Chapter XI
Forcing Computations to Converge

In this chapter several notions of forcing over E-closed structures are studied. Only set forcing is examined. The term "set forcing" refers to forcing relations that are elements of the ground structure. Forcing over an E-closed structure poses questions peculiar to E-recursion theory. How can computations be forced to converge or diverge? There does not appear to be a uniform answer. Much depends on the nature of the forcing relation under scrutiny. The two major examples studied below are countably closed, and countable chain, forcing. Both preserve E-closure, yet do so in different ways. An example of set forcing that does not preserve E-closure is given in Section 1. Note that set forcing over a Σ_1 admissible set always preserves Σ_1 admissibility.

As an application of forcing it will be shown that certain naturally enumerable sets are not E-recursively enumerable. Recall that $2^{\omega} \cap E(\omega)$ is E-recursively enumerable in ω (cf. Exercise 5.16.X). It will be seen that $2^{2^{\omega}} \cap E(2^{\omega})$ is not E-recursively enumerable in $2^{\omega}, b$ for any $b \in 2^{\omega}$ if $E(2^{\omega}) \models (\text{card } (2^{\omega}) \text{ is regular})$ (cf. Exercise 2.5).

1. Set Forcing over $L(\kappa)$

Assume $L(\kappa)$ is E-closed and not Σ_1 admissible. Thus there is a greatest cardinal in the sense of $L(\kappa)$ (cf. Part C) denoted by $\text{gc}(\kappa)$. It is convenient to think of the generic object G as a subset of $\text{gc}(\kappa)$. The primary question is: if G is generic, is $L(\kappa, G)$ E-closed? For a "no" answer, let $L(\kappa)$ be $E(\omega_1)$, and G a Levy-collapse of ω_1 to ω. Then $L(\kappa, G) = L(\kappa, b)$ for some $b \subseteq \omega$. If $L(\kappa, b)$ is E-closed, then $L(\kappa, b) = E(b)$ is Σ_1 admissible, and so $L(\kappa)$ is Σ_1 admissible, an absurdity by Exercise 5.15.X.

Thus set forcing over $E(\omega_1)$ must preserve ω_1 if it is to preserve E-closure. It turns out that preserving ω_1 does not guarantee that E-closure will be preserved, but something close to that is true. This point is discussed further at the end of the present chapter.

1.1 The Forcing Language. Suppose $G \subseteq \text{gc}(\kappa)$ as above. It follows from Proposition 2.8.X that the members of $L(\kappa, G)$ are just those sets of the form $\{e\}(a, G)$ such that $a \in L(\kappa)$ and $|\{e\}(a, G)| < \kappa$. The terms of the forcing language,

$\mathscr{L}(\kappa, G)$, are designed to name the elements of $L(\kappa, G)$. Unavoidably some do not; they refer to divergent computations. The primitive terms are $\mathscr{G}$ and $\underline{a}$ for each $a \in L(\kappa)$. A general term is of the form $\{e\}(t_1, \ldots, t_n)$, where t_i $(1 \leq i \leq n)$ is a term. By composition every term is equivalent to one of the form $\{e\}(\underline{a}, \mathscr{G})$.

The other primitives are: ε; | |; set variables $x, y, z, \ldots$ ranging over $L(\kappa, G)$; ordinal variables $\beta, \gamma, \delta, \ldots$ ranging over κ, quantifiers and propositional connectives.

The formulas of $\mathscr{L}(\kappa, \mathscr{G})$ are as follows.

(i) $|\{e\}(\underline{a}, \mathscr{G})| = \underline{\sigma}$. (i) says that the computation indicated converges and has height $\sigma(<\kappa)$. (i) is a typical ranked formula.

(ii) $(E\delta)[|\{e\}(\underline{a}, \mathscr{G})| = \delta]$. This says that the computation indicated converges and has height less than κ. A typical unranked formula.

(iii) $x \in \{e\}(\underline{a}, \mathscr{G})$. Here is a formula whose meaning is elusive. If $\{e\}(a, G)$ converges and has height $\sigma < \kappa$, then $\{e\}(\underline{a}, \mathscr{G})$ names $\{e\}(a, G)$. Otherwise $\{e\}(\underline{a}, \mathscr{G})$ does not name any member of $L(\kappa, G)$.

1.2 The Forcing Relation. Let $\mathscr{P} = \langle P, \geq \rangle$ be a notion of forcing that belongs to $L(\kappa)$; in short $\mathscr{P} \in L(\kappa)$. The elements of P, denoted by $p, q, r, \ldots$, are called forcing conditions. If $p \geq q$, then p is said to be extended by q; intuitively, q says more about $\mathscr{G}$ than p does.

$\mathscr{P}$ determines a forcing relation $\Vdash$ defined by effective transfinite recursion on $\sigma < K$. Three formal entities are defined simultaneously:

(a) $p \Vdash |\{e\}(\underline{a}, \mathscr{G})| = \underline{\sigma}$.
(b) $\mathscr{T}(p, e, a, \mathscr{G}, \sigma)$.
(c) $q \Vdash s \in \{e\}(\underline{a}, \mathscr{G})$.

(b) is a set of terms adequate for naming all the elements of $\{e\}(a, G)$ if G is generic, $p \in G$ and (a) holds. In (c), s is a member of (b), and q extends the p mentioned in (b).

At the ground level ($\sigma = 0$) take $\{e\}(\underline{a}, \mathscr{G})$ to be $\mathscr{G}$ for simplicity. Then for all p and q,

$p \Vdash |\{e\}(\underline{a}, \mathscr{G})| = \underline{0}$,
$\mathscr{T}(p, e, a, \mathscr{G}, 0) = \{\underline{\delta} \mid \delta < \mathrm{gc}(\kappa)\}$, and
$q \Vdash \underline{\delta} \in \{e\}(\underline{a}, \mathscr{G})$ iff $(q \Vdash \underline{\delta} \in \mathscr{G})$ is a ground zero forcing fact.

(Assume $\mathscr{P}$ includes a list of all ground zero forcing facts of the form $q \Vdash \underline{\delta} \in \mathscr{G}$ or $r \Vdash \underline{\gamma} \notin \mathscr{G}$.)

Above ground level ($\sigma > 0$) take e to be $2^m \cdot 3^n$ (scheme T of 2.5.X). The definition of

$$p \Vdash |\{2^m \cdot 3^n\}(\underline{a}, \mathscr{G})| = \underline{\sigma} \tag{1}$$

has three clauses:

$$(E\gamma)_{\gamma < \sigma}[p \Vdash |\{m\}(\underline{a}, \mathscr{G})| = \underline{\gamma}]; \tag{2}$$

$$p \Vdash (x)(E\tau)_{\tau < \sigma}[x \in \{m\}(\underline{a}, \mathscr{G}) \rightarrow |\{n\}(x)| = \tau]; \tag{3}$$

$$p \Vdash (\tau)_{\tau < \sigma}[|\{m\}(\underline{a}, \mathscr{G})| = \tau \vee (\mathrm{Ex})[x \in \{m\}(\underline{a}, \mathscr{G}) \ \& \ |\{n\}(x)| \geq \tau]]. \tag{4}$$

To make sense out of (2)–(4), recall some fundamental aspects of forcing:

$p \Vdash (\mathrm{Ex})\, \mathscr{F}(x)$ iff $p \Vdash \mathscr{F}(c)$ for a suitable term c;
$p \Vdash \sim \mathscr{F}$ iff $(q)_{p \geq q} \sim [q \Vdash \mathscr{F}]$;
$p \Vdash (\mathrm{x})\, \mathscr{F}$ iff $p \Vdash \sim (\mathrm{Ex}) \sim \mathscr{F}$

Then note that the γ of (2) is less than σ, hence $\mathscr{T}(p, m, a, \mathscr{G}, \gamma)$ was defined prior to stage σ. The subformula, $x \in \{m\}(\underline{a}, \mathscr{G})$, gains a meaning by letting x range over the terms in $\mathscr{T}(p, m, a, \mathscr{G}, \gamma)$. If s is such a term, then

$$q \Vdash s \in \{m\}(\underline{a}, \mathscr{G})$$

was defined prior to stage σ for all $q \geq p$.

