# *Chapter XII*
# Selection and $k$-Sections

Selection theorems, unlike so many results in recursion theory, have the virtue of being positive. They reveal uniformities not immediately apparent to the most discerning recursion theorist. They are often based on farfetched computations that converge slowly. A typical selection theorem addresses the following question. Suppose $A$ is $E$-recursively enumerable in $b$, and $A \cap x$ is nonempty. Is there a uniform method for computing an element of $A \cap x$ from $b$, $x$? More generally, since $x$ may not be effectively wellorderable, is there a uniform procedure for computing a nonempty $y \subseteq A \cap x$ from $b$, $x$? As a rule $y$ is obtained by computing an ordinal $\theta$ such that the set of all computations from $b$ of length at most $\theta$ suffices to enumerate some elements of $A \cap x$.

Let Selection ($x$) denote the following principle: there exists a partial $E$-recursive function $f$ such that for all $e \leq \omega$ and all $b$,

$$(1) \qquad (\mathrm{Ez})_{z \in x}[\{e\}(z, b)\downarrow] \leftrightarrow f(e, b)\downarrow \quad \& \quad (\mathrm{Ez})_{z \in x}[|\{e\}(z, b)| \leq f(e, b)].$$

Gandy selection, Theorem 4.1.X, is equivalent to Selection ($\omega$).

Selection ($2^\omega$) and Selection ($\omega_1$) are false. If the first were true, then $E(2^\omega)$ would be $\Sigma_1$ admissible, but the existence of Moschovakis witnesses in $E(2^\omega)$ gives rise to a $\Sigma_1^{E(2^\omega)}$ map from $\omega \times 2^\omega$ onto $E(2^\omega)$. Connections between selection and admissibility are made in Section 2 below.

Restricting the enumeration parameter $b$ of (1) can make a difference, and even more so the introduction of a *special* parameter $p$ into $f$. For example, Grilliot selection, established in Section 1, implies (1) when

$$x = 2^\omega, \quad b \in 2^{2^\omega} \quad \text{and} \quad p = 2^{2^\omega}.$$

(The $f(e, b)$ of (1) is replaced by $f(e, b, p)$.)

To sum up prematurely, a selection theorem involves a fixed set $x$ and a collection $C$ of procedures for enumerating elements of $x$. $C$ is defined in terms of Gödel numbers $e$, enumeration parameters $b$, and additional predicates $R$. A typical member of $C$ is $\{e\}^R(z, b)$, where $z$ ranges over $x$. The selection theorem asserts the existence of a uniform method of computing a nonempty $y \subseteq x$ such that $\{c\}^R(z, b)\downarrow$ for all $z \in y$. $y$ is completed from $x$, $b$, $p$; $R$, where $p$ is a special parameter independent of $b$ and $R$.

## 1. Grilliot Selection

Let $f$ and $g$ be functions with domain $x$ and range $\subseteq \omega \times (2^x \times \{2^x\})$. For simplicity assume $x$ is transitive, closed under pairing and $\omega \subseteq x$. Define $g \leq f$ by

$$(z)_{z \in x}[g(z) \text{ is an immediate subcomputation instruction of } f(z)].$$

Let $\min f = \min \{|f(z)| \mid z \in x\}$.

Assume $\min f < \infty$. It was Grilliot's idea to compute $\min f$ from $f$, $2^x$ by recursion on the ordinals less than $\min f$. The recursion equation he had in mind was

$$\text{(1)} \qquad \min f = \max_{g < f}{}^{+} (\min g).$$

An immediate obstacle to the recursion is:

$$\{g \mid g < f\} \quad \text{is not } E\text{-recursive in } f, 2^x.$$

It is overcome by approximating $\{g \mid g < f\}$ $E$-recursively in $f$, $2^x$. For each ordinal $\beta$, $g <^{\beta} f$ is the $\beta$-th approximation of $g < f$. $g <^{\beta} f$ means: for all $z \in x$, $g(z)$ is seen to be an immediate subcomputation instruction of $f(z)$ by referring to the set of all computations (from $f, 2^x$) of length at most $\beta$. $f(z)$ is of the form $\langle e, \langle u, 2^x \rangle\rangle$, but from now on $2^x$ will often be suppressed. For example, suppose $f(z)$ is $\langle 2^m \cdot 3^n, u \rangle$. Then $\langle 2^m, u \rangle$ is seen to be an immediate subcomputation instruction of $f(z)$ without any reference to computations. If $|\langle 2^m, u \rangle| < \infty$ and $y \in \{m\}(u)$, then $\langle n, y \rangle$ is seen to be an immediate subcomputation instruction of $f(z)$ by referring to computations of length $\leq |\langle 2^m, u \rangle|$. Note that Exercise 1.6 makes it safe to regard every subcomputation instruction as being of the form $\langle d, u \rangle$ for some $u \subseteq x$. It follows from Lemma 2.7.X that

$$\{g \mid g <^{\beta} f\} \leq_E f, 2^x \quad \text{uniformly in } \beta, f, 2^x.$$

If $\beta$ is too small, then the use of $<^{\beta}$ in place of $<$ in (1) may produce a false minimum, a value less than $\min f$.

Harrington and MacQueen [1976] completed Grilliot's idea. They introduced a notion of iteration that has proved valuable in the study of selection. Their idea is to apply (1) to generate a positive contribution to $\min f$. A sufficiently long sequence of such contributions adds up to $\min f$. An ordinal computed at a given stage can serve as $\beta$ at the next stage.

The proof of Theorem 1.1 makes use of a set-theoretic choice principle, $AC_x$:

$$(f)[(z)_{z \in x}(f(z) \neq \phi) \rightarrow (\mathrm{E}g)(z)_{z \in x}(g(z) \in f(z))].$$

$AC_{2^\omega}$ holds if $V$, the class of all sets, satisfies ZFC. $AC_{2^\omega}$ need not hold in $L(2^\omega)$, the natural setting for the axiom of determinateness. Nonetheless, ZF implies Theorem 1.1 holds for $x = 2^\omega$ in $L(2^\omega)$ (cf. Exercise 1.2).

**1.1 Theorem** (Grilliot 1969, Harrington & MacQueen 1976). *Let* $f: x \to \omega \times (2^x \times \{2^x\})$ *and suppose* $|f(z)| < \infty$ *for some* $z \in x$. *Then*

$$\left(\min_{z \in x} |f(z)|\right) \leq_E f, 2^x \quad \textit{uniformly.}$$

*Proof.* Note that $x \leq_E f$. Let $\theta$ be the supremum of all ordinals $\beta$ such that there exists a map of $x$ onto $\beta$. Observe that $\theta \leq_E 2^x$ thanks to the assumptions on $x$ made at the beginning of Section 1.

$t(\gamma)$ is defined by recursion on $\gamma < \theta$ with the intention that $t(\gamma)$ equal $\min f$ for all sufficiently large $\gamma$. Thus $\min f$ will be $\sup\{t(\gamma) | \gamma < \theta\}$.

*Stage* $\gamma$. Let $t^-(\gamma) = \sup\{t(\delta) | \delta < \gamma\}$.
*Case I*: $t^-(\gamma) \geq \min f$. Set $t(\gamma) = \min f$.
*Case II*: $t^-(\gamma) < \min f$. Set

(1) $$t(\gamma) = \max{}^+ \{\min g | g <^{t^-(\gamma)} f \quad \& \quad t^-(\gamma) \leq \min g\}.$$

If case II holds and there is a $g$ that satisfies the rightside condition of (1), then $t^-(\gamma) < t(\gamma)$. To find such a $g$, $AC_x$ is invoked. For each $z \in x$, the sought-after $g$ has the following properties: $g(z)$ is seen to be an immediate subcomputation instruction of $f(z)$ by examining computations of length $\leq t^-(\gamma)$; $t^-(\gamma) \leq |g(z)|$. To check there exists a suitable choice for $g(z)$, let $f(z)$ be $\langle 2^m \cdot 3^n, u\rangle$. $\langle m, u\rangle$ is seen to be an immediate subcomputation instruction of $f(z)$ without any examination of computations. If $t^-(\gamma) \leq |\langle m, u\rangle|$, then $\langle m, u\rangle$ can serve as $g(z)$. Otherwise there is a $y \in \{m\}(u)$ such that $t^-(\gamma) \leq |\langle n, y\rangle|$. If there were no such $y$, then $|f(z)|$ would be less than $\min f$.

Suppose case II holds for every $\gamma < \theta$. Then a partial map of $x$ onto $\theta$ exists, contrary to the definition of $\theta$. For each $z \in x$, let $f(z)$ be $\langle 2^{m_z} \cdot 3^{n_z}, u_z\rangle$. Call $z$ troublesome if $f(z)\uparrow$ but $|\langle m_z, u_z\rangle| < \infty$. Fix $\gamma < \theta$. If $t^-(\gamma) \geq |\langle m_z, u_z\rangle|$ for every troublesome $z$, then $t(\gamma + 1) \geq \min f$ by reasoning as in the previous paragraph. Let $z$ be troublesome and such that $t^-(\gamma) < |\langle m_z, u_z\rangle|$ and $|\langle m_z, u_z\rangle|$ has the least possible value. Then $t(\gamma) \geq |\langle m_z, u_z\rangle|$. Map $z$ to $\gamma$. $\square$

**1.2 Exercise.** Show Theorem 1.1 holds for $x = 2^\omega$ in ZF plus $V = L(2^\omega)$. (Replace $AC_x$ by a "blurred" choice principle.)

The proof of Theorem 1.1 is unaffected by the presence of an additional class $R$ (cf. Section 5.XI for $\{e\}^R(u)$). If the values of $f$ are $\langle e, u; R\rangle$, then $\min f$ can be computed as above from $f$, $2^x$; $R$.

A more sage observation concerns the role of $2^x$ in the proof of 1.1. The true power set of $x$ is not needed. For example, let $x$ be an ordinal less than some cardinal in the sense of $L(\kappa)$. Thus $TC(x) = x$ and $(2^x \cap L(\kappa)) \in L(\kappa)$. The proof of 1.1 can now be repeated with the understanding that $f, g, \ldots$ are functions in $L(\kappa)$. $\theta$ becomes $x^+$, the next cardinal after $x$ in $L(\kappa)$. If an additional class $R$ is present, then some caution is necessary. $R$ might blow up $2^x$ to a "proper class"; that is, $2^x \cap L(\kappa; R)$ might not be an element of $L(\kappa; R)$ (cf. Section 5.XI for the definition of $L(\kappa; R)$). Of course $L(\kappa; R)$ had better be $E$-closed relative to $R$. There can be trouble even if $R \subseteq x$. All goes well if $L(\kappa; R) \cap 2^x$ is an element of $L(\kappa; R)$. (These matters are discussed further in Section 5.)

The next theorem is an excellent example of the power of Grilliot's idea despite the absence of the true power set operation.

