. The proof given there is incorrect. Our first aim is to give a correct proof. For this, recall the following simple observation. Let L be a left ideal of an algebra R over a field K. Assume that the left R-module R/L is finite dimensional over K. Let R^1 be the standard extension of R to the algebra with unity. If I is the annihilator of the left R^1-module R^1/L, then the algebra R^1/I embeds into $\mathrm{End}_K(R^1/L)$ and hence it has finite dimension. Hence, L contains an ideal I of R^1 that has finite codimension.

In the proof we will also use the following lemma.

Lemma 3.1. *Let R be an algebra over a field K. Let I be a left ideal of R of finite codimension. Let M be an ideal of I of finite codimension. Then there exists an ideal W of R of finite codimension such that $W \subseteq M$.*

Proof: We can write $R = I + \sum_{i=1}^{m} a_i K$ for some $m \geq 1$ and some $a_i \in R$. Define

$$V_i = \{x \in M \mid a_i x \in M\}, \quad i = 1, \dots, m.$$

Then $a_i I x \subseteq I x \subseteq M$ for $x \in M$, so $I V_i \subseteq V_i$. It follows that $V' = \bigcap_{i=1}^{m} V_i$ is a left ideal of I. Define

$$\overline{V} = V' + \sum_{i=1}^{m} a_i V'.$$

Then $\overline{V}$ is a left ideal of the algebra R and $\overline{V} \subseteq M$. We claim that $\dim_K(R/\overline{V}) < \infty$. In order to check this, it is enough to show that $\overline{V}$ has finite codimension in M (since M has finite codimension in I and the latter has finite codimension in R). Therefore, it is sufficient to show that every V_i has finite codimension in M. Now, $V_i = \phi^{-1}(M)$, where $\phi\colon M \longrightarrow R$ is the linear map defined by $\phi(x) = a_i x$. Since M has finite codimension in R, also $\phi^{-1}(M)$ has finite codimension in M. The claim follows.

Hence, by the comment before Lemma 3.1, there exists also a two-sided ideal W of R of finite codimension such that $W \subseteq \overline{V} \subseteq M$. □

Now we can prove the following theorem, stated by Malcev in [**4**].

Theorem 3.2. *Let I be a left ideal of an algebra R over a field K. Assume that $\dim_K(R/I) < \infty$ and that there exists a positive integer n such that I embeds into the matrix ring $M_n(L)$ over a field extension L of K. Then R embeds into $M_k(L)$ for some k.*

Proof: First notice that the second part of the assumptions can be reformulated as follows: there exists an ideal M of the algebra $I \otimes_K L$ such that $I \cap M = 0$ and $\dim_L((I \otimes_K L)/M) < \infty$. Indeed, the latter is satisfied if the ideal M is defined as the kernel of the natural homomorphism $I \otimes_K L \longrightarrow M_n(L)$.

Since $I \otimes_K L$ is a left ideal of $R \otimes_K L$ and

$$\dim_L((R \otimes_K L)/(I \otimes_K L)) = \dim_K(R/I) < \infty,$$

by Lemma 3.1, there exists an ideal W of $R \otimes_K L$ such that $\dim_L((R \otimes_K L)/W) < \infty$ and $W \subseteq M$.

Since $I \cap M = 0$ and $M \subseteq I \otimes_K L$, we get $W \cap R \subseteq M \cap R = 0$. Thus R embeds into the finite dimensional L-algebra $(R \otimes_K L)/W$. It follows that R embeds into $M_k(L)$ for some k. □

In view of Theorem 2.2, it follows that the Jacobson radical of $K[S]$ is not embeddable into matrices over a field whenever S is a band with finitely many components such that $K[S]$ is not embeddable into matrices over a field.

The following general result on finite ideal extensions of semigroups is an analogue of Theorem 3.2.

Theorem 3.3. *Let S be a semigroup with an ideal I such that S/I is finite. Assume that I is a linear semigroup. Then S is a linear semigroup.*

Proof: By the hypothesis, there exist a field K and an ideal J of $K[I]$ such that I embeds into $K[I]/J$ and the latter is finite dimensional over K. Since $K[I]$ is an ideal of $K[S]$, by Lemma 3.1, there exists an ideal W of $K[S]$ of finite codimension and such that $W \subseteq J$.

Let $R = K[S]/W$. Suppose that $s - t \in W$ for some $s, t \in S$. Since $W \subseteq K[I]$, it follows that either $s = t$ or $s, t \in I$. But by the choice of J, I embeds into R under the natural map $K[S] \longrightarrow R$. Hence S embeds into the algebra R.

Then S has a faithful representation in the finite dimensional algebra R, whence applying the regular representation of the algebra R^1 obtained by adjoining an identity to R we get a faithful representation of S in some $M_n(K)$, as desired. □

We get the following immediate consequence of Theorem 3.2.