Next define

$$\mathscr{T}\{p, 2^m \cdot 3^n, \underline{a}, \mathscr{G}, \sigma\} = \{\{n\}(s) \mid s \in \mathscr{T}(p, m, a, \mathscr{G}, \gamma)\}.$$

Finally, define $q \Vdash t \in \{2^m \cdot 3^n\}(\underline{a}, \mathscr{G})$ iff for some $s \in \mathscr{T}(p, m, a, \mathscr{G}, \gamma)$,

$$t = \{n\}(s) \text{ and}$$
$$q \Vdash [s \in \{m\}(\underline{a}, \mathscr{G}) \quad \& \quad (E\tau)_{\tau < \sigma}(|\{n\}(s)| = \tau)].$$

Recall from subsection 1.1 that each general term t of $\mathscr{L}(\kappa, \mathscr{G})$ is equivalent to a term of the form $\{e\}(\underline{a}, \mathscr{G})$. It will often be convenient to replace $(\underline{a}, \mathscr{G})$ by t, as in Proposition 1.3.

1.3 Proposition. *The relations,*

$p \Vdash |\{e\}(t)| = \underline{\sigma}$,
$s \in \mathscr{T}(p, e, t, \sigma)$ *and*
$q \Vdash s \in \{e\}(t)$,
are E-recursive in σ, $\mathscr{P}$ *uniformly.*

Proof. By effective transfinite recursion on σ. The definition given in subsection 1.2 is such a recursion. At stage σ all quantifiers in the definition are bounded either by $\mathscr{P}$ or by sets defined at earlier stages. Recall the proofs of Lemmas 2.6 and 2.7 of Chapter X. □

1.4 The Tree of Possibilities. In subsection 1.2 a downward view of forcing computations to converge was adopted. This view parallels the downward view of convergence of Chapter X. It is possible to develop an upward approach based on hierarchies. The latter makes forcing over an E-closed $L(\kappa)$ look very much like forcing over L in conventional set-theoretic presentations. The downward view of computation seems closer than the upward to the nature of computation. An instruction is given and then pursued without any foreknowledge of convergence.

The upward approach deals only with convergent computations. It builds long ones by combining short ones. One advantage of proceeding downward is that it leads to the all important concept of divergence witness. Another is that most arguments for preserving E-closure in generic extensions turn on a related concept.

To study divergence witnesses in generic extensions, a forcing counterpart of $>_U$, the universal computation tree, is needed. It is denoted by $>_V$ and is called the tree of possibilities. A node on $>_V$ is a triple $\langle p, e, t\rangle$, where p is a forcing condition, $e < \omega$ and t is a term of $\mathscr{L}(\kappa, \mathscr{G})$. $>_V$ below $\langle p, e, t\rangle$ corresponds to $>_U$ below $\langle e, t\rangle$ for all generic G that satisfy p. A preliminary definition of $\langle p, e, t\rangle >_V \langle q, n, s\rangle$ is

$$p \geq q \quad \text{and} \quad q \Vdash^* [\langle e, t\rangle >_U \langle n, s\rangle]. \tag{1}$$

($\Vdash^*$ refers to weak forcing; $p \Vdash^* \mathscr{F}$ iff $p \Vdash \sim\sim \mathscr{F}$).
To make (1) precise, $>_U$ has to be expressed within the forcing language $\mathscr{L}(\kappa, \mathscr{G})$. Suppose $e = 2^m \cdot 3^n$.

$$p \Vdash [\langle 2^m \cdot 3^n, t\rangle >_U \langle m, t\rangle]. \tag{2)(a}$$

$$\text{If } p \Vdash |\{m\}(t)| = \underline{\gamma},\ s \in \mathscr{T}(p, m, t, \gamma) \text{ and}$$
$$p \Vdash s \in \{m\}(t), \quad \text{then} \quad p \Vdash [\langle 2^m \cdot 3^n, t\rangle >_U \langle n, s\rangle]. \tag{2)(b}$$

Formula (2) holds for all p, t and s, and indicates how to define $p \Vdash [\langle n, s\rangle$ is an immediate subcomputation instruction of $\langle e, t\rangle]$. The idea is to mimic the definition of immediate subcomputation instruction given in subsection 2.1.X with the help of the forcing definitions given in subsection 1.2. Then $p \Vdash a >_V b$ iff there exist $a_0, \ldots, a_n$ such that $a_0 = a$, $b = a_n$ and

$$p \Vdash [a_{i+1} \text{ is an immed. subcomp. instruc. of } a_i] \text{ for all } i < n.$$

The most pressing question about $>_V$ is: if $p \Vdash^* (E\sigma)[|\{e\}(t)| = \sigma|$, does it follow that $>_V$ below $\langle p, e, t\rangle$ is wellfounded?

An affirmative answer would be useful, since the key properties of $\Vdash$ could then be established by induction on the well-founded parts of $>_V$, as in the next lemma. The answer is yes and no. Yes for countably closed forcing, and no for countable chain condition forcing, both of which preserve E-closure.

1.5 Lemma. *Suppose* $p \Vdash^* (E\sigma)[|\{e\}(t)| = \sigma]$ *and* $>_V$ *is wellfounded below* $\langle p, e, t\rangle$. *Then*

$$p \Vdash^* |\{e\}(t)| \leq \underline{\gamma}$$

for some $\gamma \leq_E p, t, \mathscr{P}$ *(uniformly).*

Proof. γ is computed by effective transfinite recursion on $>_V$ below $\langle p, e, t\rangle$. Roughly, γ is the height of the wellfounded subtree of $>_V$ below $\langle p, e, t\rangle$. As usual

e is $2^m \cdot 3^n$ (scheme T of the proof of Lemma 2.5.X). By recursion

$$p \,|\!\vdash^* |\{m\}(t)| \leq \underline{\delta}$$

for $\delta \leq_E p, t, \mathscr{P}$, since $\langle p, e, t\rangle >_V \langle p, m, t\rangle$.

Define $\langle p', \sigma\rangle \in K$ by

$$p \geq p' \quad \& \quad \sigma \leq \delta \quad \& \quad p' \,|\!\vdash |\{m\}(t)| = \sigma.$$

K makes explicit all the values of $|\{m\}(t)|$ allowed by p. By Proposition 1.3, $K \leq_E p, t, \mathscr{P}$.