**1.3 Theorem** (Normann). *Suppose $x$ is an unbounded subset of $\rho$ and $\rho$ is a regular cardinal in the sense of $E(x)$. If $\delta < \rho$ and $C \subseteq \delta$ is nonempty and E-recursively enumerable in $x$, then some element of $C$ is E-recursive in $\delta$, $x$ uniformly.*

*Proof.* Suppose $C$ is

$$\{z \mid z < \delta \quad \& \quad \{e\}(z, x)\downarrow\}.$$

Let $f(z)$ be $\langle e, \langle z, x\rangle\rangle$ for all $z < \delta$. To find an element of $C$, it suffices to compute $\min f$ as in the proof of Theorem 1.1 with some small changes. All functions such as $f$ and $g$ now map $\delta$ into $\omega \times (\rho \times \{x\})$. $\theta$ becomes $\rho$. Since $\rho$ is regular, Hajnal's theorem implies each bounded subset of $\rho$ in $E(x)$ belongs to $L(\rho, x)$. Hence $\{g \mid g <^{\beta} f\} \in E(x)$ for all $\beta \in E(x)$, and is in fact $E$-recursive in $\beta$, $x$, $f$ uniformly. □

**1.4 Corollary** (Normann). *Suppose $W \in E(2^\omega)$ is a wellordering of $2^\omega$ whose height is a regular cardinal in the sense of $E(2^\omega)$. If $a, c \in 2^\omega$ and $C \subseteq \{b \mid b <_W a\}$ is nonempty and E-recursively enumerable in $c$, $2^\omega$, then some $b \in C$ is E-recursive in $a$, $c$, $W$ uniformly.*

Corollary 1.4 was devised by Normann for the sake of his work on Post's problem for $E(2^\omega)$, and has also proved useful in forcing arguments over $E(2^\omega)$. Not much is known about the role of regularity in 1.4. There is a model of ZFC, due to Slaman 1983, in which there is a wellordering $W$ of $2^\omega$ in $E(2^\omega)$ of singular height in the sense of $E(2^\omega)$, and in which the conclusion of 1.4 holds. It is not known if there is a model of ZFC in which there is a singular $W$ in $E(2^\omega)$ as in Slaman's model, but in which the conclusion of 1.4 fails.

In Section 5 of this Chapter, Theorem 1.3 and Corollary 1.4 will be strengthened so as to allow an additional predicate. The Hajnal argument fails in the presence of an additional predicate, and consequently the Grilliot approach also fails.

**1.5–1.6 Exercises.**

**1.5.** Show $E(\omega_{\omega_1}^L)$ does not obey less-than-$\omega_2$ selection.

**1.6.** Assume $z$ is transitive, closed under pairing, and $\omega \subseteq z$. Suppose $v \in z$. Show each immediate subcomputation instruction of $\langle e, \langle v, z\rangle\rangle$ is equivalent to one of the form $\langle d, \langle u, z\rangle\rangle$ for some $u \in z$. (cf. 2.8.X, 4.11.X and 2.7.XI.

## 2. Moschovakis Selection

Recall that Gandy selection implies $E(\omega)$ is $\Sigma_1$ admissible, or restated, $E(R(\omega))$ is $\Sigma_1$ admissible. ($R(0) = \phi$, $R(\alpha + 1) = 2^{R(\alpha)}$, and $R(\lambda)$ is $\cup \{R(\alpha) | \alpha < \lambda\}$ for limit $\lambda$.) Moschovakis combined Gandy and Grilliot selection to show $E(R(\alpha))$ is $\Sigma_1$ admissible if $\alpha$ has cofinality $\omega$ inside $E(R(\alpha))$.

$k \in E(R(\alpha))$ is said to be an $\omega$-sequence through $\alpha$ if dom $k = \omega$ and sup range $k = \alpha$.

**2.1 Theorem** (Moschovakis). *Suppose*

$$E(R(\alpha)) \models [\textit{cofinality } \alpha = \omega].$$

*Let* $k \in E(R(\alpha))$ *be an* $\omega$*-sequence through a. Assume* $f\colon R(\alpha) \to \omega \times R(\alpha) \times \{R(\alpha)\}$. *If* $|f(x)| < \infty$ *for some* $x \in R(\alpha)$, *then*

$$\left( \min_x |f(x)| \right) \leq_E f, R(\alpha), k \quad (\textit{uniformly}).$$

*Proof.* A typical value of $f$ is $\langle e, u, R(\alpha) \rangle$, where $u \in R(\alpha)$. For simplicity suppress the parameter $R(\alpha)$. The immediate subcomputation instructions of $\langle e, u \rangle$ are of the form $\langle c, v \rangle$, where $c < \omega$ and $v \in E(R(\alpha))$. The "Grilliot aspect" of the proof below requires that $v$ be construed effectively as an element of $R(\alpha)$ and that can be managed by a one-one map of $\{e\}(u)$ into $R(\alpha)$. Exercise 2.7 supplies such a map by an effective transfinite recursion on $|\{e\}(u)|$. Thus the set of immediate subcomputation instructions of $f(x)$ can be written as

$$\{f(x)(y) | y \in S_x \subseteq R(\alpha)\}.$$

For example, consider $f(x) = \langle 2^m \cdot 3^n, u \rangle$. Take 0 to be in $S_x$ and $f(x)(0)$ to be $\langle 2^m, u \rangle$. If $\{m\}(u)\downarrow$, then $f(x)(y)$ ranges over $\{\langle n, y \rangle | y \in \{m\}(u)\}$ as $y$ ranges over $S_x - \{0\}$.

The procedure for computing $\min f$ has countably many cases. If $\min f < \infty$, then at least one case yields $\min f$. Gandy selection weaves the cases into a single procedure. The intention is to compute min f by a recursion on the ordinals less than $\min f$, referred to below as the main recursion.

*Case* $\langle i, j \rangle$. Assume: $i < j$,

$$\text{(1)} \qquad \min f = |f(x_0)| \quad \text{for some } x_0 \in R(k(i)),$$
$$|f(x_0)| = \sup{}^+ \{|f(x_0)(y)| \,|\, y \in S_{x_0} \cap R(k(j))\}.$$

The method of Theorem 1.1 is used to build up $\min f$ from $\min g$ ($g < f$) via the main recursion. Now $g < f$ means:

$$\text{dom } g = R(k(i));$$
$$(x)\,[x \in \text{dom } g \to g(x) \in S_x \cap R(k(j))].$$

For each $x \in R(k(i))$, let $S_x^\beta$ be the set of all $y \in S_x \cap R(k(j))$ such that $f(x)(y)$ is seen to be an immediate subcomputation instruction of $f(x)$ by examining computations of height at most $\beta$. $g <^\beta f$ now means:

$$(x)\,[x \in R(k(i)) \rightarrow g(x) \in S_x^\beta].$$

$\Pi\{S_x^\beta \mid x \in R(k(i))\}$ can be viewed as a subset of $R(k(j))^{R(k(i))}$, hence as a small subset of $R(\alpha)$. It follows that $\{g \mid g <^\beta f\}$ is $E$-recursive in $f$, $\beta$, $R(\alpha)$ uniformly. Thus the Grilliot idea, as described in Section 1, applies to this case because $R(\alpha)$ is closed under the power set operation. Let $\theta$ be the supremum of $|W|$ for all wellfounded $W$ in $R(\alpha)$. Then $R(k(i))$ cannot be mapped onto $\theta$, and so the argument of Theorem 1.1 can be repeated with $R(k(i))$ in place of $x$.
*Case* $\langle i, 0\rangle$. Assume (1) holds. Let

$$|f(x)|_j = \sup{}^+ \{|f(x)(y)| \mid y \in S_x \cap R(k(j))\}.$$

Assume $|f(x_0)|_j < |f(x_0)|$ for all $j < \omega$. For each $j < \omega$, let $f(x)_j$ be a computation instruction such that

$$|f(x)_j| = |f(x)|_j.$$

Define $g < f$ by:

$$\operatorname{dom} g = R(k(i)) \quad \text{and}$$
$$(x)(x \in R(k(i)) \rightarrow g(x) \in \{f(x)_j \mid j < \omega\}.$$

Now proceed as in case $\langle i, j\rangle$. Observe that $\omega^{R(k(i))}$ can be construed as a small subset of $R(\alpha)$; hence $\{g \mid g <^\beta f\}$ is $E$-recursive in $f$, $\beta$, $R(\alpha)$ uniformly.

The above cases give rise to a function $t$ with domain $\omega^2$ such that: $t$ is partial $E$-recursive in $f$, $R(\alpha)$, $k$; if case $(i, j)$ holds, then $t(i, j)\downarrow$ and equals $\min f$. If $\min f < \infty$, then Gandy selection computes an $i$ and $j$ such that $t(i, j)\downarrow$. □

**2.2 Grilliot Selection Functions.** For each $\langle e, b\rangle \in E(x)$, let

$$K_{e,b}^x = \{z \mid \{e\}(z, x, b)\downarrow\} \cap TC(x).$$

$f$ is said to be a Grilliot selection function for $E(x)$ if for all $\langle e, b\rangle$ in $E(x)$:

$$K_{e,b}^x \neq \phi \rightarrow \phi \neq f(e, b) \in E(x) \cap 2^{K_{e,b}^x}.$$

Note that $f$ chooses a nonempty subset, rather than an element of, $K_{e,b}^x$. If some wellordering of $TC(x)$ is $E$-recursive in $x$, then $f$ makes it possible to choose an element of $K_{e,b}^x$.

**2.3 Lemma.** *Let $f$ be a Grilliot selection function for $E(x)$. If $f$ is partial $E$-recursive in some $b \in E(x)$, then $E(x)$ is $\Sigma_1$ admissible.*

*Proof.* Let $g$ be a many-valued $\Sigma_1^{E(x)}$ map with domain $d$. The task at hand is to find an $r \in E(x)$ so that $g(v)$ has a value in $r$ for all $v \in d$. For simplicity assume $TC(x) = (TC(x))^{<\omega}$. Let $t$ be a partial $E$-recursive (in $x$) map of $\omega \times TC(x)$ onto $E(x)$. $t$ is a restriction of a universal partial $E$-recursive function. With the help of $t$, the range of $g$ can be thought of as a subset of $TC(x)$. Let $D$ be a $\Delta_0^{ZF}$ formula such that

$$g(v) = w \quad \text{iff} \quad E(x) \models (\text{Ey})D(v, w, y).$$

$D$ may have parameters from $E(x)$. Every member of $E(x)$ is $E$-recursive in $x$, $a$ for some $a \in TC(x)$, so it is safe to assume that the only parameters are $x$ and $a$ for some $a \in TC(x)$. For each $v \in d$, the set

$$K_v = \{z \mid t(z)\downarrow \quad \& \quad D(v, (t(z))_0, (t(z))_1)\}$$

is nonempty and $E$-recursively enumerable in $v$, $x$, $a$. $f$ computes a nonempty subset of $K_v$ from $v$. Then the desired $r$ is

$$\{(t(z))_0 \mid (\text{Ev})(z \in f(v) \quad \& \quad v \in d)\}. \qquad \square$$

The next lemma is a partial converse of Lemma 2.3.

**Lemma 2.4.** *Suppose $x$ is a set of ordinals and $E(x)$ is $\Sigma_1$ admissible. Then there exists a Grilliot selection function for $E(x)$ partial $E$-recursive via some $b \in E(x)$.*

*Proof.* According to Theorem 5.8.X there is a $y \in E(x)$ such that $\kappa_r^{x,y} \notin E(x)$. Since $x$ is a set of ordinals, there is a $w \in E(x)$ such that $w$ is a relation on ordinals and $\langle x, y\rangle \leq_E w$. ($w$ encodes the $\varepsilon$-relation restricted to $TC(\{\langle x, y\rangle\})$.) By reflection there is such a $w$ $E$-recursive in $x$, $y$. Hence $\kappa_r^{x,y} = \kappa_r^w$.