Corollary 3.4. *Let $S = F \cup E$ be a band with two components E, F and such that F is an ideal of S and E is finite. Then for every field K the algebra $K[S]$ embeds into $M_k(L)$ for a field extension L of K.*

The proof of Theorem 3.3 is not constructive. Moreover, when applied to a band $S = F \cup E$ with two components, with F an ideal of S and E finite, it does not give a chance for an extension to some cases where E is infinite. These are the main motivations for the construction given in Example 3.5. First, we give a simple observation.

Suppose that $S \subseteq M_n(K)$ is any band. Assume that $S \neq \{0\}$. Let I be the ideal of S consisting of all elements of the least nonzero rank and the zero matrix, if it is in S. Then S has a nonzero ideal $F \subseteq I$ which is a rectangular band or such that $F \setminus \{0\}$ is a rectangular band. Let j be the common rank of all matrices in F. Consider the exterior power map $\phi = \Lambda^j : M_n(K) \longrightarrow M_{\binom{n}{j}}(K)$. Then $\operatorname{rank}(\phi(a)) = \binom{\operatorname{rank}(a)}{j}$ for every $a \in S \setminus \{0\}$ and in particular $\operatorname{rank}(\phi(f)) = 1$ for every $f \in F \setminus \{0\}$ (see [**6**, Lemma 1.6]). Suppose that $a, b \in S$ are such that $\phi(a) = \phi(b)$. Then $\operatorname{rank}(a) = \operatorname{rank}(b)$. If $a \neq b$ then there exists $x \in M_n(K)$ such that $\operatorname{rank}(ax) = \operatorname{rank}(a)$ but $\operatorname{rank}(bx) < \operatorname{rank}(a)$, or there exists $y \in M_n(K)$

such that $\operatorname{rank}(ya) = \operatorname{rank}(a)$ but $\operatorname{rank}(yb) < \operatorname{rank}(a)$. (It is easy to see that otherwise a, b have equal kernels as elements of $M_n(K) \cong \operatorname{End}(K^n)$ acting on the left on K^n and also as maps acting on the row vector space on the right. Then these are two $\mathcal{H}$-related idempotents of the multiplicative monoid $M_n(K)$, whence they must be equal.) It follows that $\phi(a) \neq \phi(b)$, a contradiction. Thus, $\phi_{|S}\colon S \longrightarrow M_{\binom{n}{j}}(K)$ is an embedding. Therefore, if a band S is linear, then we may assume that a minimal nonzero ideal of S consists of matrices of rank at most 1. Our construction will be of this type and it generalizes the matrix embedding of any rectangular band F given in [**3**], mentioned earlier.

Example 3.5. Let $S = F \cup E$ be a two component band with F an ideal of S and $E = \{e_{i,j} \mid i \in \{1, \dots, m\}, j \in \{1, \dots, n\}\}$ such that $e_{i,j}e_{k,l} = e_{i,l}$. We give an explicit construction of an embedding of S (and of the algebra $K_0[S]$ for any given field K_0) into the matrix algebra $M_r(L)$ for some $r \geq 1$ and a field L.

Proof: Note that $Fe_{i,j} \supseteq Fe_{k,l}e_{i,j} = Fe_{k,j}$ and $e_{i,j}F \supseteq e_{i,j}e_{k,l}F = e_{i,l}F$. Hence

$$Fe_{i,j} = Fe_{1,j} \text{ and } e_{i,j}F = e_{i,1}F$$

for all i, j.

Fix some $f_0 \in F$. Consider the maps

$$\begin{array}{rcl} \varphi\colon Ff_0 & \longrightarrow & M_{m\times n}(Ff_0) \\ ff_0 & \longmapsto & (e_{i,j}ff_0) \end{array} \quad \text{and} \quad \begin{array}{rcl} \psi\colon f_0F & \longrightarrow & M_{m\times n}(f_0F) \\ f_0f & \longmapsto & (f_0fe_{i,j}). \end{array}$$

Let K be an infinite field such that $|K| \geq |S|$.

We will define a monomorphism $\rho\colon S \to M_r(L)$ for some $r \geq 3$ and some field extension L of K such that for all $f \in F$,

$$\rho(f) = \begin{pmatrix} 0_{r_1,r_1} & \mu(ff_0)^t & \mu(ff_0)^t\eta(f_0f) \\ 0_{1,r_1} & 1 & \eta(f_0f) \\ 0_{r_2,r_1} & 0_{r_2,1} & 0_{r_2,r_2} \end{pmatrix},$$

where $0_{p,q}$ denotes the zero matrix in $M_{p\times q}(L)$ and D^t denotes the transpose of the matrix D, and r_1, r_2 are positive integers such that $r_1 + r_2 + 1 = r$. The row vector $\mu(ff_0)$ will contain encoded information about ff_0 and $\varphi(ff_0)$, and the row vector $\eta(f_0f)$ will contain the information about f_0f and $\psi(f_0f)$.