Fix $\langle p', \sigma\rangle \in K$ and $s \in T(p', m, t, \sigma)$. Then

$$p' \,|\!\vdash (E\beta)[s \in \{m\}(t) \to |\{n\}(s)| = \beta].$$

Define $q \in J(p', \sigma, s)$ by

$$p' \geq q \quad \& \quad q \,|\!\vdash s \in \{m\}(t).$$

If $q \in J(p', \sigma, s)$, then $q \,|\!\vdash^* (E\beta)[|\{n\}(s) = \beta]$. Again by 1.3,

$$J(p', \sigma, s) \leq_E p', \sigma, s, t, \mathscr{P}.$$

For each $q \in J(p', \sigma, s)$ there is by recursion some $\rho \leq_E q, s, \mathscr{P}$ such that

$$q \,|\!\vdash^* |\{n\}(s)| \leq \rho.$$

By Corollary 4.5.X to Gandy's selection principle, ρ can be construed as a partial E-recursive function of q, s and $\mathscr{P}$. As q varies over $J(p', \sigma, s)$, ρ is bounded by some

$$\rho_{J(p', \sigma, s)} \leq_E J(p', \sigma, s), s, \mathscr{P}.$$

As s varies over $\mathscr{T}(p', m, t, \sigma)$, $\rho_{J(p', \sigma, s)}$ is bounded by some

$$\rho_{\mathscr{T}(p', m, t, \sigma)} \leq_E p', \sigma, t, \mathscr{P}.$$

Finally, as $\langle p', \sigma\rangle$ varies over K, $\rho_{\mathscr{T}(p', m, t, \sigma)}$ is bounded by some $\rho_K \leq p, t, \mathscr{P}$. Let γ be the strict supremum of δ and ρ_K. □

1.6 Effective Bounding. Suppose $\mathscr{P} \in L(\kappa)$ is a notion of forcing such that: if

(1) $$p \,|\!\vdash^* (E\sigma)[|\{e\}(t)| = \sigma],$$

then $p \,|\!\vdash^* |\{e\}(t)| \leq \underline{\gamma}$ for some $\gamma \leq_E p, t, \mathscr{P}$ (uniformly). Then $\mathscr{P}$ is said to satisfy effective bounding. According to Lemma 1.5, if $>_V$ is wellfounded below $\langle p, e, t\rangle$ whenever (1) holds, then $\mathscr{P}$ satisfies effective bounding.

G is said to be $\mathscr{P}$-generic if for every sentence $\mathscr{F}$ of $\mathscr{L}(\kappa, \mathscr{G})$,

$$(\mathrm{Ep})_{p \in G}[p \Vdash \mathscr{F} \quad \text{or} \quad p \Vdash \sim \mathscr{F}].$$

G is viewed both as a subset of gc(K), and as a consistent set of forcing conditions. Thus $\delta \in G$ iff $q \Vdash \delta \in G$ for some $q \in G$. If G is $\mathscr{P}$-generic, then for every sentence $\mathscr{F}$,

$$(2) \qquad L(\kappa, G) \vDash \mathscr{F} \quad \text{iff} \quad (\mathrm{Ep})_{p \in G}[p \Vdash F].$$

(2) is proved in a standard fashion by induction on the rank and complexity of $\mathscr{F}$ (cf. Part A, Chapter IV).

1.7 Lemma. *If $\mathscr{P}$ satisfies effective bounding and G is $\mathscr{P}$-generic, then $L(\kappa, G)$ is E-closed.*

Proof. Suppose not. Then for some e and $a \in L(\kappa)$,

$$\{e\}(a, G)\downarrow \quad \text{and} \quad |\{\dot{e}\}(a, G)| \geq \kappa.$$

$T_{\langle e, \langle a, G\rangle\rangle}$ must hâve a node z such that $|T_z| = \kappa$. So it is safe to assume

$$(1) \qquad |\{e\}(a, G)| = \kappa.$$

Since G is generic there is a $p \in G$ such that p forces (1). Let e be $2^m \cdot 3^n$. Then

$$p \Vdash |\{m\}(\underline{a}, \mathscr{G})| = \underline{\sigma}$$

for some $\sigma \leq_E p, a, \mathscr{P}$ by effective bounding. p also forces

$$(2) \qquad (x)(E\beta)[x \in \{m\}(\underline{a}, \mathscr{G}) \rightarrow |\{n\}(x)| = \beta].$$

In (2) x ranges over $T(p, m, \underline{a}, \mathscr{G}, \sigma)$, which is E-recursive in p, a, $\mathscr{P}$ according to Proposition 1.3.

Now proceed as in the proof of Lemma 1.5. Obtain a bound $\rho_0 \leq_E p, a, \mathscr{P}$ on the possible values of $|\{n\}(s)|$ as s ranges over $\mathscr{T}(p, m, a, \mathscr{G}, \sigma)$. Then p forces $|\{e\}(\underline{a}, \mathscr{G})|$ to be at most $\sup^+(\rho_0, \sigma)$. The latter is less than κ, since it is E-recursive in elements of $L(\kappa)$. $\square$

In the next Section it will be shown that every countably closed notion of forcing satisfies effective bounding. The approach taken consists of proving $>_V$ is well-founded below $\langle p, e, t\rangle$ whenever 1.6(1) holds, and then applying Lemma 1.5. This approach fails for countable chain condition notions of forcing, which nonetheless will be seen to satisfy effective bounding.

2. Countably Closed Forcing

Again assume $L(\kappa)$ is E-closed and not Σ_1 admissible. Let $\mathscr{P} \in L(\kappa)$ be a notion of forcing. $\mathscr{P}$ is said to be *countably closed* if for every $(\lambda_n | p_n) \in L(\kappa)$:

$$(n)[p_n \geq p_{n+1}] \rightarrow (\mathrm{Eq})(n)[p_n \geq q].$$

For an example, assume $L(\kappa) \models [\mathrm{cf}(\mathrm{gc}(\kappa)) > \omega]$. (Recall that "gc" denotes greatest cardinal and "cf" cofinality.) Let P be the set of all $p \in L(\kappa)$ such that p is a function from some $\delta < \mathrm{gc}(\kappa)$ into 2. Then $P \in L(\kappa)$, since $P \subseteq L(\mathrm{gc}(\kappa))$. Say $p \geq q$ if $\mathrm{dom}\, p \subseteq \mathrm{dom}\, q$ and $p(x) = q(x)$ for all $x < \mathrm{dom}\, p$. $\mathscr{P}$ is countably closed because $\mathrm{gc}(\kappa)$ has uncountable cofinality in $L(\kappa)$. For q take $\cup \{p_n | n < \omega\}$.

Countably closed forcing over an E-closed structure originated in Sacks [1980] in the setting of Kleene recursion in ${}^3E$. Slaman was the first to point out the role of countable closure in Sacks [1980]. A slightly abstract account of countably closed forcing over an E-closed structure can be found in Sacks–Slaman [1987], as well as an extension of the method to class forcing.

Warning: there is an assumption not made explicit in the statement of Theorem 2.1 and needed in its proof for the application of Lemma 5.3.X. $\mathscr{P}$, and all $p \in P$, are effectively equivalent to sets of ordinals.

Theorem 2.1. *Suppose $L(\kappa)$ is E-closed, but not Σ_1 admissible, and $\mathscr{P}$ is a countably closed notion of forcing in $L(\kappa)$. Then every $\mathscr{P}$-generic extension of $L(\kappa)$ is E-closed.*

Proof. By Lemmas 1.5 and 1.7 it is enough to show $>_V$ is wellfounded below $\langle p, e, t\rangle$ when

$$p \Vdash^* (E\sigma)[|\{e\}(t)| = \sigma]. \tag{1}$$

The idea is to convert an infinite descending path, if there is one, below $\langle p, e, t\rangle$ into a $q \leq p$ that forces the existence of a Moschovakis witness to the divergence of $\{e\}(t)$.

The proof of Lemma 1.5 includes an effective procedure for computing γ from $p, e, t, \mathscr{P}$ by recursion on $>_V$ below $\langle p, e, t\rangle$. The procedure is welldefined whether or not $>_V$ is wellfounded below $\langle p, e, t\rangle$. Consequently Lemma 1.5 can be rephrased as follows.

There exists a $\{g\}$ such that if (1) holds and $>_V$ is well-founded below $\langle p, e, t\rangle$, then

$$\{g\}(p, e, t)\downarrow \quad \text{and} \quad p \Vdash^* |\{e\}(t)| \leq \{g\}(p, e, t).$$

(For simplicity, argument $\mathscr{P}$ of $\{g\}$ is suppressed.)