Suppose $C \subseteq TC(x)$ is nonempty and $E$-recursively enumerable in $x$, $a$ for some $a \in TC(x)$. $\kappa_r^{w,a} \geq \kappa_r^w$ by Lemma 5.5(iv).X. Since $C$ has an element via a computation in $E(x)$ it has one via a computation of height less than $\kappa_r^{w,a}$, hence one recursive in $w$, $a$. Thus a nonempty subset of $C$ can be computed from $w$, $a$. $\square$

**Corollary 2.5** (Moschovakis). *Suppose*

$$E(R(\alpha)) \models [\textit{cofinality } \alpha = \omega].$$

*Then $E(R(\alpha))$ is $\Sigma_1$ admissible.*

*Proof.* Combine Theorem 2.1 and Lemma 2.3. $\square$

**2.6 Admissibility of** $E(R(\alpha))$. Assume $\alpha \geq \omega$. When is $E(R(\alpha))$ $\Sigma_1$ admissible? A complete answer to this question is presently unavailable. If $\alpha$ is a successor, then $R(\alpha)^\omega \subseteq R(\alpha)$, and so $R(\alpha)$ admits Moschovakis witnesses, and hence is not $\Sigma_1$ admissible (cf. exercise 2.8).

Assume $\alpha$ is a limit. If

(1) $$E(R(\alpha)) \models [\text{cofinality } \alpha = \omega],$$

then $E(R(\alpha))$ is $\Sigma_1$ admissible by Corollary 2.5. This result can be stretched a bit. Let $g\colon \lambda \to \alpha$ be a cofinality sequence for $\alpha$ in $E(R(\alpha))$. The proof of Theorem 2.1 will work with $\omega$ replaced by $\lambda$ if Gandy selection is replaceable by less-than-$\lambda^+$ selection. (cf. Exercise 2.11.) The latter principle implies: there exists a uniform method of computing an element of a subset of $\lambda$, if that subset is nonempty and $E$-recursively enumerable in some $b \in E(R(\alpha))$.

If cofinality $\alpha > \omega$ in $V$, then $E(R(\alpha))$ admits Moschovakis witnesses and hence is not $\Sigma_1$ admissible. All that is actually needed here is: the $\Sigma_2^{E(R(\alpha))}$ cofinality of $\alpha$ is greater than $\omega$.

Suppose $\Sigma_2^{E(R(\alpha))}$ cofinality of $\alpha$ equals $\omega$ and (1) is false. This is a dead zone as far as the question of $\Sigma_1$ admissibility of $E(R(\alpha))$ is concerned.

### 2.7–2.11 Exercises

**2.7.** Find a recursive function $h$ such that for all $x$ and $u$ the following holds. If $u \in TC(x)$ and $\{e\}(TC(x), u)\downarrow$, then $\{h(e)\}(TC(x), u)\downarrow$ and its value is a one-one map of $\{e\}(TC(x), u)$ into $TC(x)$.

**2.8.** Show $E(R(\beta + 1))$ is not $\Sigma_1$ admissible for all $\beta \geq \omega$. Show $E(R(\omega_{\omega_1}))$ is not $\Sigma_1$ admissible.

**2.9** (Grilliot). Assume $\alpha$ is a successor and $P(x, y)$ is an $E$-recursively enumerable (in $R(\alpha)$) relation on $R(\alpha)$. Show

$$(\mathrm{Ey})\,[y \in R(\alpha - 1) \quad \& \quad P(x, y)]$$

is an $E$-recursively enumerable (in $R(\alpha)$) relation on $R(\alpha)$.

**2.10.** Let $\alpha$ be $\omega_{\omega_1}^L$. Show there is no uniform method for selecting an element of a nonempty $E$-recursively enumerable (in $\alpha$) subset of $\omega_1^L$.

**2.11.** Suppose $\gamma$ is the greatest cardinal in the sense of $E(\gamma)$, and is singular in $E(\gamma)$. Assume less-than-(cofinality $\gamma)^+$-selection holds as defined at the beginning of Section 4.XI. Show there is a uniform method for selecting an element of a nonempty $E$-recursively enumerable (in $p$) subset of $\gamma$, where $p \in E(\gamma)$.

## 3. Plus-One Theorems

In the manner of Kleene 1959 define

$$tp(0) = \omega,$$
$$tp(\alpha + 1) = 2^{tp(\alpha)}, \quad \text{and}$$
$$tp(\lambda) = \cup \{tp(\alpha) \mid \alpha < \lambda\} \quad \text{(limit } \lambda).$$

Note that $tp(\alpha)$ is transitive and $tp(\alpha) \subseteq tp(\beta)$ if $\alpha \leq \beta$. The elements of $tp(\alpha)$ are referred to as the *objects* of type $\alpha$. $tp(\alpha)$ is equivalent to $R(\alpha + \omega)$. Suppose $0 < \delta < \alpha + 2$; the *δ-section* of a type $\alpha + 2$ $F$ is:

(1) $$\delta - sc(F) = \{z \mid z \in tp(\delta) \quad \& \quad z \leq_E tp(\alpha); F\}.$$

Note the semicolon (;) preceding $F$ in (1). It indicates that $F$ is present as an additional class. Thus "$z \leq_E tp(\alpha); F$" means $z = \{e\}^F(tp(\alpha))$ for some $e$ (cf. Section 5.XI for further details of "; $F$".)

For all $\alpha$, ${}^{\alpha+2}E$ is the function from $tp(\alpha + 1)$ into 2 defined by:

$$^{\alpha+2}E(x) = \begin{matrix} 1 & \text{if } x \neq \varnothing \\ 0 & \text{if } x = \varnothing. \end{matrix}$$

${}^{\alpha+2}E$ is equivalent to a type $\alpha + 2$ object. In the language of Kleene 1959 an object $F$ of type $\alpha + 2$ is said to be *normal* if ${}^{\alpha+2}E$ is recursive (in the sense of Kleene) in $F$. $E$-recursion theory has normality built in; equality is an $E$-recursive predicate. An important result of Kleene 1959 is:

(2) $$1 - sc({}^2E) = \text{HYP}.$$

It is reasonable to think of $1 - sc({}^3E)$ as a continuation of the hyperarithmetic hierarchy of reals, since $E(tp(\delta))$ has a hierarchic structure imparted to it by Proposition 2.10.X. There are difficulties with this point of view, because some of the reals in $1 - sc({}^3E)$ result from computations of uncountable length. It is natural to ask, as Hinman and Shoenfield did, if there exists a type 2 object $F$ whose 1-section is the same as that of ${}^3E$. An affirmative answer means that type 3 objects are not essential to the definition of $1 - sc({}^3E)$. In addition it follows that the members of $1 - sc({}^3E)$ can be arranged in a hierarchy that resembles HYP, since $E(\omega; {}^2E)$ is similar in form to $E(\omega; F)$. $F$ can be regarded as a jump operator, and then $1 - sc({}^3E)$ is the result of iterating the $F$ jump through the ordinals $E$-recursive in $F$.

Moschovakis has asked if it is possible to conclude anything about the type of $F$ from the constitution of its $\delta$-sections ($\delta < \alpha + 2 =$ type of $F$). The answer appears to be: "very little".

**3.1 Theorem** (Sacks 1974, 1977). *Let $F$ be an object of type $\alpha + 2$. Assume: $\delta < \alpha + 2$; $\delta$ is a successor; and $\alpha$ is a successor if $\delta > 1$. Then there exists a $G$ of type $\delta + 1$ such that*

$$\delta - sc(F) = \delta - sc(G).$$

The proof of Theorem 3.1 splits in two: $\delta > 1$ and $\delta = 1$. Proposition 3.5 and Theorem 3.6 cover $\delta = 1$.

**3.2 Plus-One for $\delta > 1$.** The proof is an application of Grilliot selection to an otherwise straightforward forcing construction. $G$ is going to be a function from $tp(\delta) \times tp(\delta - 1)$ into 2. Strictly speaking $G$ is not of type $\delta + 1$, but $G$ is certainly equivalent to an object of type $\delta + 1$. For each such $G$ and $x \in tp(\delta)$, let

$$g(x) = \{y \mid y \in tp(\delta - 1) \quad \& \quad G(x, y) = 0\}. \tag{1}$$

Thus $g$ maps $tp(\delta)$ into $tp(\delta)$. For any set $b$ and class $C$ the structure $L(\beta, b; C)$ is defined by recursion on $\beta$.

$L(0, b; C) = TC(b)$.

$L(\beta + 1, b; C) =$ set of all sets first order definable over $L(\beta, b; C)$; "$x \in C$" is added to the atomic formulas of ZF.

$L(\lambda, b; C) = \cup \{L(\beta, b; C) \mid \beta < \lambda\}$.

For example, if $x \in L(\beta, tp(\delta - 1); G)$, then $g(x) \in L(\beta + 1, tp(\delta - 1); G)$.

Let $\gamma$ be the ordertype of a wellfounded relation whose field is $tp(\delta - 1)$, and which is $E$-recursive in $tp(\alpha)$; $F$. Let $\lambda_{\delta-1}^{tp(\alpha); F}$ be the sup of all such $\gamma$'s. $\kappa_0^{tp(\delta-1); G}$ is going to be $\lambda_{\delta-1}^{tp(\alpha); F}$. The plan is to construct $L(\kappa_0^{tp(\delta-1); G}, tp(\delta - 1); G)$ in countably many steps.

$p, q, r, \ldots$ are forcing conditions. $p$ specifies an object of the form $L(\beta, t(\delta - 1);$ $H)$, where $H$ maps $tp(\delta) \times tp(\delta - 1)$ into 2. The specification includes the diagram of $L(\beta, tp(\delta - 1); H)$, the set of all sentences true in $L(\beta, tp(\delta - 1); H)$ in the language of ZF with names for $H$ and all the members of $L(\beta, tp(\delta - 1); H)$. $p$ of course does not specify all of $H$, but only the partial object

$$L(\beta, tp(\delta - 1); H) \cap H. \tag{2}$$

Denote (2) by $H_p$. If $p$ specifies $L(\beta, tp(\delta - 1); H)$, then

$$L(\beta, tp(\delta - 1); H) = L(\beta, tp(\delta - 1); H_p).$$

A forcing condition $p$ must have the following properties.

(3) (i) $p$ specifies $L(\beta_p, tp(\delta - 1); H_p)$.

(ii) $p$ is first order definable over $L(\delta, tp(a); F)$ for some $\delta \leq_E tp(\alpha)$; $F$. The first order definition takes only $tp(\alpha)$ and $F$ as parameters; thus it is specifiable by a Gödel number $< \omega$. (Hence $p \leq_E tp(\alpha)$; $F$.)

(iii) $p$ has a *top*; namely, for some $e$

$$|\{e\}^{H_p}(tp(\delta - 1))| = \beta_p. \tag{4}$$

The presence of a partial, rather than total, object in (4) is legitimate. The computation of $\{e\}^{H_p}(tp(\delta - 1))$ calls values of $H_p$ only for arguments in $L(\beta_p,$ $tp(\delta - 1); H_p)$.

Clause (3)(iii) insures that $\beta_p \leq_E tp(\delta-1); H_p$.

