Aberth, 5, 194 ad hoc computability [see computability, counterexamples] adjoint (definition), 125-126 Ahlfors, 61 analytic continuation, 60-64 analytic functions, 50-51, 59-64 [see also computability] axioms (for computability) Banach space, 1, 3, 5, 11, 77–82 [Note. The axioms are used in the proof of virtually every theorem from Chapter 2 onwards. This is understood. We do not list all the pages, since such a list would grow so long that it would cease to be informative.] Banach space (statement), 81 computability structure, 1, 3, 5, 77–82, 85-87 [Note. Again, this notion permeates the book. Only the most important references are listed.] — computability structure (definition), 80 — Hilbert space [same as for a Banach space]

Banach/Mazur [see computability]
Banach space [see axioms, computability,

bounded operators, 1-2, 93-94, 96, 123,

128, 150-184 [see also First Main

First Main Theorem]

Banach space (definition), 8 Bishop, 4, 192, 194

Blum, 192

Theorem?

bounded operators (definition), 96 Bridges, 194 Brouwer, 4, 192

C, 8 [see also computability, Chapter 0] C^n , 8 [see also computability] C^{∞} , 8 [see also computability] C_o , 8 [see also computability] Caldwell, 12, 26, 62 [see also computability] Cauchy integral formula, 12, 60 characteristic function (definition), 8 Closed Graph Theorem (classical, noneffective), 97, 108 closed operators, 93-94, 96-100 [see also First Main Theorem] closed operators (definition), 96-97 Closure Criterion (First/Second), 98-100, 105, 108, 110, 116 CompNorm (in proof of the Second Main Theorem), 165–166 compact operators, 123, 129, 133, 136 comparisons (between real numbers/ rationals), 14-15, 23 Composition Property, 81 computability [For the underlying notion of computability on a Banach space, see "axioms, Banach space." The derived notions, for standard Banach spaces and related topics, are listed below. Theorems are listed elsewhere.] ad hoc, 5, 80, 90–92, 124, 134–142, 146-147 — analytic functions, 59–60 — Banach/Mazur, 28, 64–65 — Banach space [see axioms]

- Cⁿ, 50-59, 104-105, 117
- $--C^{\infty}$, 50, 54–57, 60
- $-C_o$, 84, 91, 107–108, 111, 118–119
- Caldwell/Pour-El, 12, 24–28 [see also Definition B and Chapter 0 computabity]
- Chapter 0 (an alternative designation for computability in the sense of Definitions A and B), 50, 79, 82–83, 94–95, 104–106, 115–116, 118–120, 149, 161–162
- complex numbers 14, 27
- continuous functions [same as Chapter 0 computability]
- Definition A, 12, 25–28, 28 (again in Section 4), 33, 36–37, 40, 44
- Definition B, 12, 25–28, 33, 36–37, 39, 44
- Definitions A and B (equivalence of), 36-37, 44-49
- double sequence, 18, 80
- energy norm, 95–96, 116–118
- Grzegorczyk/Lacombe, 4-5, 12, 24-28 [see also Definition A and Chapter 0 computability]
- Hilbert space [see axioms]
- inner products, 136-138
- intrinsic, 79, 82-85, 90-91
- $-L^p$, 5, 83–85, 94–95, 107–114
- $-\ell^p$, 85, 94, 107–111
- $-L^{\infty}$, 89–90
- $-\ell_o^{\infty}$, 85, 107–108, 110
- open sets, 193
- operators [see effectively determined operators]
- real numbers, 11, 13–17, 20
- rectangles, 25, 27
- sequences of rational numbers, 12, 14, 24
- sequences of real numbers, 11–12, 17–24
- sequences on a Banach space [see axioms]
- sequential (for continuous functions), 25, 28–29, 32, 34–35, 40, 51, 55, 64, 67–68, 71–72
- Sobolev spaces, 5, 95–96
- structure [see axioms]
- $-(0, \infty), 27-28$

continuous functions [see computability] convolution, 69–70

counterexamples (involving)

ad hoc computability, 90–92, 134– 139, 146

- analytic continuation for noncompact domains/sequences of functions, 62– 63
- converse parts (iii) and (iv) of the Second Main Theorem, 188–191
- creation and destruction of eigenvalues, 130–132
- derivatives, 51, 55, 58-59, 104-105
- derivatives of a sequence of functions, 59
- eigenvectors [see Eigenvector Theorem]
- entire functions, 62
- Fourier series, 105, 110
- Intermediate Value Theorem for sequences of functions, 42
- isometry of nonequivalent computability structures, 146
- $-L^{\infty}$, 89–90
- non closed operators, 105
- non normal operators, 132–133
- noncompact domains, 58, 62
- noncomputable real numbers, 12, 20, 129–130, 132–133, 135
- noneffective convergence, 11–12, 16, 19–20, 22–23, 105
- norm of an effectively determined operator, 129
- separable but not effectively separable Banach spaces, 88
- sequences of eigenvalues, 129, 189–191
- sequences of *n*-th derivatives, 55
- sequences of real numbers, 19–20, 22–24
- sequences of step functions, 112
- sequentially computable continuous functions, 67
- unbounded operators [see First Main Theorem]
- wave equation, 68–69, 72–73, 115–116, 120