Observe that if $\{g\}(p, e, t)\downarrow$, then $>_V$ is wellfounded below $\langle p, e, t\rangle$. This is so because the computation of the value of $\{g\}(p, e, t)$ assigns an ordinal rank to each node $\langle p', e', t'\rangle$ of $>_V$ below $\langle p, e, t\rangle$. That rank is $\{g\}(p', e', t')$. In short, $\{g\}$ cannot finish if there is an ∞ descending path below $\langle p, e, t\rangle$.

Suppose $\{g\}(p, e, t)\uparrow$. The plan is to compare $>_V$ below $\langle p, e, t\rangle$ with $>_U$ below $\langle g, \langle p, e, t\rangle\rangle$. There is a witness, $\lambda n|z_n$, to the divergence of $\{g\}(p, e, t)$ inside $L(\kappa)$ by Theorem 5.7.X. In addition

$$z_0 = \langle g, \langle p, e, t\rangle\rangle \quad \text{and} \quad z_n >_U z_{n+1}.$$

The proof of Lemma 1.5 defines $\{g\}$ in such a way that $\{g\}(p, e, t)\uparrow$ if and only if there exists some $\langle q', m', s\rangle$ such that

(2) $$\{g\}(q', m', s)\uparrow, \quad \text{and}$$

(3) $$\langle q', m', s\rangle \text{ lies immediately below } \langle p, e, t\rangle \text{ in } >_V.$$

It follows that z must contain some information of the sort expressed by (2) and (3). Thus there is an n and a $\langle q', m', s\rangle$ such that

$$z_n = \langle g, \langle q', m', s\rangle\rangle$$

and $\langle q', m', s\rangle$ satisfies (2) and (3). Continuing in this fashion extracts a sequence $\lambda r|w_r$ from $\lambda n|z_n$ such that $w_0 = \langle p, e, t\rangle$ and for all r:

$$w_r >_V w_{r+1},$$
$$w_r \text{ is of the form } \langle p_r, e_r, t_r\rangle,$$
$$\{g\}(p_r, e_r, t_r)\uparrow,$$
$$(\mathrm{En})[z_n = \langle g, \langle p_r, e_r, t_r\rangle\rangle].$$

Since $\mathscr{P}$ is countably closed, there is a q such that $p_r \geq q$ for all r. Then q weakly forces $\lambda r|\langle e_r, t_r\rangle$ to be a Moschovakis witness to $\{e\}(t)\uparrow$. $\square$

2.2 Non-enumerability. A typical application of countably closed forcing is the proof that certain naturally enumerable sets are not E-recursively enumerable. For example, it follows from Theorem 2.3 that $E(\omega_1)$ is not E-recursively enumerable in any $b \in E(\omega_1)$. The case for regarding $E(\omega_1)$ as naturally enumerable is as follows. Begin with ω_1 and iterate first order definability. At a limit stage λ, look back and see if there is an x already enumerated and an e such that $|\{e\}(x)| = \lambda$. If so, collect everything already enumerated and continue. Otherwise stop. The procedure is predicative in nature, because $|\{e\}(x)| = \lambda$ only if all the immediate subcomputations of $\{e\}(x)$ converge and have been enumerated prior to stage λ. This way of laying out $E(\omega_1)$ is close in spirit to a standard way of laying out all finite computations in classical recursion theory.

Theorem 2.4, together with Corollary 4.6.X, explain why $E(\omega)$ is E-recursively enumerable. Recall that $\mathrm{gc}(\kappa)$ is the greatest cardinal in the sense of $L(\kappa)$, if there is one.

2.3 Theorem (Sacks 1986). *Assume $L(\kappa)$ is not Σ_1 admissible and*

$$L(\kappa) \models [\mathrm{gc}(\kappa) \textit{ is regular}].$$

Then $2^{\mathrm{gc}(\kappa)} \cap L(\kappa)$ is not E-recursively enumerable in any $b \in L(\kappa)$.

Proof. Recall the forcing conditions defined at the beginning of Section 2. Suppose (in hope of a contradiction) that for some $b \in L(\kappa)$,

$$2^{\mathrm{gc}(\kappa)} \cap L(\kappa) = \{x \mid \{e\}(b, x)\downarrow\}.$$

By Theorem 2.1 the null condition ϕ weakly forces $L(\kappa, \mathscr{G})$ to be E-closed. By Theorem 5.7.X, ϕ weakly forces $L(\kappa, \mathscr{G})$ to admit divergence witnesses. Thus ϕ weakly forces $(E\delta)[|\{e\}(t)| = \delta \vee (\mathrm{Ew})(w \in L(\delta)\ \&\ w$ is a divergence witness to $\{e\}(t)\uparrow)]$ for every term t. Hence *either* (1) or (2) holds:

(1) $$\phi \Vdash^* (E\sigma)[|\{e\}(\underline{b}, \mathscr{G})| = \sigma].$$

(2) There exist q, w and $\delta < \kappa$ such that

$$q \Vdash^* [w \in L(\underline{\delta}, \mathscr{G}) \quad \& \quad w \text{ witnesses } \{e\}(\underline{b}, \mathscr{G})\uparrow].$$

If (1) holds, then there will be a $G \subseteq \mathrm{gc}(\kappa)$ such that $\{e\}(b, G)\downarrow$ but $G \notin L(\kappa)$. If (2) holds, then there will be a $G \subseteq \mathrm{gc}(\kappa)$ such that $G \in L(\kappa)$ but $\{e\}(b, G)\uparrow$. Assume (1) holds. According to Theorem 2.1

(3) $$\phi \Vdash^* |\{e\}(\underline{b}, \mathscr{G})| < \underline{\lambda}$$

for some limit $\lambda < L(\kappa)$. The language $\mathscr{L}_\lambda(\mathscr{G})$ consists of sentences of rank at most λ and has enough expressive power to give a complete account of the computation of $\{e\}(b, G)$. A more detailed definition of $\mathscr{L}_\lambda(\mathscr{G})$ follows. As usual, only scheme T is considered. Let $e = 2^m \cdot 3^n$.

(4) (i) $(E\sigma)_{\sigma < \underline{\lambda}}[|\{e\}(\underline{b}, \mathscr{G})| = \sigma]$ belongs to $\mathscr{L}_\lambda(\mathscr{G})$.

(ii) If $\rho < \lambda$ and $p \Vdash |\{2^m \cdot 3^n\}(t)| = \rho$, then all of the following belong to $\mathscr{L}_\lambda(\mathscr{G})$:

$$(x)(E\tau)_{\tau < \underline{\rho}}[x \in \{m\}(t) \to |\{n\}(x)| = \tau];$$
$$(\tau)_{\tau < \underline{\rho}}(\mathrm{Ex})[x \in \{m\}(t) \to |\{n\}(x)| \geq \tau];$$
$$(E\sigma)_{\sigma < \underline{\rho}}[|\{m\}(t)| = \sigma].$$

(x ranges over the terms in $\mathscr{T}(p, 2^m \cdot 3^n, t, \rho)$.)

(iii) Every instance of every subformula of every sentence in $\mathscr{L}_\lambda(\mathscr{G})$ belongs to $\mathscr{L}_\lambda(\mathscr{G})$.

Proposition 1.3 implies $L_\lambda(G) \in L(\kappa)$. Let $f \in L(\kappa)$ be a map from $\mathrm{gc}(\kappa)$ onto $L_\lambda(\kappa)$.

There exists an $S \subseteq \mathrm{gc}(\kappa)$ such that $S \notin L(\kappa)$ but $(S \cap \delta) \in L(\kappa)$ for all $\delta < \mathrm{gc}(\kappa)$. S is needed in the construction of G to insure that $G \notin L(\kappa)$. If κ is countable, then S is any ω-sequence through $\mathrm{gc}(\kappa)$ not in $L(\kappa)$. If $\mathrm{gc}(\kappa)$ is a regular L-cardinal, then S exists because κ is not Σ_1 admissible, hence less than the next L-cardinal after $\mathrm{gc}(\kappa)$. Otherwise S is obtained by a Jensenian fine structure argument such as Theorem 4.1 of Sacks [1986].