Let $F_1 = \bigcup_{i=1}^m e_{i,1}F$, $F_2 = \bigcup_{j=1}^n Fe_{1,j}$, $\overline{F_1} = F \setminus F_1$ and $\overline{F_2} = F \setminus F_2$.

Let $f \in F$. We define $i(e_{k,l}f)$ to be the least positive integer such that $e_{k,l}ff_0 \in e_{i(e_{k,l}f),1}F$. Note that if $ff_0 \in F_1$ then

$$ff_0 = e_{i(f),1}e_{1,1}ff_0 \text{ and } e_{k,l}ff_0 = e_{i(e_{k,1}f),1}e_{1,1}ff_0.$$

Thus $(e_{1,1}ff_0, i(f), i(e_{1,1}f), i(e_{2,1}f), \dots, i(e_{m,1}f))$ contains all the information about ff_0 and $\varphi(ff_0)$ if $ff_0 \in F_1$. If $ff_0 \in \overline{F_1}$ then

$$e_{k,l}ff_0 = e_{i(e_{k,l}f),1}e_{1,l}ff_0.$$

Thus $(e_{1,1}ff_0, \dots, e_{1,n}ff_0)$ and $(i(e_{k,l}f)) \in M_{m\times n}(\mathbb{N})$ contain all the information about $\varphi(ff_0)$ if $ff_0 \in \overline{F_1}$. In order to encode the elements $i(e_{k,l}f)$ we use commuting indeterminates $x_{i(e_{k,l}f)}$ over K, thus we need the indeterminates $x_1, \dots, x_m$. To encode the elements $e_{1,l}ff_0$ we choose any injective map $\widetilde{\mu}\colon e_{1,1}F \longrightarrow K \setminus \{0\}$. To encode the elements $ff_0 \in \overline{F_1}$ we choose any injective map $\mu_0\colon \overline{F_1} \longrightarrow K \setminus \{0\}$.

Let $L_1 = K(x_1, \dots, x_m)$ be the field of fractions of the polynomial ring $K[x_1, \dots, x_m]$.

Let $\mu\colon Ff_0 \longrightarrow L_1^{m(n+1)+2}$ be the map defined by

$$\mu(ff_0) = (0, x_{i(f)}\widetilde{\mu}(e_{1,1}ff_0), x_{i(e_{1,1}f)}\widetilde{\mu}(e_{1,1}ff_0), \dots \\ \dots, x_{i(e_{m,1}f)}\widetilde{\mu}(e_{1,1}ff_0), 0, \dots, 0)$$

for all $ff_0 \in F_1$, and

$$\mu(ff_0) = (\mu_0(ff_0), 0, \dots, 0, \mu_{1,1}(ff_0), \dots, \mu_{1,n}(ff_0), \\ \mu_{2,1}(ff_0), \dots, \mu_{2,n}(ff_0), \dots, \mu_{m,1}(ff_0), \dots, \mu_{m,n}(ff_0))$$

for all $ff_0 \in \overline{F_1}$, where $\mu_{k,l}(ff_0) = x_{i(e_{k,l}f)}\widetilde{\mu}(e_{1,l}ff_0)$. Looking at the two first components of $\mu(ff_0)$ it is easy to see that μ is an injective map.

Let $f \in F$. We define $j(fe_{k,l})$ to be the least positive integer such that $f_0fe_{k,l} \in Fe_{1,j(fe_{k,l})}$. Note that if $f_0f \in F_2$ then

$$f_0f = f_0fe_{1,1}e_{1,j(f)} \text{ and } f_0fe_{k,l} = f_0fe_{1,1}e_{1,j(fe_{1,l})}.$$

Thus $(f_0fe_{1,1}, j(f), j(fe_{1,1}), j(fe_{1,2}), \dots, j(fe_{1,n}))$ contains all the information about f_0f and $\psi(f_0f)$ if $f_0f \in F_2$. If $f_0f \in \overline{F_2}$ then

$$f_0fe_{k,l} = f_0fe_{k,1}e_{1,j(fe_{k,l})}.$$

Thus $(f_0fe_{1,1}, \dots, f_0fe_{m,1})$ and $(j(fe_{k,l})) \in M_{m\times n}(\mathbb{N})$ contain all the information about $\psi(f_0f)$ if $f_0f \in \overline{F_2}$. To encode the $j(fe_{k,l})$ we use commuting indeterminates $y_{j(e_{k,l})}$ over K, thus we need the indeterminates $y_1, \dots, y_n$. To encode the elements $f_0fe_{k,1}$ we choose any injective map $\widetilde{\eta}\colon Fe_{1,1} \longrightarrow K \setminus \{0\}$. To encode the elements $f_0f \in \overline{F_2}$ we choose any injective map $\eta_0\colon \overline{F_2} \longrightarrow K \setminus \{0\}$.