Cook, 192

Cutland, 7

D

 $\mathcal{D}(T)$ [see domain] Davis, 7 degrees of unsolvability, 192 density (effective), 82, 85–87 density (not necessarily effective), 82, 85, 88

differentiation, 11, 40, 50–59, 60–62, 104–

105

Dirichlet norm [see computability, energy norm] distributions, 99–100, 108, 116 domain of an operator, 96–104, 125–126 Dunford/Schwartz, 110	G Gödel, 78 Grzegorczyk, 4, 5, 12, 25, 28, 104, 192–193 [see also computability] Grzegorczyk hierarchy, 192
effective — convergence, 11–24, 34, 37, 44–49, 56, 72, 81, 86–87, 105–106, 162 — convergence (definitions), 14, 18, 34, 81 — generating set, 78–79, 82–87, 93–94, 101, 104, 127 [Note. This notion permeates the book. Only the most important references are listed.] — generating set (definition), 82 — uniform continuity, 25–27, 29, 32, 34–35, 40, 50, 53–55, 64, 67–68, 71–72 — uniform continuity (definition), 25–27 effectively — determined operator, 2, 123–124, 127–129, 132–134, 138–139, 150, 158–160, 184–185, 187–188 — determined operator (definition), 127 — separable, 78–79, 82, 88–89, 128, 138, 141–142 — separable (definition), 82 eigenvalue, 2, 123–124, 126–127 [see also Second Main Theorem] eigenvalue (definition), 127 eigenvector, 2, 123–124, 126–127 [see also Eigenvector Theorem] eigenvector Theorem (statement), 133–134 elementary functions, 21 energy norm [see computability] exponential time, 192 extremal points, 148	H Hahn Banach Theorem (in Addendum on problems), 194 Halmos, 129, 151–153, 157 heat equation, 3, 70, 118–119 Hellwig, 116 Herbrand, 78 higher order recursion theory, 192 Hilbert space [see Second Main Theorem, Eigenvector Theorem] Hilbert space (definition), 8 Hilbert transform, 109 I I I ^q [see computable rectangle] InEq (in proof of the Second Main Theorem), 178 injection operator, 107–108 Insertion Property, 81 integration, 12, 33, 35, 37–40 intrinsic computability [see computability] isometries, 125, 145–148 [see also counterexamples] K Karp, 192 KdV equation, 194 Kirchhoff's formula, 12, 33, 73, 115–116 Kleene, 7, 25 Ko, 192 Kreisel, 2, 5, 13, 41, 193
F Feferman, 192 Feigenbaum's constant, 194 First Main Theorem 1-4, 69, 77, 93-96, 101-120 First Main Theorem (statement), 101 Fourier series, 83, 94-95, 105-106, 108-111 Fourier transforms, 3, 95, 108-111 Friedman, 192	L L^p , 8 [see also computability] ℓ^p , 8 [see also computability] L^∞ [see computability, counterexamples] ℓ_0^∞ [see computability] Lachlan, 5 Lacombe, 4, 5, 12–13, 25, 28, 41, 104, 193 [see also computability] Laplace's equation, 3, 70, 119 Limit Axiom [see axioms] Linear Forms Axiom [see axioms]

polynomial space, 192 linear independence [see Effective polynomial time, 192 Independence Lemma Post, 78 linear span (definition), 78 potential equation [see Laplace's equation] Loomis, 151-153 Pour-El, 50, 64–65, 68, 105–106, 111, 118– 120, 194 [see also Caldwell] primitive recursive functions, 192 M [-M, M] (in proof of the Second Main problems, 192-194 Theorem), 160 Markov, 78, 192 Mazur, 5, 17 [see also computability, Banach/Mazur] quantum mechanics, 2, 124, 126-127 Metakides, 5, 194 modulus of convergence, 16 monotone convergence, 20 real analytic functions, 64 Moschovakis, 5 Mostowski, 5, 24 real numbers [see computability] recursive function (description in terms of Mycielski, 28 Myhill, 5, 50–53, 105 Turning machines), 6 recursive set (definition), 7 recursive topology, 193 recursively enumerable nonrecursive set, 6-Nagy [see Riesz/Nagy] 7, 15, 22, 52, 56, 58, 62–63, 90, 102, Navier-Stokes equation, 194 104, 113, 129-130, 135, 146, 189 Nerode, 5, 194 recursively enumerable nonrecursive set non normal operators, 194 [see also (definition), 7 recursively enumerable set (definition), 6-7 counterexamples] nonlinear analysis, 194 recursively inseparable pair of sets, 7, 42, 65 Norm Axiom [see axioms] recursively inseparable pair of sets normal operators, 123, 126, 132-133, 157, (definition), 7 Rice, 5, 11-12, 16-17 184–187 [see also Second Main Theorem Richards, [see Pour-El] normal operators (definition), 126 Riemann Mapping Theorem (in Addendum Not an eigenvalue! (in proof of the Second on problems), 193 Main Theorem), 168–169, 176–177 Riemann surfaces, 193 Riesz Convexity Theorem (classical), 110 Riesz/Nagy, 125-129, 151-153, 157, 190 0 Robinson, 5, 17 operational calculus (for the Spectral Rogers, 5, 7 Theorem), 152–153, 161–162 operators [see bounded, unbounded, closed, self-adjoint, normal, compact; \mathcal{S} [see axioms, computability structure] see also effectively determined operator Sanin, 192 orthonormal basis, 136-137, 140-141 Schwartz [see Dunford/Schwartz] overlapping intervals, 149, 162, 166–167, Second Main Theorem, 1–4, 77, 123–124, 174 128-130, 149-191 Second Main Theorem (statement), 128 see saw construction, 44 self-adjoint operators, 2, 123, 125-126, P = NP problem, 192 128-132 [see also Second Main pairing function (definition), 7 Theorem, Eigenvector Theorem partial derivatives, 58, 72 self-adjoint operators (definition), 126 Petrovskii, 115