The desired G is generic with respect to every sentence of $\mathscr{L}_\lambda(\mathscr{G})$ and encodes S. The characteristic function of G is a union of forcing conditions p_δ of $(\delta < \mathrm{gc}(\kappa))$ defined by recursion on δ.

(5)
$$p_0 = \phi. \quad p_\gamma = \cup \{p_\beta\} | \beta < \gamma\} \quad \text{if } \gamma \text{ is a limit.}$$
$$q_\delta = \text{least } q \geq p_\delta \quad \text{such that} \quad q \Vdash f_\delta \quad \text{or} \quad q \Vdash {\sim} f_\delta.$$
$$p_{\delta+1} = p_\delta{}^\frown\langle 0\rangle \quad \text{if} \quad \delta \in S, \quad p_\delta{}^\frown\langle 1\rangle \quad \text{if} \quad \delta \notin S.$$

$q^\frown\langle i\rangle = q \cup \langle \mathrm{dom}\, q, i\rangle$.) Observe that

(6) $\quad p_\delta$ is a forcing condition (i.e. $p_\delta \in L(\kappa)$), and

(7) $\quad \{p_\gamma | \gamma < \delta\} \leq_E S \cap \delta, \mathrm{gc}(\kappa), \lambda, f, \delta$ (uniformly in δ).

The proof of (6) and (7) is a simultaneous induction on δ. When δ is a limit, then (6) follows from (7) because $\mathrm{gc}(\kappa)$ is regular in $L(\kappa)$ and $(S \cap \delta) \in L(\kappa)$. When δ is a successor, then (7) follows from Proposition 1.3.

Hence G is well defined. By (3) and (4) $\{e\}(b, G)\!\downarrow$. $G \notin L(\kappa)$ because $S \notin L(\kappa)$ and

(8)
$$S \leq_E G, \mathrm{gc}(\kappa), \lambda, f.$$

(8) is a consequence of

(9)
$$\langle p_\delta, S \cap \delta\rangle \leq_E G, \mathrm{gc}(\kappa), \lambda, f, \delta \quad \text{(uniformly in } \delta),$$

and (9) is derived by recursion on δ. (In fact, S and G have the same E degree modulo $\mathrm{gc}(\kappa), \lambda, f$.)

Now assume (2) holds. Let λ be a limit greater than δ. Repeat the construction of G given in (5) with p_0 replaced by q and S by 0. Then G is generic with respect to all sentences of $\mathscr{L}_\lambda(\mathscr{G})$, and so $\{e\}(b, G)\!\uparrow$. According to (7), $G \leq_E \mathrm{gc}(\kappa), \lambda, f$, hence $G \in L(\kappa)$. $\square$

The proof of Theorem 2.3 relied heavily on the Σ_1 inadmissibility of $L(\kappa)$. The next result assumes Σ_1 admissibility and generalizes the classic result of Kleene: $2^\omega \cap E(\omega)$ is E-recursively enumerable in ω.

2.4 Theorem. *If z is a set of ordinals and $E(z)$ is Σ_1 admissible, then $E(z)$ is E-recursively enumerable in some element of $E(z)$.*

Proof. $E(z) = L(\kappa, z)$ for some κ by Proposition 2.10.X. According to Theorem 5.8.X, there is a $y \in E(z)$ such that $\kappa_r^{z,y} \geq \kappa$. Any $u \in L(\kappa, z)$ can be encoded by some v, a set of ordinals, in $L(\kappa, z)$. Hence there is a $v \in L(\kappa, z)$ such that $v \subseteq \kappa$ and $z, y \leq_E v$. By reflection it is safe to assume $v \leq_E z, y$. Since $v \equiv_E z, y$ it follows that $\kappa_0^v = \kappa_0^{y,z}$ and $\kappa_r^v = \kappa_r^{z,y}$. Hence

$$\kappa_r^{v,x} \geq \kappa \quad \text{for all } x$$

by Lemma 5.5.X(iv).

Assume $x \in E(z)$ with the intention of computing $O(x)$ from v, x. $O(x)$ is the unique δ such that

$$\delta < \kappa \leq \kappa_r^{v,x} \quad \text{and} \quad x \in L(\delta + 1, z) - L(\delta, z).$$

By reflection $O(x) \leq_E v, x$. Gandy selection (Theorem 4.1.X) provides an e such that for all $x \in E(z)$,

$$\{e\}(v, x)\downarrow \quad \text{and} \quad \{e\}(v, x) = O(x).$$

$E(z)$ is enumerated as follows. Assume x is arbitrary. Try to compute $\{e\}(v, x)$. If $(e)(v, x)\downarrow$, then enumerate x if $E(z)$ is not a subset of $L(\{e\}(v, x), z)$. □

A partial converse to Theorem 2.4 exists (Sacks [1986]). If γ is an ordinal and $E(\gamma)$ is E-recursively enumerable in some $b \in E(\gamma)$, then $E(\gamma)$ is Σ_1 admissible. The argument does not relativize to a set z of ordinals.

Further results on the enumerability of $L(\kappa)$, when $L(\kappa)$ is E-closed, are given in Sacks [1986]. A complete resolution of the problem is available when $L(\kappa)$ is not Σ_1 admissible. The admissible case is open.

Suppose $L(\kappa)$ is E-closed and not Σ_1 admissible. Then (i) and (ii) are equivalent.

(i) $2^{\mathrm{gc}(\kappa)} \cap L(\kappa)$ is E-recursively enumerable in some $b \in L(\kappa)$.

(ii) $L(\kappa) \models [\omega < \text{cofinality } (\mathrm{gc}(\kappa)) < \mathrm{gc}(\kappa)]$; or $L(\kappa) \models [\omega = \text{cofinality } \mathrm{gc}(\kappa)]$ and every amenable subset of $\mathrm{gc}(\kappa)$ belongs to $L(\kappa)$.

(A set $Z \subseteq \mathrm{gc}(\kappa)$ is *amenable* (Jensen) if $(Z \cap \delta) \in L(\mathrm{gc}(\kappa))$ for all $\delta < \mathrm{gc}(\kappa)$.)

The cofinality ω case is handled by a variant of Green's compactness theorem [1974]. The uncountable singular cofinality case is managed by fine structure results of S. Friedman [1981].

2.5–2.6 Exercises

2.5. Suppose

$$E(2^\omega) \models [\mathrm{card}(2^\omega) = \omega_1].$$

Show $E(2^\omega) \cap 2^{2^\omega}$ is not E-recursively enumerable in any $b \in 2^\omega$. Normann selection, Corollary 1.4.XII, makes it possible to replace $[\mathrm{card}\,(2^\omega) = \omega_1]$ by $[\mathrm{card}\,(2^\omega) \text{ is regular}]$.