Let $L = L_1(y_1, \dots, y_n)$ be the field of fractions of the polynomial ring $L_1[y_1, \dots, y_n]$.

Let $\eta\colon f_0F \longrightarrow L^{n(m+1)+2}$ be the map defined by

$$\eta(f_0f) = (0,\dots,0, y_{j(fe_{1,1})}\widetilde{\eta}(f_0fe_{1,1}),\dots \\ \dots, y_{j(fe_{1,n})}\widetilde{\eta}(f_0fe_{1,1}), y_{j(f)}\widetilde{\eta}(f_0fe_{1,1}),0)$$

for all $f_0f \in F_2$, and

$$\eta(ff_0) = (\eta_{1,1}(f_0f),\dots,\eta_{m,1}(f_0f),\eta_{1,2}(f_0f),\dots,\eta_{m,2}(f_0f),\dots \\ \dots,\eta_{1,n}(f_0f),\dots,\eta_{m,n}(f_0f),0,\dots,0,\eta_0(f_0f))$$

for all $f_0f \in \overline{F_2}$, where $\eta_{k,l}(f_0f) = y_{j(fe_{k,l})}\widetilde{\eta}(f_0fe_{k,1})$. Looking at the two last components of $\eta(f_0f)$ it is easy to see that η is an injective map.

Let I_n denote the identity matrix in $M_n(L)$, a_i denote the element of $M_{1\times m}(L)$ with 1 in position i and zeros elsewhere, b_j denote the element of $M_{n\times 1}(L)$ with 1 in position j and zeros elsewhere, and $E_{i,j} \in M_{m\times n}(L)$ denote the matrix whose (i,j)-entry is 1, while all the other entries are 0.

Let $C_j = (E_{1j}, E_{2,j},\dots,E_{mj}) \in M_{m\times(mn)}(L)$. Let

$$D_i = \begin{pmatrix} E_{i,1} \\ E_{i,2} \\ \vdots \\ E_{i,n} \end{pmatrix} \in M_{(mn)\times n}(L).$$

We define a function $\rho\colon S \longrightarrow M_{2mn+m+n+5}(L)$ by

$$\rho(e_{i,j}) = \begin{pmatrix}
0 & 0 & 0_{1,m} & 0_{1,mn} & 0 & 0_{1,mn} & 0_{1,n} & 0 & 0 \\
0 & 0 & a_i & a_iC_j & 0 & 0_{1,mn} & 0_{1,n} & 0 & 0 \\
0_{m,1} & 0_{m,1} & I_m & C_j & 0_{m,1} & 0_{m,mn} & 0_{m,n} & 0_{m,1} & 0_{m,1} \\
0_{mn,1} & 0_{mn,1} & 0_{mn,m} & 0_{mn,mn} & 0_{mn,1} & 0_{mn,mn} & 0_{mn,n} & 0_{mn,1} & 0_{mn,1} \\
0 & 0 & 0_{1,m} & 0_{1,mn} & I_1 & 0_{1,mn} & 0_{1,n} & 0 & 0 \\
0_{mn,1} & 0_{mn,1} & 0_{mn,m} & 0_{mn,mn} & 0_{mn,1} & 0_{mn,mn} & D_i & D_ib_j & 0_{mn,1} \\
0_{n,1} & 0_{n,1} & 0_{n,m} & 0_{n,mn} & 0_{n,1} & 0_{n,mn} & I_n & b_j & 0_{n,1} \\
0 & 0 & 0_{1,m} & 0_{1,mn} & 0 & 0_{1,mn} & 0_{1,n} & 0 & 0 \\
0 & 0 & 0_{1,m} & 0_{1,mn} & 0 & 0_{1,mn} & 0_{1,n} & 0 & 0
\end{pmatrix}$$

for all $i = 1,\dots,m$ and all $j = 1,\dots,n$, and

$$\rho(f) = \begin{pmatrix}
0_{mn+m+2,mn+m+2} & \mu(ff_0)^t & \mu(ff_0)^t\eta(f_0f) \\
0_{1,mn+m+2} & I_1 & \eta(f_0f) \\
0_{mn+n+2,mn+m+2} & 0_{mn+n+2,1} & 0_{mn+n+2,mn+n+2}
\end{pmatrix}$$

for all $f \in F$.

Since μ and η are injective, it follows that ρ is injective. It is easy to see that $\rho(fg) = \rho(f)\rho(g)$ for all $f,g \in F$ and that $\rho(e_{i,j}e_{k,l}) = \rho(e_{i,j})\rho(e_{k,l})$.