Define $p \geq q$ to be: $\beta_p \leq \beta_q$ and $H_q \upharpoonright p = H_p$.

The forcing relation $\Vdash$ treats the logical connectives in standard fashion. In addition:

$$p \Vdash \{e\}^{\mathscr{G}}(\underline{z})\downarrow \quad \text{iff} \quad L(\beta_p, tp(\delta-1); H_p) \models \{e\}^{\mathscr{G}}(\underline{z})\downarrow.$$

($z \in L(\beta_p, tp(\delta-1); H_p)$, $\mathscr{G}$ is interpreted as $H_p$ and $\underline{z}$ as $z$.)

$$p \Vdash \{e\}^{\mathscr{G}}(\underline{z})\uparrow \quad \text{iff} \quad (q)_{p \geq q} \sim [q \Vdash \{e\}^{\mathscr{G}}(\underline{z})\downarrow].$$

A generic sequence $\{p_n | n < \omega\}$ is constructed by stages. $\cup\{p_n | n < \omega\}$ will specify $L(\kappa_0^{tp(\delta-1);G}, tp(\delta-1); G)$.

*Stage 0.* $p_0 = \varnothing$.

*Stage* $n+1$. Let $e = (n)_1$.

*Case 1*: $n$ is even. The purpose of this case is to settle whether or not $\{e\}^{\mathscr{G}}(tp(\delta-1))$ converges.

*Subcase 1a:* there is a $p \leq p_n$ such that

$$p \Vdash \{e\}^{\mathscr{G}}(tp(\delta-1))\downarrow. \tag{4b}$$

Such a $p$ can be chosen to be $p_{n+1}$ via Gandy selection, since (4b) is $E$-recursively enumerable.

*Subcase 1b*: otherwise. Then $p_{n+1}$ is $p_n$.

*Case 2*: $n$ is odd. Let $z_e$ be the $e$-th member of the $\delta$-section of $F$. The purpose of this case is to add $z_e$ to the $\delta$-section of $G$. According to (3)(iii) there is an $m$ such that

$$|\{m\}^{H_{p_n}}(tp(\delta-1))| = \beta_{p_n}. \tag{5}$$

Let $x_n$ denote $L(\beta_{p_n}, tp(\delta-1); H_{p_n})$. (5) implies $x_n$ is $E$-recursive in $tp(\delta-1); H_{p_n}$. As in Exercise 3.8, there is a $y_n \subseteq tp(\delta-1)$ such that $y_n \notin x_n$ and $y_n$ is first order definable over $x_n$. The definition in question is lightface; its only parameter is $tp(\delta-1)$ and it treats $H_{p_n}$ as an additional predicate. Extend $p_n$ to $p_{n+1}$ so that

$$\beta_{p_{n+1}} = \beta_{p_n} + 1, \quad \text{and} \tag{6}$$
$$H_{p_{n+1}}(y_n, y) = 0 \leftrightarrow y \in z_e.$$

Then $z_e$ can be computed from $tp(\delta-1); H_{p_{n+1}}$ with the aid of (5) and (6).

Let $p_\infty = \lim_n p_n$, $\beta_\infty = \sup \beta_{p_n}$ and $H_\infty = \cup\{H_{p_n} | n < \omega\}$. Thus $p_\infty$ specifies $L(\beta_\infty, tp(\delta-1); H_\infty)$. Clause (ii) of (3) implies $\beta_\infty \leq \lambda_{\delta-1}^{tp(\alpha);F}$. To see that $\beta_\infty \geq \lambda_{\delta-1}^{tp(\alpha);F}$, let $z_e$ be a wellfounded relation on $tp(\delta-1)$ $E$-recursive in $tp(\alpha); F$. Case 2 insures that $z_e \leq_E tp(\delta-1); H_{p_n}$ for some $n$. Case 1 guarantees that $|z_e|$, the ordertype of $z_e$, is computed from $tp(\delta-1); H_{p_n}$ for some $n$. Hence $|z_e| \leq \beta_n$ for some $n$.

Let $G = H_\infty$ on domain $H_\infty$, and 0 elsewhere. Then $\kappa_0^{t(\delta-1);G} \geq \beta_\infty$, and by case 2, the $\delta$-section of $F$ is the set of all $z \in tp(\delta)$ such that $z \leq_E tp(\delta-1); G$ via a computation of height $<\beta_\infty$.

It remains only to show $\kappa_0^{t(\delta-1);G} \leq \beta_\infty$. Suppose not. Then for some $d$,

$$\infty > |\{d\}^G(tp(\delta-1))| \geq \beta_\infty.$$

Let

(7a) $\quad \gamma_0 = |\{d\}^G(tp(\delta-1))|$, and let $r$ specify $L(\gamma_0, tp(\delta-1); G)$.

Observe that $r$ satisfies clauses (i) and (iii) of (3) (with $r$ in place of $p$), and also (7b) and (8).

(7b) $\quad r$ is first order definable over $L(\delta, tp(\alpha); F)$ for some $\delta$.

(8) $\quad r \leq_E b, tp(\alpha); F \quad$ for some $b \in tp(1)$.

$\kappa_0^{tp(\alpha);F}$ can serve as $\delta$ in (7), partly because any object of the form

$$L(\gamma, tp(\delta-1); G) \quad (\gamma < \kappa_0^{tp(\delta-1);G})$$

can be construed as an object of type $\alpha$. $H_\infty$ was defined over $\kappa_0^{tp(\alpha):F}$. The $b$ of (8) is a map from $\omega$ into $\omega$ such that for all $n$, $p_n = \{b(n)\}^F(tp(\alpha))$. $r$ is not quite a forcing condition; nonetheless it makes sense to say

(9) $$r \Vdash \{\underline{d}\}^{\mathscr{G}}(t(\delta-1))\downarrow.$$

Fix $n$ so that $(n)_1 = d$ and $n$ is even. Let $Q$ be the set of all $q \leq p_n$ that satisfy (3)(i) and (3)(iii) (with $q$ in place of $p$), and (7) (with $q$ in place of $r$). Then

$$L(\kappa_0^{b,tp(\alpha);F}, b, tp(\alpha); F) \cap Q \neq \varnothing$$

It follows from Lemma 3.3 (simple reflection), a consequence of Grilliot selection, that

$$\kappa_r^{tp(\alpha);F} \geq \kappa_0^{b,tp(\alpha);F}.$$

Consequently there is a $q \in Q$ such that $q \leq_E tp(a); F$. One such $q$ can be found by first minimizing the $\delta$ of (7b) and then minimizing the Gödel number of the required first order definition. But then subcase (1a) of stage $n+1$ guarantees that

$$|\{d\}^G(tp(\delta-1))| < \beta_\infty. \quad \square$$

**Lemma 3.3** (Sacks 1974). *Assume $\alpha$ is a successor and $F$ is an object of type $\alpha+2$. Then*

$$\kappa_r^{tp(\alpha);F} \geq \kappa_0^{a,tp(\alpha);F}$$

*for all $a \in tp(\alpha-1)$.* *(Simple reflection)*

*Proof.* Theorem 1.1 (Grilliot selection) is easily revised to show: let $f$ map $tp(\alpha-1)$ to values of the form $\{e\}^F(y, tp(\alpha))$ $(y \in tp(\alpha))$ and suppose $|f(z)| < \infty$ for some $z \in tp(\alpha-1)$; then

$$\left(\min_z |f(z)|\right) \leq_E f, tp(\alpha); F.$$

Let $\mathscr{S}$ be a $\Sigma_1^{\mathrm{ZF}}$ sentence whose only parameters are $tp(\alpha)$ and $F$. Suppose

$$(\mathrm{E}a)\,[a \in tp(\alpha-1) \quad \& \quad L(\kappa_0^{a,tp(\alpha);F}, tp(\alpha); F) \models \mathscr{S}].$$

Thus there is an $a \in tp(\alpha-1)$ and a $\delta \leq_E a, tp(\alpha); F$ such that

$$L(\delta, tp(a); F) \models \mathscr{S}.$$

Grilliot selection implies the least such $\delta$ is $E$-recursive in $tp(\alpha); F$. □

**3.4 Plus-One for $\delta = 1$.** Suppose $Z$ is a countable subset of $tp(1)(=2^\omega)$. $Z$ is said to be an *abstract 1-section* if there is a $\Sigma_1$ admissible set $A$ satisfying $\Sigma_1$ dependent choice and such that $Z = 2^\omega \cap A$. Note that $Z$ is an abstract 1-section iff $Z_A$, the set of sets coded by reals in $Z$, satisfies $\Sigma_1$ dependent choice and hence is $\Sigma_1$ admissible (cf. Exercise 3.9). Proposition 3.5 and Theorem 3.6 imply the plus-one theorem for $\delta = 1$.

**3.5 Proposition.** *If $F$ is an object of type $\alpha+2$, then $1-sc(F)$ is an abstract 1-section.*

*Proof.* Let $w$ be a real and $y$ a set. Let "$w$ codes $y$" mean: $w$ codes the diagram of $\langle TC(\{y\}), \varepsilon\rangle$. Then the relation, $w$ codes $y$, is $E$-recursively enumerable.

Let $A$ be the set of sets coded by reals in $1-sc(F)$. Suppose

$$\langle A, \varepsilon\rangle \models (\mathrm{E}y)D(\underline{a}, y)$$

for some $a \in A$ and $\Sigma_1^{\mathrm{ZF}}$ $D$. The set

$$(1) \qquad \{w \mid w \in 1-sc(F) \quad \& \quad w \text{ codes } y \quad \& \quad D(\underline{a}, y)\}$$

is $E$-recursively enumerable in $a, tp(\alpha); F$ uniformly. By Gandy selection there is a method for computing an element of (1) from $a, tp(\alpha); F$ uniformly. The method, iterated $\omega$ times, establishes $\Sigma_1$ DC in $A$. □

**3.6 Theorem** (Sacks 1970). *If $Z$ is an abstract 1-section, then there exists a type 2 $G$ such that $1-sc(G) = Z$.*

*Proof.* $G$ is going to be a function from $2^\omega \times \omega$ into 2. The construction of $G$ roughly resembles that of Section 3.2. The reflection argument that concludes 3.2 will be replaced by an application of $\Sigma_1$ dependent choice within $Z_A$, the set of all sets coded by reals in $Z$. Consequently the details of the forcing argument now matter more than they did in 3.2. As before a forcing condition $p$ has the following properties.