2.6. Suppose $L(\kappa)$ is E-closed but not Σ_1 admissible. Let $\mathscr{P}$ be a countably closed, set notion of forcing in $L(\kappa)$. (Heed the warning just before Theorem 2.1.) Show for all $a \in L(\kappa)$

$$(p)(\mathrm{Eq})(p \geq q \quad \& \quad q \Vdash [\kappa_r^{\mathscr{G},a} \leq \kappa_r^{\mathscr{P},p,a}]).$$

3. Enumerable Forcing Relations

Assume $L(\kappa)$ is E-closed but not Σ_1 admissible. Let $\mathscr{P} \in L(\kappa)$ be a set notion of forcing as in Section 1. Recall the concept of effective bounding from subsection 1.6: if

(1) $$p \Vdash^* (E\sigma)[|\{e\}(\underline{a}, \mathscr{G})| = \sigma],$$

then $p \Vdash^* |\{e\}(\underline{a}, \mathscr{G})| \leq \underline{\gamma}$ for some $\gamma \leq_E p, \underline{a}, \mathscr{P}$ (uniformly). According to Lemma 1.7, $L(\kappa, G)$ is E-closed if G is $\mathscr{P}$-generic and $\mathscr{P}$ satisfies effective bounding. Another consequence of effective bounding is: relation (1) is E-recursively enumerable on $L(\kappa)$. More intuitively, the forcing relation restricted to "r.e. sentences" is "r.e.". This last follows from effective bounding via Corollary 4.4.X. The purpose of the present section is to prove the converse. The result clarifies the nature of forcing over E-closed structures, but does not help to show any particular notion of forcing preserves E-closure.

The notion of recursive enumerability on $L(\kappa)$ is boldface; any member of $L(\kappa)$ can occur as a parameter. For simplicity it is assumed in Theorem 3.1 that the only parameter needed for the enumeration of relation (1) above is $\mathscr{P}$. This assumption is in agreement with the definition of effective bounding, which requires γ to be computed from p, a, $\mathscr{P}$.

Throughout the present section it is taken for granted that G, $\mathscr{P}$, and all $p \in P$, are equivalent to sets of ordinals in some uniform, effective fashion. Thus it is safe to apply 5.3–5.6 of Chapter X.

3.1 Theorem (Sacks & Slaman). *Assume $L(\kappa)$ is E-closed but not Σ_1 admissible. Let $\mathscr{P} \in L(\kappa)$ be a set notion of forcing. Then (i) $\equiv$ (ii).*

(i) $\mathscr{P}$ satisfies effective bounding.

(ii) The relation, $p \Vdash^* (E\sigma)[|\{e\}(\underline{a}, \mathscr{G})| = \sigma]$, is E-recursively enumerable on $L(\kappa)$.

Proof. (i) implies (ii) by Corollary 4.4.X.

Assume (ii) holds. For simplicity suppress $\mathscr{P}$. Suppose in hope of a contradiction that (1) holds

(1) $$p \Vdash^* (E\sigma)[\sigma = |\{e\}(\underline{a}, \mathscr{G})| \geq \kappa_r^{\underline{a},\underline{p}}].$$

Gandy selection implies there is an e^* such that:

$$\{e^*\}(p, a, G)\downarrow \quad \text{iff} \quad (\text{Ee})[\infty > |\{e\}(a, G)| \geq \kappa_r^{\underline{a}, \underline{p}}].$$

Gandy chooses the shortest computation, so (1) is equivalent to $p \Vdash^* (E\sigma)[|\{e^*\}(\underline{p}, \underline{a}, \mathscr{G})| = \sigma]$. Then (ii) implies

(2) (1) is E-recursively enumerable on $L(\kappa)$.

Let f be such that on $L(\kappa)$:

(3) $\{f\}(p, e, b)\downarrow$ iff

$$p \Vdash^* (E\sigma)[|\{e\}(\underline{b}, \mathscr{G})| = \sigma].$$

There exists a recursive function t such that

$$\{t(c)\}(\langle p, a\rangle, G) \simeq \begin{cases} 0 \text{ if some } w \leq_E a, p, G \text{ is a} \\ \qquad \text{witness to } \{f\}(p, c, \langle p, a\rangle)\uparrow, \\ \text{undefined otherwise.} \end{cases}$$

Let e_0 be a fixed point of t. Then

(4) $\{e_0\}(\langle p, a\rangle, G)\downarrow$ iff some $w \leq_E a, p, G$ is a witness to $\{f\}(p, e_0, \langle p, a\rangle)\uparrow$.

To see that

(5) $$\{f\}(p, e_0, \langle p, a\rangle)\uparrow,$$

assume otherwise. Then by (3),

$$p \Vdash^* (E\sigma)[|\{e_0\}(\langle \underline{p}, \underline{a}\rangle, \mathscr{G})| = \sigma].$$

Choose a generic $G_0 \in p$. Then $\{e_0\}(\langle p, a\rangle, G_0)\downarrow$, and by (4), $\{f\}(p, e_0, \langle p, a\rangle)\uparrow$. So (5) is true. It follows from Lemma 5.3.X that there is a witness w to (5) such that $w \leq_E \kappa_r^{a,p}$. Consider a generic $G_1 \in p$. By (1),

$$\kappa_r^{a,p} < \kappa_r^{a,G_1} \leq \kappa_r^{p,a,G_1}.$$

By reflection there is a witness w to (5) E-recursive in p, a, G_1. By (4),

(6) $$\{e_0\}(\langle p, a\rangle, G_1)\downarrow.$$

Note that (6) holds thanks to a computation of height $h \leq_E \kappa_r^{a,p} < \kappa$. Thus

(7) $$|\{e_0\}(\langle p, a\rangle, G_1)| < \kappa.$$

Since (7) holds for *all* generic $G_1 \in p$, it follows that

$$p \Vdash^* (E\sigma)[|\{e_0\}(\langle \underline{p}, \underline{a} \rangle, \mathscr{G})| = \sigma].$$

But then (3) contradicts (5) with $\langle p, a \rangle$ in place of b. Hence supposition (1) was false, and so

(8) $$\text{not}\,[p \Vdash^* (E\sigma)(\sigma = |\{e\}(\underline{a}, \mathscr{G})| \geq \kappa_r^{\underline{a},\underline{p}})]$$

is proved for all e, p and a.

Now (i) can be proved. Assume

$$p \Vdash^* (E\sigma)[|\{e\}(\underline{a}, \mathscr{G})| = \sigma].$$

By Lemma 3.2 below, it suffices to show

(9) $$p \Vdash^* |\{e\}(\underline{a}, \mathscr{G})| < \kappa_r^{\underline{a},\underline{p}}.$$

Suppose (9) fails in order to contradict (8). So there is a $q \leq p$ such that

(11) $$q \Vdash^* |\{e\}(\underline{a}, \mathscr{G})| \geq \kappa_r^{\underline{a},\underline{p}}.$$

For $q \leq p$, the negation of (11) is equivalent to

(12) $$(E\gamma)[\gamma < \kappa_r^{a,p} \quad \& \quad (\mathrm{Er})_{q \geq r}(r \Vdash |\{e\}(\underline{a}, \mathscr{G})| = \underline{\gamma})].$$

By Proposition 1.3, "$r \Vdash |\{e\}(\underline{a}, \mathscr{G})| = \underline{\gamma}$" is E-recursive in γ. By reflection, (12) is E-recursively enumerable in a, p. (Recall that parameter $\mathscr{P}$ is being suppressed.) Hence (11), for $q \leq p$, is co-E-recursively enumerable in a, p. Kechris's basis theorem (5.1.X) supplies a $q_0 \leq p$ such that (11) holds and $\kappa_r^{a,p,q_0} \leq \kappa_r^{a,p}$. But then

$$q_0 \Vdash^* (E\sigma)(\sigma = |\{e\}(\underline{a}, \mathscr{G})| \geq \kappa_r^{\langle \underline{a}, \underline{p} \rangle, \underline{q_0}}),$$

which contradicts (8) with p replaced by q_0 and a by $\langle a, p \rangle$. $\square$

The next lemma relates bounds on κ_r to bounds on κ_0 in generic extensions. It is a useful general fact about forcing over E-closed structures.