piecewise linear functions, 83, 112

sequences [see axioms, computability,

counterexamples]

Shepherdson, 5 Shohat/Tamarkin, 87 Shore, 5, 194 Shub [see Blum] Simpson, 5 Smale [see Blum] Soare, 7 Sobolev spaces [see computability] spectral measure, 149, 151-152, 166, 170-Specker, 5, 11-13, 16, 41 Spectral Theorem (classical, noneffective), 149, 151-157, 170 spectrum, 123-124, 126-127 [see also Second Main Theorem spectrum (definition), 126 SpThm (in proof of the Second Main Theorem), 153–157, 172 standard functions, 27 step functions, 5, 79, 84-85, 112-114 subrecursive hierarchies, 192

T

Tamarkin [see Shohat/Tamarkin] Taylor series, 60-61

Theorems [The redundant terms "computability/noncomputability" have largely been omitted in the indexed list which follows.]

- Analytic Continuation Theorem, 60
- Closure Under Effective Uniform Convergence, 34
- Compact Operators, 129
- Composition of Functions, 28-31
- Creation and Destruction of Eigenvalues, 130
- Differentiation Theorem for C^1 , 51, 104
- Differentiation Theorem for C^2 , 53
- Differentiation Theorem for the Sequence of n-th Derivatives, 55
- Effective Density Lemma, 86
- Effective Independence Lemma, 142
- Effective Modulus Lemma, 65
- Eigenvector Theorem, 133-134
- Entire Function Theorem, 62
- Expansion of Functions, 33
- Fejer's Theorem, 106
- First Main Theorem, 101
- Fourier Series (effective convergence of), 105
- Fourier Series and Transforms (L^p-computability of), 110
- Heat Equation, 119

- Integration Theorems, 35, 37–39, 104
- Intermediate Value Theorem, 41
- L^p -Computability for Varying p, 107
- Laplace's Equation, 119
- Max-Min Theorem, 40
- Mean Value Theorem, 44
- Non-Normal Operators (noncomputable eigenvalues), 132
- Operator Norm, 129
- Patching Theorem, 32
- Plancherel Theorem, 111
- Potential Equation [see Laplace's Equation]
- Real Closed Field, 44
- Riemann-Lebesgue Lemma, 111
- Second Main Theorem, 128
- Second Main Theorem (converse parts), 189–190
- Second Main Theorem (for normal operators), 184
- Second Main Theorem (for unbounded operators), 188
- Sequence of Eigenvalues (non-computability of), 129
- Sequentially Computable but Not Computable Continuous Functions, 67
- Stability Lemma, 87
- Step Functions, 114
- Stieltjes, Hamburger, Carleman Theorem, 87
- Third Main Theorem [see Eigenvector Theorem]
- Translation Invariant Operators, 71
- Waiting Lemma, 15
- Wave Equation Theorem (energy norm), 118
- Wave Equation Theorem (uniform norm), 116
- Wave Equation Theorem (weak solutions), 73
- Weierstrass Approximation Theorem, 45, 86
- Wiener Tauberian Theorem, 87 translation, 69-70 translation invariant operators, 51, 69-73

triangle functions (in proof of the Second

Main Theorem), 149, 162–165, 167 Turing, 78

Turing machine, 6

I

unbounded operators, 1–2, 93–94, 123, 157, 187–188 [see also First Main Theorem]

uniform convergence, 12, 18-19, 33-34, 37, 44-49, 56, 72, 86, 105-106, 162 Uniformity in the Exponent Lemma, 158-160

W

wave equation, 3, 51, 65, 68-70, 72-73, 115-118, 120 [see also counter-examples, Theorems] weak solutions, 73

weak topologies, 99-100, 108, 116 well understood functions, 112-114

X $\{x_n\}$ (in proof of the Second Main Theorem), 160

Z Zgymund, 110