(1) (i) $p$ specifies $L(\beta_p, \omega; H_p)$, and $L(\beta_p, \omega; H_p) \models (x)$ ($x$ is countable).
(ii) $p \in Z_A$.
(iii) $p$ has a *top*; namely, for some $m$

$$|\{m\}^{H_p}(\omega)| = \beta_p.$$

The second half of (i) can be deleted (cf. Exercise 3.11). The forcing relation $\Vdash$ is defined as in 3.2 with some extra clauses for bounding scheme situations.

$$p \Vdash \{2c \cdot 3d\}^{\mathscr{G}}(\omega)\downarrow \quad \text{iff} \tag{2}$$

$$p \Vdash \{c\}^{\mathscr{G}}(\omega)\downarrow \quad \text{and} \tag{3}$$

$$p \Vdash (x)[x \in \{c\}^{\mathscr{G}}(\omega) \rightarrow \{d\}^{\mathscr{G}}(\omega, x)\downarrow]. \tag{4}$$

By Exercise 2.7 it is safe to regard $\{c\}^G(\omega)$ as a subset of $\omega$. Thus if (3) holds, then (4) means:

$$(q)_{p \geq q}(i)_{i \in \omega}(\mathrm{E}r)_{q \geq r}[r \Vdash i \notin \{c\}^{\mathscr{G}}(\omega) \quad \text{or} \quad r \Vdash \{d\}^{\mathscr{G}}(\omega, i)\downarrow]. \tag{5}$$

The generic sequence $\{p_n | n < \omega\}$ is constructed by stages as in 3.2.
*Stage 0.* $p_0 = \varnothing$.
*Stage* $n+1$. let $e = (n)_1$.
*Case 1*: $n$ is even.
*Subcase 1a*: there is a $p \leq p_n$ such that

$$p \Vdash \{e\}^{\mathscr{G}}(\omega)\downarrow.$$

Any such $p$ will do for $p_{n+1}$.
*Subcase 1b*: otherwise. If $e$ is not of the form $2^c \cdot 3^d$ let $p_{n+1}$ be $p_n$. Otherwise there is a $q$ such that either

$$q \Vdash \{c\}^{\mathscr{G}}(\omega)\uparrow \tag{6}$$

or for some $i < \omega$,

$$q \Vdash \{c\}^{\mathscr{G}}(\omega)\downarrow \quad \text{and} \tag{7}$$

$$q \Vdash i \in \{c\}^{\mathscr{G}}(\omega) \quad \text{and} \quad q \Vdash \{d\}^{\mathscr{G}}(\omega, i)\uparrow.$$

Any such $q$ will do for $p_{n+1}$.

*Case 2*: $n$ is odd. Let $z_e$ be the $e$-th element of $Z$. Let $x_n$ denote $L(\beta_{p_n}, \omega; H_{p_n})$. Since $p_n$ has a top, $\beta_{p_n}$ is the length of a computation from $\omega; H_{p_n}$. As in Exercise 3.8, there is a $y_n \subseteq \omega$ such that $y_n \notin x_n$ and $y_n$ is first order definable over $x_n$. The definition in question is lightface; its only parameter is $\omega$ and it treats $H_{p_n}$ as an additional atomic predicate. Extend $p_n$ to $p_{n+1}$ so that

$$(8) \qquad \beta_{p_{n+1}} = \beta_{p_n} + 1, \quad \text{and} \quad H_{p_{n+1}}(y_n, y) = 0 \leftrightarrow y \in z_e.$$

Then $z_e$ can be computed from $\omega; H_{p_{n+1}}$.

Let $p_\infty = \lim_n p_n$, $\beta_\infty = \sup \beta_{p_n}$ and $H_\infty = \cup \{H_{p_n} | n < \infty\}$. Thus $p_\infty$ specifies $L(\beta_\infty, \omega; H_\infty)$. Let $\lambda_0^Z$ be the least ordinal not in $Z_A$. Clause (ii) of (1) implies $\beta_\infty \leq \lambda_0^Z$. To check that $\beta_\infty \geq \lambda_0^Z$, let $z_e \in Z_A$ be a wellfounded relation on $\omega$. Case 2 insures $z_e \leq_E \omega; H_{p^n}$ for some $n$. Case 1 guarantees that $|z_e|$, the ordertype of $z_e$, is computed from $\omega; H_{p_n}$ from some $n$, hence $|z_e| \leq \beta_{p^n}$ for some $n$.

Let $G = H_\infty$ on the domain of $H_\infty$, and 0 elsewhere. Then $\kappa_0^{\omega; G} \geq \beta_\infty$, and by case 2,

$$2^\omega \cap L(\beta_\infty, \omega; G) = Z_A.$$

It need only be shown that $\kappa_0^{\omega; G} \leq \beta_\infty$. Suppose not. Then there is an $e$ such that

$$(9) \qquad |\{e\}^{\mathscr{G}}(\omega)| = \beta_\infty.$$

Since $\beta_\infty$ is a limit, $e$ can be presumed to be $2^c \cdot 3^d$. Go to stage $n + 1$ for $n$ even and $(n)_1 = e$. If subcase (1b) holds, then either (6) or (7) holds (with $p_{n+1}$ in place of $q$), contrary to (9). Hence subcase (1a) holds. Thus

$$p_{n+1} \Vdash |\{c\}^{\mathscr{G}}(\omega)| \leq \beta_{p_{n+1}},$$

and (5) holds with $p_{n+1}$ in place of $q$. The matrix of (5) is $\Sigma_1$ over $Z_A$. Since $\Sigma_1$ *DC* holds in $Z_A$, there is a function $\lambda_i | r_i$ in $Z_A$ such that $p_{n+1} = r_0$ and

$$(10) \qquad (i)_{i\in\omega}[r_i \geq r_{i+1} \quad \& \quad (r_{i+1} \Vdash i \notin \{c\}^{\mathscr{G}} \quad \text{or} \quad r_{i+1} \Vdash \{d\}^{\mathscr{G}}(\omega, i)\downarrow)].$$

Let $r_\infty = \cup \{r_i | i < \omega\}$. $r_\infty$ specifies $L(\beta_{r_\infty}, \omega; H_{r_\infty})$. $r_\infty$ is an excellent candidate for a condition that forces

$$|\{e\}^{\mathscr{G}}(\omega)| \leq \max{}^+(\beta_{p_{n+1}}, \beta_{r_\infty}) < \beta_\infty,$$

thereby contradicting (9). The only difficulty is the possibility that $r_\infty$ does not have a top. It is handled by "minimizing" $\beta_{r_i}$. For each $i$, require that no proper initial segment of $r_{i+1}$ satisfy the matrix of (10). A *proper initial segment* of $r_{i+1}$ is a condition that specifies a proper initial segment of $L(\beta_{r_{i+1}}, \omega; H_{r_{i+1}})$. It follows that $\lambda_i | r_i$ is $\Sigma_1$ definable over $L(\beta_{r_\infty}, \omega; H_{r_\infty})$ but is not a member of it. Therefore

$L(\beta_{r_\infty}, \omega; H_{r_\infty})$ is not $\Sigma_1$ admissible, hence not $E$-closed by Gandy selection (cf. Exercise 3.10). In other words $r_\infty$ has a top. $\square$

**3.7 Extended Plus-One.** Suppose $1 \leq \delta < \alpha + 2$ and $F$ is a type $\alpha + 2$ object. The extended $\delta$-section of $F$ is denoted by $ex - \delta - sc(F)$, and equals

$$\{z \mid z \in tp(\delta) \quad \& \quad (\mathrm{E}b)_{b \in tp(\delta - 1)} \quad (z \leq_E b, tp(\alpha); F)\}.$$

If $\delta = 1$, then $ex - \delta - sc = \delta - sc$. The extended 2-section of ${}^3E$ is simply $2^{2^\omega} \cap E(2^\omega)$ and conveys the same information as $E(2^\omega)$. It is hard not to ask: does there exist a type 3 $G$ such that

$$ex - 2 - sc(G) = ex - 2 - sc({}^4E)?$$

Harrington has obtained a negative answer by assuming $AD$. On the other hand a positive answer follows from $2^\omega = \omega_1$. Normann has obtained a positive answer if (i) $2^\omega = \omega_2$ and Martin's axiom holds, or if (ii) there is a wellordering $W$ of $2^\omega$ in $E(2^{2^\omega})$ such that $|W|$ is regular inside $E(2^{2^\omega})$. Slaman has found a generic extension of $L$ in which: (a) the answer is yes, and (b) there is a wellordering $V$ of $2^\omega$ in $E(2^{2^\omega})$ such that $|V|$ is a singular cardinal in $E(2^{2^\omega})$.

At this writing it is not known if there is a generic extension of $L$ in which the answer is no and some wellordering of $2^\omega$ belongs to $E(2^{2^\omega})$.

Hoole (1982) defines an abstract extended 2-section and shows every such is the extended 2-section of a type 3 object.

### 3.8–3.12 Exercises

**3.8.** Suppose $\beta = |\{e\}(x)|$. Show there is a $y \subseteq TC(\{x\})$ such that $y \notin L(\beta, TC(\{x\}))$ and $y$ is first order definable over $L(\beta, TC(\{x\}))$ via a definition whose only parameter is $x$.

**3.9.** Let $w$ be a real and $y$ a set. $w$ is said to code $y$ if $w$ is a binary relation isomorphic to $\langle TC(\{y\}), \varepsilon\rangle$. Suppose $Z$ is an abstract 1-section. Show $Z_A$, the set of all sets coded by reals in $Z$, satisfies $\Sigma_1$ $DC$:

$$(x)(\mathrm{E}y)R(x, y) \rightarrow (\mathrm{E}f)(n)_{n<\omega} R(f(n), f(n+1)),$$

where $R$ is $\Sigma_1$ over $Z_A$.

**3.10.** Suppose the structure $L(\beta, \omega; H)$ is $E(\omega; H)$; show it is $\Sigma_1$ admissible.

**3.11.** Show that the second half of 3.6(1)(i) follows from the first half and 3.6(1)(ii)–(iii). (Forcing condition simplification.)

**3.12.** Let $D_2^1$ be the set of all lightface $\Delta_2^1$ subsets of $\omega$. Show that $D_2^1$ is the 1-section of some type 2 object. (Also true for $D_n^1$ when $n > 2$.)

# 4. Harrington's Plus-Two Theorem

Assume $\delta < \alpha+2$, $\delta$ is a successor, and $F$ is a type $\alpha+2$ object. The *$\delta$-envelope* of $F$ is denoted by $\delta - en(F)$, and is defined (following Moschovakis) to be

$$(1) \qquad \{z \mid z \in tp(\delta) \quad \& \quad z \text{ is } E\text{-r.e. in } tp(\delta-1), tp(\alpha); F\}.$$

Note the occurrence of "$tp(\delta-1)$" on the extreme right of (1); it is superfluous if $\delta \leq_E tp(\alpha)$; $F$; thus it is omitted if $\delta < \omega$. The $\delta$-envelope carries the same information about $F$ as the complete $E$-recursively enumerable (in $tp(\delta-1), tp(\alpha); F$) subset of $tp(\delta-1)$. Early on Moschovakis observed that the 1-envelope of ${}^3E$ is not the 1-envelope of any type 2 object (cf. Exercise 4.5). It follows from Harrington's plus-two theorem that the 1-envelope of ${}^4E$ is the 1-envelope of a type 3 object. His argument requires Theorem 4.1, a reflection result analogous to Corollary 8.2.III.

Some of the notions occurring in 4.1 and its applications need elaboration. $P(X)$ is a formula of class-set theory whose only class variable is the free variable $X$. $P(X)$ is built up from the atomic formulas of ZF, and additional formulas of the form $y \in X$, by means of set quantifiers and logical connectives.

Suppose $\gamma < \kappa^{tp(\alpha);F}$ and $a \in tp(\alpha)$. $\gamma$ is said to be $E$-constructive in $a, tp(\alpha); F$ if

$$\gamma = |\{e\}^F(a, tp(\alpha))|$$

for some $e$. Constructive in $a, tp(\alpha); F$ implies recursive in $a, tp(\alpha); F$, but not conversely (cf. Exercise 4.6). Of course each recursive ordinal is less than some constructive.