3.2 Lemma. *Assume $L(\kappa)$ is E-closed but not Σ_1 admissible. Let $\mathscr{P} \in L(\kappa)$ be a set notion of forcing. If*

$$p \Vdash^* |\{e\}\,(\underline{a}, \mathscr{G})| \leq \kappa_r^{p,a,\mathscr{P}},$$

then

$$p \Vdash^* |\{e\}\,(\underline{a}, \mathscr{G})| \leq \gamma$$

for some $\gamma \leq_E p, a, \mathscr{P}$.

Proof. By Lemma 5.5.X,

(1) $$\kappa_r^{p,a,\mathscr{P}} \leq \kappa_r^{p,a,q,\mathscr{P}}$$

for all $q \in P$.

Assume $e = 2^m \cdot 3^n$. Then

(2) $$(q)_{p \geq q}(\mathrm{Er})_{q \geq r}(E\delta)[r \Vdash |\{m\}(\underline{a}, \mathscr{G})| = \delta].$$

Clearly the δ of (2) is less than $\kappa_r^{p,a,\mathscr{P}}$. It follows from (1) and reflection that the r and δ of (2) can be taken to be E-recursive in p, a, q, $\mathscr{P}$. By Corollary 4.5.X to Gandy selection, r and δ can be taken to be partial E-recursive functions of p, a, q, $\mathscr{P}$. As q ranges over all possible extensions of p, δ is bounded by some $\delta_0 \leq_E p, a, \mathscr{P}$. Hence

$$p \Vdash^* |\{m\}(\underline{a}, \mathscr{G})| \leq \underline{\delta}_0.$$

Fix $q \leq p$ and suppose

(3) $$q \Vdash |\{m\}(\underline{a}, \mathscr{G})| = \underline{\delta}$$

for some δ. Then $\delta \leq_E q, p, a, \mathscr{P}$ by Proposition 1.3, since $\delta \leq \delta_0$. Let K_q^t be the set of all $r \leq q$ such that

$$r \Vdash t \in \{m\}(\underline{a}, \mathscr{G}) \qquad (t \in \mathscr{T}(q, m, a, \mathscr{G}, \delta)).$$

For each $r \in K_q^t$, there is an $s \leq r$ such that

$$s \Vdash |\{n\}(t)| = \underline{\sigma} \quad \text{for some} \quad \sigma < \kappa_r^{p,a,\mathscr{P}}.$$

The argument given above for δ_0 is now repeated three times. As q ranges over K_q^t, σ is bounded by some $\sigma_0 \leq_E t, q, p, a, \mathscr{P}$. As t ranges over $\mathscr{T}(q, m, a, \mathscr{G}, \delta)$, σ_0 is bounded by some $\sigma_1 \leq_E q, p, a, \mathscr{P}$. As q ranges over all extensions of p that satisfy (3), σ_1 is bounded by some $\sigma_2 \leq_E p, a, \mathscr{P}$.

Let γ be $\max(\sigma_2, \delta_0) + 1$. □

3.3 Exercise. Formulate and prove a theorem similar to 3.1 for class forcing.

4. Countable-Chain-Condition Forcing

Let $L(\kappa)$ be E-closed and $\mathscr{P} \in L(\kappa)$ be a set notion of forcing. If $p, q \in P$ and

$$\sim(\mathrm{Er})[p \geq r \quad \& \quad q \geq r],$$

then p and q are said to be *incompatible*. A set of mutually incompatible conditions is called an antichain. $\mathscr{P}$ is said to satisfy the *countable chain condition* (c.c.c.) in $L(\kappa)$ if every antichain in $L(\kappa)$ is countable in $L(\kappa)$. In a moment it will be shown that c.c.c. forcing preserves E-closure. It will be seen from the proof that c.c.c. forcing differs radically from countably closed forcing. The former preserves the κ_r spectrum, but the latter does not (cf. Corollary 4.4 and Exercise 2.6).

The standard argument from set theory that finitely supported, iterated c.c.c. forcing is c.c.c. works without change inside $L(\kappa)$. An application of this fact to a problem in E-recursion is given in Section 5.

The next theorem proves more than is promised above. Let ρ be a regular cardinal in the sense of $L(\kappa)$. $\mathscr{P}$ is said to satisfy the *ρ-chain condition* in $L(\kappa)$ if every antichain in $L(\kappa)$ has cardinality less than ρ. (Note that the ω_1-chain condition is equivalent to the countable chain condition.) Selection plays a part in the study of forcing relations that fulfill the ρ-chain condition. Let γ be a cardinal in the sense of $L(\kappa)$. $L(\kappa)$ is said to obey *less-than-γ* selection if there exists a partial E-recursive (in γ and possibly another parameter) function f such that for all $e < \omega$, $\delta < \gamma$ and $p \in L(\kappa)$:

$$(\mathrm{Ex})_{x<\delta}[\{e\}(p, x)\downarrow] \rightarrow [f(e, \delta, p)\downarrow \quad \& \quad \{e\}(p, f(e, \delta, p))\downarrow].$$

Thanks to Gandy selection, $E(\omega_1^L)$ obeys less-than-ω_1 selection (cf. Exercise 4.6). In contrast $E(\omega_{\omega_1}^L)$ does not obey less-than-ω_2 selection (cf. Exercise XII.1.5). A theorem of Normann implies $E(\gamma)$ obeys less-than-γ selection when γ is a regular cardinal of L (cf. Theorem XII.1.3).

To grasp more quickly the proof of Theorem 4.1, assume $L(\kappa)$ is $E(\omega_1)$ and $\rho = \omega$. Then only cases 0 and 2 apply.

4.1 Theorem (Sacks 1986). *Let $L(\kappa)$ be E-closed and $\mathscr{P} \in L(\kappa)$ a set notion of forcing. Assume $\mathscr{P}$ satisfies the ρ-chain condition, and $L(\kappa)$ obeys less-than-ρ selection. If there exist r and δ such that*

$$p \geq r \quad \text{and} \quad r \Vdash |\{e\}(t)| = \underline{\delta},$$

then there exist such r and δ E-recursive in p, t, $\mathscr{P}$, ρ (and background parameters).

Proof. Let $c \leq \kappa$ be such that t, $\mathscr{P}$, $\rho \in L(c)$, and gc exists, the greatest cardinal in the sense of $L(c)$. All cofinalities below are in the sense of $L(c)$. For simplicity $\mathscr{P}$ and all background parameters such as gc or those needed for less-than-ρ selection are suppressed. (As usual $\mathscr{P}$ and all $p \in P$ come effectively coded as relations on ordinals.) Define

$$\min(p, e, t) \simeq \min_{\delta}(\mathrm{Er})_{p \geq r}[r \Vdash |\{e\}(t)| = \underline{\delta}].$$

An effective transfinite recursion on $\min(p, e, t)$, henceforth called the main recursion, will show that $\min(p, e, t)$ is E-recursive in p, t, ρ uniformly. Proving the theorem is equivalent to computing $\min(p, e, t)$ by Proposition 1.3.

Assume $e = 2^m \cdot 3^n$ and $\min(p, e, t)$ is defined. Then $\min(p, m, t)$ is defined and, by the main recursion, equals some $\gamma_0 \leq_E p, t, \rho$. Let

$$\mathcal{T}(p, m, t, \gamma_0) = \{t_i \mid i < \theta\}$$

for some $\theta \leq_E p, t, \rho$ such that $\theta \leq \text{gc}$.