Let $f \in F$. Since $f_0 f = f_0 e_{i,j} f$, we have $\eta(f_0 f) = \eta(f_0 e_{i,j} f)$. Thus, in order to show that $\rho(e_{i,j} f) = \rho(e_{i,j})\rho(f)$, it is sufficient to see that

$$\begin{pmatrix} 0_{1,1} & 0_{1,1} & 0_{1,m} & 0_{1,mn} \\ 0_{1,1} & 0_{1,1} & a_i & a_i C_j \\ 0_{m,1} & 0_{m,1} & I_m & C_j \\ 0_{mn,1} & 0_{mn,1} & 0_{mn,m} & 0_{mn,mn} \end{pmatrix} \mu(ff_0)^t = \mu(e_{i,j} f f_0)^t.$$

Suppose that $ff_0 \in \overline{F_1}$. Then

$$\begin{pmatrix} 0_{1,1} & 0_{1,1} & 0_{1,m} & 0_{1,mn} \\ 0_{1,1} & 0_{1,1} & a_i & a_i C_j \\ 0_{m,1} & 0_{m,1} & I_m & C_j \\ 0_{mn,1} & 0_{mn,1} & 0_{mn,m} & 0_{mn,mn} \end{pmatrix} \mu(ff_0)^t = \begin{pmatrix} 0 \\ x_{i(e_{i,j}f)}\widetilde{\mu}(e_{1,j}ff_0) \\ x_{i(e_{1,j}f)}\widetilde{\mu}(e_{1,j}ff_0) \\ x_{i(e_{2,j}f)}\widetilde{\mu}(e_{1,j}ff_0) \\ \vdots \\ x_{i(e_{m,j}f)}\widetilde{\mu}(e_{1,j}ff_0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $e_{k,1} e_{i,j} = e_{k,j}$, we have

$$\begin{pmatrix} 0_{1,1} & 0_{1,1} & 0_{1,m} & 0_{1,mn} \\ 0_{1,1} & 0_{1,1} & a_i & a_i C_j \\ 0_{m,1} & 0_{m,1} & I_m & C_j \\ 0_{mn,1} & 0_{mn,1} & 0_{mn,m} & 0_{mn,mn} \end{pmatrix} \mu(ff_0)^t = \mu(e_{i,j} f f_0)^t,$$

in this case.

Suppose that $ff_0 \in F_1$. Then

$$\begin{pmatrix} 0_{1,1} & 0_{1,1} & 0_{1,m} & 0_{1,mn} \\ 0_{1,1} & 0_{1,1} & a_i & a_i C_j \\ 0_{m,1} & 0_{m,1} & I_m & C_j \\ 0_{mn,1} & 0_{mn,1} & 0_{mn,m} & 0_{mn,mn} \end{pmatrix} \mu(ff_0)^t = \begin{pmatrix} 0 \\ x_{i(e_{i,1}f)}\widetilde{\mu}(e_{1,1}ff_0) \\ x_{i(e_{1,1}f)}\widetilde{\mu}(e_{1,1}ff_0) \\ x_{i(e_{2,1}f)}\widetilde{\mu}(e_{1,1}ff_0) \\ \vdots \\ x_{i(e_{m,1}f)}\widetilde{\mu}(e_{1,1}ff_0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $e_{k,1}e_{i,1} = e_{k,1}$, we have

$$\begin{pmatrix} 0_{1,1} & 0_{1,1} & 0_{1,m} & 0_{1,mn} \\ 0_{1,1} & 0_{1,1} & a_i & a_iC_j \\ 0_{m,1} & 0_{m,1} & I_m & C_j \\ 0_{mn,1} & 0_{mn,1} & 0_{mn,m} & 0_{mn,mn} \end{pmatrix} \mu(ff_0)^t = \mu(e_{i,j}ff_0)^t,$$

in this case.

Hence $\rho(e_{i,j}f) = \rho(e_{i,j})\rho(f)$ for all $f \in F$. Similarly one can verify that $\rho(fe_{i,j}) = \rho(f)\rho(e_{i,j})$ for all $f \in F$. Therefore ρ is an injective homomorphism of semigroups.