Suppose $A \in \delta - en(F)$. Thus

$$(1) \qquad A = \{b \mid \{e\}^F(b, tp(\delta-1), tp(\alpha); F)\downarrow\}$$

for some $e$. $A$ can be thought of as being enumerated by stages. At stage $\sigma$ enumerate all elements of $tp(\delta-1)$ that belong to $A$ according to (1) by virtue of computations of length $\sigma$; the set of all such elements is $A_\sigma$. Thus $A$ is enumerated without repetitions at stages $E$-constructive in $b, tp(\delta-1), tp(\alpha); F$ as $b$ ranges over $A$. For all $\gamma$ define

$$A_{<\gamma} = \cup\{A_\sigma \mid \sigma < \gamma\}.$$

Let $\kappa_{\delta-1}^{tp(\delta-1), tp(\alpha); F}$ be the supremum of all ordinals $E$-constructive in $b, tp(\delta-1), tp(\alpha); F$ as $b$ ranges over $tp(\delta-1)$. Thus

$$A = A_{\kappa_{\delta-1}^{tp(\delta-1), tp(\alpha); F}}.$$

**Theorem 4.1** (Harrington 1973). *Assume $\delta \leq \alpha$, $\delta$ and $\alpha$ are successors, and $F$ is a type $\alpha+2$ object. Suppose $A \in \delta - en(F)$ and $b \in tp(\delta-1)$.* If

$$\{e\}^F(A, b, tp(\delta-1), tp(\alpha); F)\downarrow$$

*then*

$$\{e\}^F (A_{<\gamma}, b, tp(\delta-1), tp(\alpha); F)\downarrow$$

*for some* $\gamma$ *E-constructive in* $(c, tp(\delta-1), tp(\alpha); F)$ *for some* $c \in tp(\delta-1)$ (*further reflection*).

If Theorem 4.1 seems too farfetched, let $\alpha$ and $\delta$ be 1, $F$ be ${}^3E$, and $A$ the complete, $E$-recursively enumerable (in $2^\omega$) subset of $\omega$. Then 4.1 implies

$$\kappa_r^{2^\omega} \geq \kappa_0^{2^\omega, A} > \kappa_0^{2^\omega}.$$

Suppose $P(X)$ is a $\Sigma_1$ formula of class-set theory whose only class variable is the free variable X, and whose only parameters are $X$ and $2^\omega$. If

$$L(\kappa_0^{2^\omega}, 2^\omega) \models P(A),$$

then 4.1 implies

$$L(\gamma, 2^\omega) \models P(A_{<\gamma})$$

for some $\gamma$ $E$-constructive in $2^\omega$ (cf. Corollary 8.2.III) $A$ functions as an additional atomic predicate above. The integers in A can be regarded as notations for ordinals cofinal in $\kappa_0^{2^\omega}$, hence $P(A)$ can be rewritten as a $\Delta_0$ formula. Then the truth-value of $P(A_{<\gamma})$ in $L(\kappa_0^{2^\omega}, 2^\omega)$ can be checked in $L(\gamma, 2^\omega)$.

**4.2 Proof of Theorem 4.1.** Let $W$ be the set of all $x$ such that $x$ is a wellfounded relation whose field is a type $\delta$ object. Since $\alpha > 0$, $W \leq_E tp(\alpha)$. For each $x \in W$, $|x|$ is the ordinal height of $x$. If $\beta < |x|$, then $x_\beta$ is the set of elements of the field of $x$ of height $\beta$.

Let $\mathcal{O}_{\delta-1}$ be the set of ordinals $E$-constructive in $d$, $tp(\delta-1)$, $tp(\alpha)$; $F$ as $d$ ranges over $tp(\delta-1)$, and let $t$ be an order preserving map of $\mathcal{O}_{\delta-1}$ onto an initial segment of ordinals.

A function $f$ with domain $W$ is computed from $b$, $tp(\delta-1)$, $tp(\alpha)$; $F$ as follows. Three possibilities occur.

(i) $(Ec)[c \in (A - (\text{field of } x)]$.
(ii) $(E\beta)[x_\beta \neq A_{t^{-1}(\beta)}]$.
(iii) There is a $\sigma \leq |x|$ such that: $x_\beta = A_{t^{-1}(\beta)}$ for all $\beta < \sigma$; and

(1) $$\{e\}(A_{<\sigma}, b, tp(\delta-1), tp(\alpha); F)\downarrow.$$

As in Exercise 2.9, Grilliot selection implies (i), (ii) and (iii) are predicates (of $x$) $E$-recursively enumerable in $b$, $tp(\delta-1)$, $tp(\alpha)$; $F$. At least one of the three is true, and one such is provided by Gandy selection. Define $f(x)$ to be the length of the computation of the one selected.

Let $\gamma_0 = \sup\{f(x) | x \in W\}$. Let $x_0$ be the wellfounded relation on $A$ induced by the enumeration of $A$ by stages. Then $f(x_0)$ was defined by (iii). Hence (1) holds for some $\sigma \leq \gamma_0$. Since $\gamma_0 \leq_E b$, $tp(\delta-1)$, $tp(\alpha)$; $F$, there is a $\sigma_0 \geq \sigma$ such that $\sigma_0$ is $E$-constructive in $c$, $tp(\delta-1)$, $tp(\alpha)$; $F$ for some $c \in tp(\delta-1)$, and $A_{<\sigma} = A_{<\sigma_0}$. □

**4.3 Theorem** (Harrington 1973). *Suppose $\delta < \alpha$, $\delta$ and $\alpha$ are successors, and $F$ is a type $\alpha + 2$ object. Then there exists a type $\delta + 2$ object $G$ such that*

$$\delta - en(G) = \delta - en(F).$$

*Proof.* $G$ will map $tp(\delta + 1) \times tp(\delta - 1)$ into 2. $G$ is defined by stages that correspond to the stages of the enumeration of $C^{\delta-1;F}$, the complete $E$-recursively enumerable (in $tp(\delta - 1)$, $tp(\alpha)$; $F$) subset of $tp(\delta - 1)$. At stage $\gamma$ all elements of $tp(\delta - 1)$ that belong to $C^{\delta-1;F}$ by virtue of computations of length $\gamma$ are enumerated in $C^{\delta-1;F}$. Stage $\gamma$ is said to *exist* if there is a computation of length $\gamma$ that puts an element in $C^{\delta-1;F}$. As $C^{\delta-1;F}$ is enumerated, the graph of $G$, restricted to

(1) $$L(\kappa_{\delta-1}^{tp(\delta);G}, tp(\delta); G),$$

is enumerated in $tp(\delta - 1)$, $tp(\alpha)$; $F$. ($\kappa_{\delta-1}^{tp(\delta);G}$ is the supremum of all ordinals $E$-constructive in $b$, $tp(\delta)$; $G$ for all $b \in tp(\delta - 1)$.) Thus the external aspects of the construction insure that $C^{\delta-1;G}$, the complete $E$-recursively enumerable (in $tp(\delta)$; $G$) subset of $tp(\delta - 1)$, belongs to the $\delta$-envelope of $F$, and hence that

(2) $$\delta - en(G) \subseteq \delta - en(F).$$

Along the way steps are taken to insure that $C^{\delta-1;F}$ belongs to the $\delta$-envelope of $G$. Thus the internal aspects of the construction guarantee the converse of (2).

Two difficulties arise: undershoot and overshoot. Undershoot means there is a $\gamma$ such that stage $\gamma$ exists but $G$ is already defined on (1), that is, $C^{\delta-1;G}$ is already defined. Undershoot is possible because $G$ is of lower type than $F$. Overshoot means the enumeration of $C^{\delta-1;F}$ comes to end before $G$ is defined on (1), that is, before $C^{\delta-1;G}$ is defined. Both of these difficulties are overcome with the help of Harrington's reflection theorem (4.1).

Suppose stage $\gamma$ exists. Let

$$\gamma^- = \sup\{\sigma \mid \sigma < \gamma \quad \& \quad \text{stage } \sigma \text{ exists}\}.$$

The construction so far has produced

(3) $$L(\beta_{\gamma^-}, tp(\delta); G) = \bigcup_{\sigma < \gamma} L(\beta_\sigma, tp(\delta); G)$$

and $G$ restricted to (3). As in Exercise 3.8, if (5) holds, there is a $y(\beta_{\gamma^-}) \subseteq tp(\delta)$ such that $y(\beta_{\gamma^-})$ is first order definable over, but not a member of, (3). The definition in question has $tp(\delta)$ as its only parameter and treats $G$ as an additional atomic predicate; it is "complete $\Sigma_1$", hence uniform in every sense. $G(y(\beta_{\gamma^-}), b)$ is not yet defined for any $b \in tp(\delta - 1)$.

Let

(4) $$C_\gamma^{\delta-1;F} = \{b \mid b \in C^{\delta-1;F} \quad \& \quad |b| = \gamma\}.$$

($|b|$ is the length of the computation that puts $b$ in $C^{\delta-1;F}$.)

*Case 1*: $\gamma^- < \gamma$ and stage $\gamma^-$ exists. Assume (to be checked at end of each stage)

(5) $\beta_{\gamma-}$ is $E$-constructive in $b$, $tp(\delta-1)$; $G$ for some $b \in tp(\delta-1)$.

Define

$$G(y(\beta_{\gamma-}), b) = \begin{matrix} 1 & \text{if} & b \in C_\gamma^{\delta-1;F}, \\ 0 & \text{if} & b \notin C_\gamma^{\delta-1;F}. \end{matrix} \tag{6a}$$

Let $\beta_\gamma = \beta_{\gamma-} + \omega$, and extend $G$ trivially to $L(\beta_\gamma, tp(\delta); G)$ by setting $G$ equal to 0 whenever it needs to be defined. It follows from (5) and (6a) that the members of $C_\gamma^{\delta-1;F}$ can be enumerated at stage $\beta_\gamma$ of the enumeration of $C^{\delta-1;G}$.
*Case 2*: case 1 does not hold. Let $G_0$ be the trivial extension of $G$ from (3) to all of $tp(\delta-1) \times tp(\delta-1)$.

*Subcase 2a*: $\beta_{\gamma-} = \kappa_{\delta-1}^{tp(\delta);G_0}$ (undershoot). Whether or not subcase 2a holds can be computed from $\beta_{\gamma-}$, $G_0$, $tp(\alpha)$. The hierarchy

$$L(\kappa_{\delta-1}^{tp(\delta);G_0}, tp(\delta); G_0) \tag{6b}$$

can be viewed as a $tp(\delta)$ object. There is a partial $E$-recursive (in $G_0$) map of $tp(\delta)$ onto (6b). Hence (6b) is encodable by an element of $tp(\delta+1)$. Thus Subcase 2a holds iff there is an element of $tp(\delta+1)$ that encodes (6b) and 2a holds within (6b). In short $tp(\delta+1)$ has the power to compare $\beta_{\gamma-}$ with $\kappa_{\delta-1}^{tp(\delta);G_0}$. Since $\delta < \alpha$, $tp(\alpha)$ also has the power. Define $G(y(\beta_{\gamma-}), b)$ as in (6a), and then trivially extend $G$ to $G_1$ defined everywhere. Now Theorem 4.1 can be used to show

$$\beta_{\gamma-} < \kappa_{\delta-1}^{tp(\delta);G_1}. \tag{7}$$

Fix $b_0 \in C_\gamma^{\delta-1;F}$. The construction assigns non-zero values to $G$ only when required by (6a). $C^{\delta-1;F}$ is enumerated without repetitions. Thus

$$\beta_{\gamma-} = \mu\tau[G(y(\tau), b_0) = 1]. \tag{8}$$

Let $A$ denote $C^{\delta-1;G_1}$. $e$ is such that the computation of

$$\{e\}(A, b_0, tp(\delta-1), tp(\alpha); G_1) \tag{9}$$

proceeds as follows. Enumerate $A$ $E$-recursively in $tp(\delta)$; $G$. Let $\rho$ be the supremum of all computations that put elements in $A$. The value of (9), if it converges, is

$$\mu\tau_{\tau \leq \rho}[G(y(\tau), b_0)] = 1.$$

It follows from (8) that (9) converges to $\beta_{\gamma-}$. Theorem 4.1 implies (9) yields the same result when $A$ is replaced by $A_{<\sigma}$ for some $\sigma$ $E$-constructive in $c$, $tp(\delta)$; $G_1$ for some $c \in tp(\delta-1)$; $G_1$. Hence (7) holds. Let $\beta_\gamma = \sigma + \omega$, and let $G$ be the restriction of $G_1$ to $L(\beta_\gamma, tp(\delta-1); G_1)$. Then (7) is true when $G_1$ is replaced by $G$; moreover $\beta_\gamma$ is $E$-constructive in $c$, $tp(\delta)$; $G$.

*Subcase 2b*: $\beta_{\gamma-} < \kappa_{\delta-1}^{tp(\delta); G_0}$. Hence there is a $\tau \geq \beta_\gamma^-$ such that $\tau$ is $E$-constructive in $b$, $tp(\delta)$; $G_0$ for some $b \in tp(\delta-1)$. Let $\tau_\gamma$ be the least such $\tau$. Define $G$ to agree with $G_0$ on $L(\tau_\gamma, tp(\delta); G_0)$, and define $G(y(\tau_\gamma), b)$ as in (6a). Let $\beta_\gamma = \tau_\gamma + \omega$ and extend $G$ trivially to $L(\beta_\gamma, tp(\delta); G)$.

To finish the construction, define

$$\beta_\infty = \sup\{\beta_\gamma \mid \gamma \text{ exists}\}$$

and extend $G$ trivially to all of $tp(\delta+1) \times tp(\delta-1)$.

Let $C^{\delta-1; G}_{<\beta_\infty}$ be that part of $C^{\delta-1; G}$ enumerated via computations of height less than $\beta_\infty$. The outward form of the construction implies $C^{\delta-1; G}_{<\beta_\infty}$ is $E$-recursively enumerable in $tp(\delta-1)$, $tp(\alpha)$; $F$. The only remaining problem is to show $C^{\delta-1; G}_{<\beta_\infty}$ is $C^{\delta-1; G}$ (overshoot). Suppose not. Thus there is a $\tau \geq \beta_\infty$ such that $C^{\delta-1; G}_\tau \neq \varnothing$. Let $\tau_\infty$ be the least such $\tau$ and choose $b_1 \in C^{\delta-1; G}_{\tau_\infty}$. Theorem 4.1 shows $b_1 \in C^{\delta-1; G}_{<\beta_\gamma}$ for some $\gamma$ that exists. There is an $e$ such that

$$(10) \qquad \{e\}(C^{\delta-1; F}, b_1, tp(\delta-1), tp(\alpha); F)$$

is computed as follows: enumerate all of $C^{\delta-1; F}$; enumerate $C^{\delta-1; G}_{<\beta_\infty}$ simultaneously; look for the least $\tau$ such that $b_1 \in C^{\delta-1; G}_\tau$. According to Theorem 4.1, (10) converges when $C^{\delta-1; F}$ is replaced by $C^{\delta-1; F}_{<\gamma}$ for some $\gamma$ that exists. Furthermore $\{e\}$ can require $C^{\delta-1; F}_{<\gamma}$ to be a fairly good imitation of $C^{\delta-1; F}$; in particular $\gamma^- = \gamma$. Consequently case 2b holds at stage $\gamma$ and $b_1 \in C^{\delta-1; G}_{<\beta_\gamma}$. □

Harrington noted, and Moldstad 1977 proved, that the plus-one and plus-two theorems can be combined. For example, there is a type 3 $G$ such that $1-en(G) = 1-en({}^4E)$ and $2-sc(G) = 2-sc({}^4E)$.

No satisfactory plus-three theorem is known.

## 4.4–4.6 Exercises

**4.4** (Harrington). Define

$$\tau_0 = \kappa_0^{2^\omega}$$
$$\tau_{n+1} = \kappa_0^{\tau_n, 2^\omega} \quad (n < \omega).$$

Show $\kappa_r^{2^\omega} > \lim_n \tau_n$.

**4.5** (Moschovakis). Show the 1-envelope of ${}^3E$ is not the 1-envelope of any type 2 object.

**4.6** Show that an ordinal $E$-recursive in $a$, $tp(\alpha)$; $F$ need not be $E$-constructive in $a$, $tp(\alpha)$; $F$. ($a \in tp(\alpha)$ & $\alpha$ a successor & $F$ of type $\alpha+2$.)

## 5. Selection with Additional Predicates

The proofs of Normann selection (Theorem 1.3) and Moschovakis selection (Theorem 2.1) made use of a "power set" hypothesis. In the Normann case, Hajnal's theorem was invoked to show $2^\rho \cap E(\{x, \rho\})$ was an element of $E(\{x, \rho\})$. In the Moschovakis case strong use was made of the fact that

$$R(k(j))^{R(k(i))} \in R(\alpha)$$

if $k(j) \leq k(i) < \alpha$. In both cases the need for a "power set" hypothesis stemmed from Grilliot's approach to selection: compute min $f$ by computing min $g$ for all $g$ "below" $f$. In this section min $A$ is computed by considering subcomputation instructions of computation instructions in $A$. Thus functions (or subsets) are replaced by elements, and so there is no need to know some class of subsets of a set is a set.

The elemental approach to Moschovakis selection is well exemplified by the proof of Theorem 5.1. Let $x$ be a set of ordinals. Then $E(x)$ has a greatest cardinal (in the sense of $E(x)$) denoted by gc($E(x)$). In general $\text{gc}(E(x)) \leq \sup x$. For simplicity assume $\text{gc}(E(x)) = \sup x$. Let $A \subseteq \sup x$ be $E$-recursively enumerable in $x$. Thus

$$(1) \qquad A = \{a \mid a < \sup x \quad \& \quad \{e\}(a, x)\downarrow\}.$$

If $A \neq \phi$, then define

$$\min A \text{ to be } \min\{|\{e\}(a, x)| \mid a \in A\}.$$

Assume

$$E(x) \models [\text{cofinality gc}(E(x)) = \omega].$$

The proof of 5.1 uses the elemental approach to compute min $A$ from $x$ and an $\omega$-sequence through gc($E(x)$). Note one last time that there might be a cardinal $\rho$ (in the sense of $E(x)$) such that $2^\rho \cap E(x)$ is not a member of $E(x)$. In that event there is little hope for Grilliot's approach.

Some *notational conventions* will be helpful in the proof of 5.1. Let $a, b, c, \ldots$ be ordinals less than $\sup x$. $a, b, c, \ldots$ will be used to denote (or encode) computation instructions, that is, nodes on the universal computation tree $<_U$. If the node is of the form $\langle e, \langle a, x\rangle\rangle$, then it can be readily encoded as an ordinal less than sup $x$. An immediate subcomputation instruction of $\langle e, \langle a, x\rangle\rangle$ is of the form $\langle c, v\rangle$ for some $c < \omega$ and $v \in E(x)$. As in the beginning of the proof of Theorem 2.1, $v$ can be construed effectively as an element of sup $x$ (cf. Exercise 2.7), and so $\langle c, v\rangle$ can be encoded as an element of sup $x$.

As an example of the above usage,

$$|a| = \sup{}^+ \{|b| \mid b <_U a\}.$$

**5.1 Theorem** (Sacks & Slaman 1987). *Let x be a set of ordinals. Suppose in E(x) there is a strictly ascending sequence* $\{k_i \mid i < \omega\}$ *of cardinals (in the sense of E(x)) such that*

$$\sup x = \sup\{k_i | i < \omega\}.$$

*If* $A \subseteq x$ *is nonempty and E-recursively enumerable in x, then*

$$\min A \leq_E x, \{k_i \mid i < \omega\} \quad \textit{(uniformly)}.$$

*Proof.* The predicate, $b$ is an immediate subcomputation instruction of $a$, is not $E$-recursive in $x$. A slight modification, $b$ is an immediate subcomputation instruction of $a$ via $\beta$, is. The essential clauses are:

$\langle m, u\rangle$ is an immediate subcomputation instruction of $\langle 2^m \cdot 3^n, u\rangle$ via $\beta$;
if $|\{m\}(u)| \leqslant \beta$ and $v \in \{m\}(u)$, then $\langle n, v\rangle$ is immediate subcomputation of $\langle 2^m \cdot 3^n, u\rangle$ via $\beta$.

The theorem is proved by effective transfinite recursion on min A, henceforth called the *main recursion.* There are countably many cases, at least one of which yields min A. They are woven into one procedure by Gandy selection.
*Case* $\langle i, j\rangle$: $i < j$ and there is an $a_0 \in A \cap k_i$ such that

$\min A = \min(A \cap k_i) = |a_0|$, and

$\min A = \sup^+ \{|b| \mid b < k_j \;\&\; b$ is an immediate subcomputation instruction of $a_0\}$.

In this case min $A$ is computed by a recursion of length $k_{j+1}$. Fix $\alpha < k_{j+1}$ and assume $\beta(\gamma)$ has been computed for $\gamma < \alpha$. Let

$$\beta^-(\alpha) = \sup\{\beta(\gamma) \mid \gamma < \alpha\}.$$

*Subcase 1*: $\beta^-(\alpha) + 1 \geq \min(A \cap k_i)$. Define $\beta(\alpha)$ to be $\min(A \cap k_i)$.

*Subcase 2*: $\beta^-(\alpha) + 1 < \min(A \cap k_i)$. Let $Z_\alpha$ be the set of all $b$ such that $b < k_j$ & $b\downarrow$ & $\beta^-(\alpha) < |b|$ & $b$ is an immediate subcomputation instruction of some $a < k_i$ via $\beta^-(\alpha)$.

To see $Z_\alpha$ is nonempty consider the $a_0$ mentioned in the case hypothesis. Suppose $a_0$ is $\langle 2^m \cdot 3^n, u\rangle$. If $\beta^-(\alpha) < |\langle m, u\rangle|$, then $\langle m, u\rangle \in Z_\alpha$. This last assertion assumes that the encoding of nodes on $<_U$ by elements of $\sup x$ has the property that $\langle m, u\rangle \in k_j$ if $\langle 2^m \cdot 3^n, u\rangle \in k_j$. Suppose $\beta^-(\alpha) \geq |\langle m, u\rangle|$. Then for every $v \in \{m\}(u)$, $\langle n, v\rangle$ is an immediate subcomputation instruction of $a_0$ via $\beta^-(\alpha)$. Hence the case, and subcase, hypotheses provide a $b$ in $Z_\alpha$.