Let K_0 be any field. Choosing μ and η in this construction in an appropriate way, we also get an embedding of the algebra $K_0[S] \hookrightarrow M_n(L)$ for a field extension L of K_0. Indeed, let X, Y, X' and Y' be pairwise disjoint sets of commuting indeterminates over K_0 such that $|X|, |Y|, |X'|, |Y'| \geq |S|$. Choose any injective maps $\widetilde{\mu}\colon e_{1,1}F \longrightarrow X$, $\widetilde{\eta}\colon Fe_{1,1} \longrightarrow Y$, $\mu_0\colon \overline{F_1} \longrightarrow X'$ and $\eta_0\colon \overline{F_2} \longrightarrow Y'$. Let $K = K_0(X \cup Y \cup X' \cup Y')$ be the field of fractions of the polynomial ring $K_0[X \cup Y \cup X' \cup Y']$. Then, applying the above construction, it is easy to see that $\rho(S)$ is K_0-linearly independent. Hence we get an algebra embedding $K_0[S] \hookrightarrow M_n(L)$. $\square$

Our second aim in this section is to discuss certain natural embeddability criterions for a band S, formulated in terms of annihilators associated to S. This approach seems justified because linearity of S implies certain finiteness conditions on annihilators. Let T be a semigroup. Let $s, t \in T^1$ and let I be an ideal of T. Then we may form the right annihilator $\mathrm{r}_I(s-t) = \{x \in I \mid sx = tx\}$, which is a right ideal of T, and similarly $\mathrm{l}_I(s-t)$, the left annihilator in I. Also, we define $\mathrm{r}_I(s) = \{(x,y) \in I \times I \mid sx = sy\}$. This is a restriction to I of a right congruence on T, whence a right congruence on I. A symmetric definition yields a left congruence $\mathrm{l}_I(s)$ on I. Recall that if $T \subseteq M_n(L)$ is a linear semigroup then every chain of annihilator (one-sided) ideals of the algebra $R = \mathrm{lin}_L(T) \subseteq M_n(L)$ has length bounded by n. This justifies the following conjecture. Let S be a band with finitely many components. Assume that every chain of one-sided ideals of the form $\mathrm{r}_S(s-t), \mathrm{l}_S(s-t), s, t \in S^1$, is of bounded length and every chain of one-sided congruences of the form $\mathrm{r}_S(s), \mathrm{l}_S(s), s \in S$, has bounded length. Then S (or maybe also $K[S]$ for any field K) embeds into matrices over a field L.

The information about the structure of a 2-component band $S = E \cup F$ is contained in properties of the action of E on Ff_0 by left multiplication and the action of E on f_0F by right multiplication. This is clear because $e_{ij}f = (e_{ij}ff_0)(f_0f)$ and hence $e_{ij}f$ is determined by f_0f and by $e_{ij}ff_0$; so by f and by the left action of e_{ij} on Ff_0. Similarly for fe_{ij}. So, necessary and sufficient conditions for embeddability of S must be in terms of conditions on these two actions: $\phi\colon E \longrightarrow T$, where T is the semigroup of all maps $Ff_0 \longrightarrow Ff_0$ and $\phi(e)$ is the left multiplication by e, and $\psi\colon E \longrightarrow T'$, where T' is the semigroup of all maps $f_0F \longrightarrow f_0F$ and $\psi(e)$ is the right multiplication by E. This approach is used below.

Lemma 3.6. *Let S be a semigroup with an ideal I and a subsemigroup E such that $S = I \cup E$ is a disjoint union and E is a rectangular band. If I is linear and there are only finitely many one-sided congruences $\mathrm{r}_I(x)$, $\mathrm{l}_I(x)$ and one-sided ideals $\mathrm{r}_I(1-x)$, $\mathrm{l}_I(1-x)$, where $x \in E$, then S is linear.*

Proof: Let $e \in E$. Assume that $\{\mathrm{r}_I(1-e) \mid e \in E\}$ is a finite set consisting of m elements. Let $e_1, \dots, e_m \in E$ be elements such that $\{\mathrm{r}_I(1-e) \mid e \in E\} = \{\mathrm{r}_I(1-e_1), \dots, \mathrm{r}_I(1-e_m)\}$. Consider the map $E \longrightarrow \{1, \dots, m\}$ defined by $e \mapsto i_e$, where $\mathrm{r}_I(1-e) = \mathrm{r}_I(1-e_{i_e})$. Now, for all $f \in I$, we have that $ef \in \mathrm{r}_I(1-e) = \mathrm{r}_I(1-e_{i_e})$, thus $ef = e_{i_e}ef$.

Assume that $\{\mathrm{r}_I(e) \mid e \in E\}$ is a finite set consisting of n elements. Let $e'_1, \dots, e'_n \in E$ be elements such that $\{\mathrm{r}_I(e) \mid e \in E\} = \{\mathrm{r}_I(e'_1), \dots, \mathrm{r}_I(e'_n)\}$. Consider the map $E \longrightarrow \{1, \dots, n\}$ defined by $e \mapsto j_e$, where $\mathrm{r}_I(e) = \mathrm{r}_I(e'_{j_e})$. Now, for all $f \in I$, we have that $(f, ef) \in \mathrm{r}_I(e) = \mathrm{r}_I(e'_{j_e})$, and thus $e'_{j_e}f = e'_{j_e}ef$. Therefore, multiplying by e on the left, we get $ee'_{j_e}f = ee'_{j_e}ef = ef$ (the latter equality holds because $xyz = xz$ for all $x, y, z \in E$).