$Z_\alpha$ is $E$-recursively enumerable in $\beta^-(\alpha)$, $x$, $k_i$, $k_j$; and $\min Z_\alpha < \min A$, hence the main recursion can be applied to compute $\min Z_\alpha$. Define $\beta(\alpha)$ to be $\min Z_\alpha$.

$\sup\{\beta(\alpha) \mid \alpha < k_{j+1}\} \leq_E x, k_i, k_j$, so it need only be shown that $\beta(\alpha) = \min(A \cap k_i)$ for all sufficiently large $\alpha$. Suppose not. Then subcase 2 holds for all

$\alpha$. For each $\alpha$, there is a $b_\alpha \in Z_\alpha \cap k_j$ such that

$$|b_\alpha| = \beta(\alpha) \quad \text{and} \quad (\gamma)_{\gamma < \alpha}\, [\beta(\gamma) < |b_\alpha|].$$

The map $\alpha \mapsto b_\alpha$ is an injection of $k_{j+1}$ into $k_j$.

For the sake of case $\langle i, 0 \rangle$, define

$$\gamma(j, a) = \sup{}^+ \{|b| \,|\, b < k_j \quad \& \quad b \text{ is an immediate subcomputation instruction of } a\} \tag{1}$$

with the understanding that $\gamma(j, a)$ has a value iff $b\!\downarrow$ for all $b < k_j$ an immediate subcomputation instruction of $a$.

Define $\gamma(j, a, \beta)$ as in (1) with $b$ restricted to immediate subcomputation instructions of $a$ via $\beta$. $\gamma(j, a, \beta)$ is a partial $E$-recursive approximation of $\gamma(j, a)$.
*Case* $\langle i, 0 \rangle$: $\min A = \min (A \cap k_i)$ and
(*a*) $[a \in A \cap k_i \;\&\; |a| = \min A \to (j)_{j<\omega}\, (\gamma(j, a) < \min A)]$.
In this case $\min(A \cap k_i)$ is computed by a recursion of length $k_{i+1}$. Fix $\alpha < k_{i+1}$ and suppose $\beta(\gamma)$ has been computed for all $\gamma < \alpha$.

*Subcase 1*: $\beta^-(\alpha) + 1 \geq \min(A \cap k_i)$. Define $\beta(\alpha)$ to be $\min(A \cap k_i)$.

*Subcase 2*: $\beta^-(\alpha) + 1 < \min(A \cap k_i)$. Let $Y_\alpha$ be the set of $\langle j, a \rangle$ such that

$$j < \omega \quad \& \quad a < k_i \quad \& \quad \beta^-(\alpha) < \gamma(j, a, \beta^-(\alpha)).$$

$Y_\alpha$ is nonempty by the same sort of argument used in subcase 2 of case $\langle i, j \rangle$ to show $Z_\alpha$ is nonempty. $Y_\alpha$ is $E$-recursively enumerable in $\beta^-(\alpha), x, \{k_j \,|\, j < \omega\}$. Hence the main recursion can compute

$$\beta(\alpha) = \min\{\gamma(j, a, \beta^-(\gamma)) \,|\, \langle j, a \rangle \in Y_\alpha\}.$$

For all sufficiently large $\alpha$, $\beta(\alpha)$ is $\min(A \cap k_i)$. Suppose not. Then subcase 2 holds for all $\alpha$. For each $\alpha$ there is a $\langle j, a \rangle \in \omega \times k_i$ such that

$$\beta(\alpha) = \gamma(j, a, \beta^-(\alpha)) \quad \& \quad (\gamma)_{\gamma < \alpha}\, (\beta(\gamma) < \beta(\alpha)).$$

Each such $\langle j, a \rangle$ is associated with at most two $\alpha$'s. But there are $k_{i+1}$ $\alpha$'s associated with only $\omega \times k_i$ $\langle j, a \rangle$'s, an impossibility. The two $\alpha$'s, $\alpha_0$ and $\alpha_1$, occur as follows. Suppose $a$ is $\langle 2^m \cdot 3^n, u \rangle$. $\beta^-(\alpha_0)$ is less than $|\langle m, u \rangle|$. Later, when $\beta^-(\alpha_1) \geq |\langle m, u \rangle|$, $\langle n, v \rangle$ is an immediate subcomputation instruction of $a$ via $\beta^-(\alpha_1)$ for all $v \in \{m\}(n)$.

An effective selection procedure is defined above for case q uniformly in $q$. If $\min A < \infty$, then case $q$ converges for some $q$. For all $q$, if $q$ converges, then it converges to $\min A$. If some $q$ converges, then Gandy selection computes one. □

The proof of Theorem 5.1 establishes more than has been stated. $\{k_i \,|\, i < \omega\}$ can be replaced by a singular sequence of length $\rho > \omega$, if there is an effective method $M$

for selecting elements of nonempty subsets of $\rho$ $E$-recursively enumerable in $x$. $M$ replaces Gandy selection at the end of the proof of 5.1.

$E(x)$ can be replaced by $E(x; R)$ in the statement of Theorem 5.1 without any noteworthy changes in the proof of 5.1. Easy relativization to $R$ is typical of the elemental approach to selection. The assumption that $x$ is a set of ordinals can be relaxed somewhat. The essential point to remember about the proof of 5.1 is the non-injectibility of $k_{i+1}$ into $k_i$.

**5.2 Corollary.** *If $x$ is a set of ordinals and*

$$E(x) \models [\textit{cofinality of greatest cardinal} = \omega],$$

*then $E(x)$ is $\Sigma_1$ admissible.*

*Proof.* Theorem 5.1 and Lemma 2.3. □

Now the elemental method is used to relativize Normann selection (Theorem 1.3) to an additional predicate.

**5.3 Theorem** (Griffor & Normann 1982). *Let $\rho$ be a regular cardinal in the sense of $E(\rho; R)$. If $\delta < \rho$ and $C \subseteq \delta$ is nonempty and E-recursively enumerable in $\rho$; $R$, then some element of $C$ is E-recursive in $\delta$, $\rho$; $R$ (uniformly).*

*Proof.* Let $a, b, c, \ldots$ denote ordinals less than $\rho$. Recall the remarks about notational conventions made immediately prior to Theorem 5.1. They apply to the present situation. Thus $a, b, c, \ldots$ will be used to denote (or encode) computation instructions, that is, nodes on $<_{U;R}$. If the node is of the form $\langle e, \langle a, x\rangle\rangle$ then it can be encoded by a slight modification of $x$. If $b$ is a node of the form $\langle 2^m \cdot 3^n, u\rangle$, then its immediate predecessors are:

$$\langle m, u\rangle;$$
$$\langle n, v\rangle \text{ for all } v \in \{m\}(u) \quad \text{if} \quad \{m\}(u)\downarrow.$$

$\langle m, u\rangle$ will be denoted by $b_0$, and the $\langle n, v\rangle$'s, if they exist, by $b_1, b_2, \ldots, b_\alpha, \ldots$ $(\alpha < \rho)$. (If $\{m\}(u)$ has cardinality less than $\rho$, then some of the $b_\alpha$'s are dummies; the listing, $b_\alpha$ $(\alpha < \rho)$, is derived from the one-one map mentioned in Exercise 2.7).

Let $C = \{a \mid a < \delta \;\&\; \{e\}^R(a)\downarrow\}$. The intent is to compute

$$\min C = \min\{|\{e\}^R(a)| \mid a < \delta\}$$

by a recursion on $\min C$ henceforth called the main recursion. For simplicity let $a$ denote node $\langle e, a\rangle$ of $<_{U;R}$. Thus $\min C$ is $\min\{|a| \mid a \in C\}$.

$\beta(\alpha)$ is computed by recursion on $\alpha < \rho$. Let $\beta^-(\alpha)$ be $\sup\{\beta(\gamma) \mid \gamma < \alpha\}$.

*Case 1*: $\min C \leq \beta^-(\alpha) + 1$. Define $\beta(\alpha) = \min C$.

*Case 2*: $\min C > \beta^-(\alpha)+1$. Consider an arbitrary $a < \delta$. If $|a_0| > \beta^-(\alpha)$, then define $t(a, \alpha) = 0$. Suppose $|a_0| \leq \beta^-(\alpha)$. Then for all $\beta < \rho$, $a_\beta$ is seen to be an immediate subcomputation instruction of $a$ via a computation of length at most $\beta^-(\alpha)$. Let

$$t(a, \alpha) \simeq \mu\beta[|a_\beta| > \beta^-(\alpha)].$$

It is intended that $t(a, \alpha)$ be defined and equal to $\beta$ iff

$$(\gamma)_{\gamma<\beta}(a_\gamma\downarrow \quad \& \quad |a_\gamma| \leq \beta^-(\alpha)), \quad a_\beta\downarrow \quad \text{and} \quad |a_\beta| > \beta^-(\alpha).$$

$t(a, \alpha)$ is defined for some $a \in C$, since $\min C = |a|$ for some $a \in C$. Use the main recursion to compute

(1) $$\beta(\alpha) = \min_{a<\delta} |a_{t(a,\alpha)}|.$$

If case 1 holds for some $\alpha$, then $\min C = \sup\{\beta(\alpha) \mid \alpha < \rho\}$.

Suppose case 2 holds for all $\alpha$. Let $a^\alpha$ be the least $a$ that satisfies (1). $\{\langle\alpha, a^\alpha\rangle \mid \alpha < \rho\}$ is $E$-recursive in $x, \rho; R$ thanks to the main recursion. The regularity of $\rho$ in $E(x; R)$ implies $a^\alpha$ equals some fixed $a^\infty$ for all $\alpha \in Z$, an unbounded subset of $\rho$. Hence

$$(\alpha)_{\alpha\in Z}(\gamma)[\gamma < t(a^\infty, \alpha) \rightarrow |a_\gamma| < a_{t(a^\infty,\alpha)}],$$

$$\begin{aligned} |a^\infty| &= \sup\nolimits^+_{\gamma<\rho} |a^\infty_\gamma| = \sup\nolimits^+_{\gamma\in Z} |a^\infty_{t(a^\infty,\gamma)}| \\ &= \sup\nolimits^+_{\gamma\in Z} \beta(\gamma) \ = \sup\nolimits^+_{\gamma<\rho} \beta(\gamma), \end{aligned}$$

and so $\min C \leq \sup^+_{\gamma<\rho} \beta(\alpha)$. □

Theorem 5.3 extends Normann's result on the extended plus-one hypothesis ((ii) of subsection 3.7) from ${}^4E$ to an arbitrary type 4 object. It also extends his work on Post's problem for $E(2^\omega)$ to $E(2^\omega; R)$ (cf. Chapter XIII).

There appears to be little left to discover of a general nature about selection. Suppose $\rho$ is a cardinal in the sense of $E(\rho)$, and $E(\rho)$ is not $\Sigma_1$ admissible. If $\rho$ is regular in $E(\rho)$, then selection is false for unbounded $E$-recursively enumerable subsets of $\rho$, but true for bounded ones. If $\rho$ is singular in $E(\rho)$, then selection holds for $E$-recursively enumerable subsets of $\rho$ iff it holds for subsets of $\rho_0$, the cofinality of $\rho$ in $E(\rho)$.

According to Exercise 2.10, selection fails for subsets of $\omega_1^L$ $E$-recursively enumerable in $\omega_{\omega_1}^L$.

The study of forcing over $E$-closed structures leads to new, highly specialized selection results, e.g. Theorem 4.1.XI.