Note that for all $e \in E$ and all $f \in I$ we have

$$ef = e_{i_e}ef = e_{i_e}ee'_{j_e}f = e_{i_e}e'_{j_e}f.$$

Let $\phi\colon E \longrightarrow T$, where T is the semigroup of all maps $I \longrightarrow I$ and $\phi(e)$ is the left multiplication by e. Then $\phi(E) \subseteq \{\phi(e_i)\phi(e'_j) \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$. Thus $\phi(E)$ is finite in this case.

Let $\psi\colon E \longrightarrow T$, where T is the semigroup of all maps $I \longrightarrow I$ and $\psi(e)$ is the right multiplication by e. As above, one shows that $\psi(E)$ is also finite.

Let $E_0 = \{(\phi(e), \psi(e)) \mid e \in E\} \subseteq \phi(E) \times \psi(E)$. Then E_0 is a finite rectangular band, as a homomorphic image of E (the operation in E_0 is defined by the rule $(\phi(e), \psi(e))(\phi(e'), \psi(e')) = (\phi(ee'), \psi(ee'))$). It is easy to see that $S_0 = E_0 \cup I$ is a semigroup with respect to the operation extending the given operations on E_0 and I and on the remaining pairs of elements defined by the rules: $(\phi(e), \psi(e))f = ef$ and $f(\phi(e), \psi(e)) =$

fe for $e \in E$, $f \in I$. By Theorem 3.3, there exists a monomorphism $\rho\colon S_0 \longrightarrow M_k(L)$ for some field L and some positive integer k . Next, let $\rho'\colon E \longrightarrow M_3(L)$ be an embedding (we know that it exists if L is big enough). Then define $\delta\colon S \longrightarrow M_k(L) \times M_3(L)$ by $\delta(f) = (\rho(f), 0)$ for $f \in I$ and $\delta(e) = (\rho((\phi(e), \psi(e))), \rho'(e))$ for $e \in E$. Then δ is a homomorphism and it is injective. $\square$

Let S be a band. Let E, E' be components of S. It is known that EE' and $E'E$ are subsets of the same component. We write $E' \preceq E$ in case $EE' \subseteq E'$. Thus $\preceq$ is a partial order on the set of all components of S. For any component E of S, we denote by $\mathcal{C}_E$ the set of all components E' of S such that $E \not\preceq E'$ and by I_E the subband of S defined by

$$I_E = \bigcup_{E' \in \mathcal{C}_E} E'.$$

Note that I_E is an ideal of S, whence also of the band $I_E \cup E$.

Theorem 3.7. *Let S be a band with finitely many components. If for any component E of S there are finitely many $\mathrm{r}_{I_E}(x)$, $\mathrm{l}_{I_E}(x)$ (one-sided congruences in I_E) and $\mathrm{r}_{I_E}(1-x)$, $\mathrm{l}_{I_E}(1-x)$ (one-sided ideals in $I_E \cup E$), where $x \in E$, then S is linear.*

Proof: Let $E_1, \ldots, E_r$ be the components of S. We may assume that $I_{E_r} = E_1 \cup \cdots \cup E_{r-1}$. Then $S = I_{E_r} \cup E_r$ and the result follows by induction on r using Lemma 3.6. $\square$

It was observed in [**3**] that, if a band S is linear, then for every subset $T \subseteq S$ there exists a finite subset $X \subseteq T$ such that $\mathrm{l}(X) = \mathrm{l}(T)$ and $\mathrm{r}(X) = \mathrm{r}(T)$. Here $\mathrm{l}(X)$ denotes the left congruence on S defined by $(a, b) \in \mathrm{l}(X)$ if $(a - b)X = 0$ in $K[S]$. And $\mathrm{r}(X)$ is defined dually. One can ask whether the converse is true. In this direction we have the following result.

Proposition 3.8. *Let S be a band with finitely many components. Assume that $\mathrm{r}_S(f) = \mathrm{r}_S(F)$, $\mathrm{l}_S(f) = \mathrm{l}_S(F)$ for every component F of S and for every $f \in F$. Then, for any field K, the semigroup algebra $K[S]$ embeds in $M_n(L)$ for some field extension L of K and some positive integer n.*

Before proving the assertion, recall that a band S is normal if $zxyz = zyxz$ for all $x, y, z \in S$, see [**7**]. We claim that the annihilator conditions in the above proposition are equivalent to saying that S is a normal band.

Indeed, let S be a normal band with a component S_γ. Assume that $f, f' \in S_\gamma$, $e \in S$. Using normality of S twice, we get

$$ef'f = ef'fef'f = eff'ef'f = eff'f'ef = ef \tag{5}$$

(the last equality follows because $(ef)f'f'(ef)$, ef are in the same component). By symmetry, we also get

$$ff'e = fe. \tag{6}$$

Furthermore, if $(e,g) \in \mathrm{l}_S(f)$ for some $g \in S$ then by (5) we get

$$ef' = eff' = gff' = gf'.$$

Hence $\mathrm{l}_S(f) = \mathrm{l}_S(S_\gamma)$. Similarly, one shows that $\mathrm{r}_S(f) = \mathrm{r}_S(S_\gamma)$.

On the other hand, if S satisfies the hypothesis of Proposition 3.8, then $(ef' - e)f' = 0$ implies that $(ef' - e)f = 0$, for all $e, f, f' \in S$ with f, f' in the same component of S. Hence (5) follows and similarly (6) follows. Let $x, y, z \in S$. Then, applying (5) and (6) we get

$$xyzx = x(xzy)(xyz)x = xzyx,$$

since xzy and xyz are in the same component of S. Therefore S is a normal band, which proves our claim.

Now we are ready for the proof of Proposition 3.8.

Proof: Assume that S satisfies the hypothesis. Let $\{S_\gamma \mid \gamma \in \Gamma\}$ be the set of all components of S. For every $\gamma \in \Gamma$ choose some $e_\gamma \in S_\gamma$. Let S^0_γ be the semigroup obtained by adjoining a zero element to S_γ. For $a \in S$ define $\phi_\gamma(a) = ae_\gamma a$ if $ae_\gamma a \in S_\gamma$ and $\phi_\gamma(a) = 0$ otherwise. We claim that $\phi_\gamma \colon S \longrightarrow S^0_\gamma$ is a homomorphism. Let $a, b \in S$. Then the elements $ae_\gamma abe_\gamma b$ and $abe_\gamma ab$ are in the same component of S. So $\phi_\gamma(ab) = 0$ if and only if $\phi_\gamma(a)\phi_\gamma(b) = 0$. Therefore we may assume that $ae_\gamma a, be_\gamma b, abe_\gamma ab \in S_\gamma$. Moreover, in this case,

$$abe_\gamma ab = (abe_\gamma)(e_\gamma ab)b = abe_\gamma b$$

by (6), because $abe_\gamma, e_\gamma ab \in S_\gamma$, and

$$abe_\gamma b = a(be_\gamma)(e_\gamma b) = ae_\gamma b$$

by (5) because $be_\gamma, e_\gamma b \in S_\gamma$. We also get

$$ae_\gamma abe_\gamma b = (ae_\gamma)(abe_\gamma)b = ae_\gamma b$$

because $ae_\gamma, abe_\gamma \in S_\gamma$. Therefore $\phi_\gamma(ab) = \phi_\gamma(a)\phi_\gamma(b)$.

This leads to the homomorphism $\phi \colon S \longrightarrow \prod_{\gamma\in\Gamma} S^0_\gamma$ defined by $\phi(s) = (\phi_\gamma(s))_{\gamma\in\Gamma}$. Since we know that for every γ there is an embedding $\chi_\gamma \colon K[S_\gamma] \longrightarrow M_3(L_\gamma)$ for a field extension L_γ of K, it follows that ϕ defines an algebra homomorphism $\psi = (\psi_\gamma)_{\gamma\in\Gamma} \colon K[S] \longrightarrow \prod_{\gamma\in\Gamma} M_3(L_\gamma)$, defined by $\psi_\gamma(s) = \chi_\gamma\phi_\gamma(s)$ for $s \in S$. As Γ is finite, the latter algebra embeds into $M_n(L)$ for some positive integer n and any field L containing all L_γ, $\gamma \in \Gamma$. Suppose $\psi(z) = 0$ for some nonzero $z = \sum_{s\in S} z_s s \in K[S]$. Let $\gamma \in \Gamma$ be a maximal element such that $\mathrm{supp}(z) \cap S_\gamma \neq \emptyset$. Let $z_\gamma = \sum_{s\in S_\gamma} z_s s$. By the construction of ϕ we get $\psi_\gamma(z) = \psi_\gamma(z_\gamma)$. Note that, if $s \in S_\gamma$ then $\phi_\gamma(s) = se_\gamma s = s$. Since χ_γ is an embedding, $\psi_\gamma(z_\gamma) = 0$ leads to a contradiction. This shows that ψ is an embedding. □

We note that the linearity of a normal band with finitely many components follows from the results in [**2**]. However, the embeddability of $K[S]$ into $M_n(L)$ for a field L containing K cannot be proved in this way.

Acknowledgements. The first author is grateful for the warm hospitality of the Institute of Mathematics of Warsaw University, where a part of this work was done.

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Rebut el 20 de setembre de 2007.