For each $i < \theta$, Q_i will be a set of conditions that settles the fate of t_i. For all $q \in Q_i$:

(1a) $$p \geq q;$$

(1b) $$q \Vdash t_i \in \{m\}(t) \quad \text{or} \quad q \Vdash t_i \notin \{m\}(t);$$

(1c) $$\text{if} \quad q \Vdash t_i \in \{m\}(t), \text{ then } q \Vdash (E\delta)[|\{n\}(t_i)| = \delta].$$

Initially all Q_i's are empty. The final Q_i's will be such that

(2) $$\bigcap_{i < \theta} Q_i \neq \varnothing.$$

Intuitively (2) means there is a condition r such that for all i, the conditions in Q_i are *dense* in r. More precisely, (2) means

$$(\mathrm{E}r)(r_1)_{r \geq r_1}(i)_{i < \theta}(\mathrm{E}q)(q \in Q_i \ \&\ r_1, q \text{ are compatible}).$$

The Q_i's are built up simultaneously by recursion on $\beta \leq \theta$, henceforth called the beta-recursion. During stage β conditions are added to Q_i for various $i < \beta$. At the end of stage β

$$\cap \{Q_i \mid i < \beta\} \neq \varnothing.$$

Stage β. There are four cases.

Case 0: $\beta < \rho$. A subrecursion of length ρ adds conditions to Q_i for various $i < \beta$. At stage $\beta_\gamma (\gamma < \rho)$ at most one q is added to at most one Q_i.

Stage β_γ. Suppose $\cap \{Q_i \mid i < \beta\} = \varnothing$ after all the additions prior to stage β, and prior to stage β_γ; otherwise go to the next stage. It is safe to assume each Q_i already contains all $q \leq p$ such that

$$q \Vdash t_i \notin \{m\}(t),$$

because all such q's can be added to Q_i at stage β_0. Q_i can be treated as if it were a condition by taking

$$Q_i \geq r \text{ to mean } (q_1)_{r \geq q_1} (\mathrm{E}q)(q \in Q_i \ \&\ q_1, q \text{ are compatible}).$$

Thus (2) means $(\mathrm{E}r)(i)\ (Q_i \geq r)$, and for all $i < \beta$ the steps taken at stage β_0 imply

$$(p - Q_i) \Vdash^* t_i \in \{m\}(t)$$

unless $p - Q_i$ is not a condition (that is, there is no $q \le p - Q_i$). Since $\min(p, e, t)$ is defined, there is an $r \le p$ such that for all i,

$$r \Vdash^* [t_i \in \{m\}(t) \rightarrow (E\delta)(|\{n\}(t_i)| = \delta)].$$

But $p = \cup \{p - Q_i | i < \beta\}$, so there must be an $i < \beta$ such that

$$\min(p - Q_i, n, t_i) \tag{3}$$

is defined. According to the main recursion, (3), if defined, is E-recursive in $p - Q_i$, t_i, ρ uniformly. Note that $p - Q_i$ and t_i are E-recursive in certain parameters and i uniformly. Consequently less-than-ρ selection makes it possible to compute an $i < \rho$ for which (3) is defined; call it i_0 and let δ_0 be the value of (3) when $i = i_0$. From δ_0 a q_0 can be computed via Proposition 1.3 such that

$$p - Q_{i_0} \ge q_0 \quad \& \quad q_0 \Vdash t_{i_0} \in \{m\}(t) \quad \& \quad q_{i_0} \Vdash \{n\}(t_{i_0})| \le \underline{\delta}_0.$$

Add q_0 to Q_{i_0}.

For each $i < \beta$, the conditions added to Q_i as γ increases from 0 to ρ, are mutually incompatible. Since $\beta < \rho$, it follows from the ρ-chain condition that for all sufficiently large γ, nothing is added to any $Q_i (i < \beta)$. Thus for all sufficiently large γ, $\cap \{Q_i | i < \beta\}$ is nonempty at stage β_γ, hence nonempty at the end of stage β.

Case 1: $\mathrm{gc} > \beta \ge \rho > \mathrm{cf}(\beta)$. Similar to Case 0. Let β^+ be the least cardinal greater than β in the sense of $E(c)$. A subrecursion of length β^+ adds conditions to Q_i for various $i < \beta$. Since $\rho > \mathrm{cf}(\beta)$, there is a $W \subseteq \beta$ such that $\sup W = \beta$ and the ordertype of W is less than ρ. Define

$$Q_i^- = \cap \{Q_j | j < i\}.$$

Then $\cap \{Q_i^- | i \in W\} = \cap \{Q_j | j < \beta\}$. Stage β_γ $(\gamma < \rho \cdot \beta^+)$ acts directly on at most one Q_i^- for some $i \in W$. The restriction of i to W makes it possible to use less-than-ρ selection as in case 0.

The augmentation of some Q_i^- $(i \in W)$ at stage β_γ is performed with the aid of procedure P_i developed at stage i of the beta-recursion. This makes sense because $i < \beta$. P_i, given an empty Q_i^-, delivered a non-empty Q_i^-. P_i did its work by adding conditions to various Q_j's $(j < i)$. P_i is effective because it is based on the main recursion. P_i converged because: if each Q_j is extended to a maximal antichain $Q_j^m \le p$, then $\cap \{Q_j^m | j < i\}$ must weakly force $(E\delta)(j)_{j<i}[|\{n\}(t_i)| = \delta]$. P_i slowly builds up the antichains and stops as soon as their intersection is nonempty.

P_i^+ is a slight extension of P_i. It proceeds in precisely the same manner as P_i but is allowed to begin with a nonempty Q_i^-, denoted by start (Q_i^-). P_i^+ stops when

$$\text{finish } (Q_i^-) - \text{start } (Q_i^-) \ne \varnothing.$$

Thus P_i^+ is said to *augment* Q_i^-. P_i^+ converges iff such an augmentation is possible, that is iff

$$\cap\{Q_j^m \mid j < i\} - \mathrm{start}(Q_i^-) \neq \varnothing.$$

Stage β_γ (for case 1). Suppose $Q_\beta^- = \varnothing$ after all the additions and augmentations prior to stage β, and prior to stage β_γ; otherwise go to the next stage. The argument given in Case 0 that (3) is defined now shows $P_i^+(Q_i^-)$ is defined for some $i \in W$. Thanks to the main recursion, $P_i^+(Q_i^-)$, if defined, is E-recursive in certain parameters and i uniformly. Consequently less-than-ρ selection computes such an i; call it i_0. Augment $Q_{i_0}^-$ to $P_{i_0}^+(Q_{i_0}^-)$.

Now consider the action of stage β_γ as γ increases from 0 to β^+. $Q_j(j < \beta)$ is always an antichain, so additions are made to Q_j at a set of stages of cardinality less than ρ. Hence for all sufficiently large γ, nothing is added to any $Q_j(j < \beta)$, and no Q_i^- $(i \in W)$ is augmented. Thus for all sufficiently large γ, Q_β^- is nonempty at stage β_γ, hence nonempty at the end of stage β.

Case 2: $\mathrm{cf}(\beta) \geq \rho$. According to the β-recursion, $Q_i^- \neq \varnothing$ for all $i < \beta$. It follows from the ρ-chain condition that $Q_\beta^- \neq \varnothing$.

Case 3: $\mathrm{gc} = \beta$ & $\rho > \mathrm{cf}(gc)$. By Exercise 2.11.XII, there is a uniform method for choosing an element of a nonempty E-recursively enumerable (in p) subset of β, for $p \in E(c)$. Proceed as in Case 0.

At the end of the β-recursion, the end of stage θ, $Q_\theta^- \neq \varnothing$. In other words there is an $r_0 \leq Q_\theta^-$ such that

$$r_0 \Vdash^* |\{e\}(t)| \leq \underline{\delta}_0,$$

where δ_0 is the $\sup^+$ of γ_0 and all the values of (3) computed during the β-recursion. The value of $\min(p, e, t)$, needed for the main recursion, is computable from δ_0 with the aid of Proposition 1.3. $\square$

4.2 Countable Chain Selection. The proof of Theorem 4.1 (ρ-chain condition forcing) should be contrasted with that of theorem 2.1 (countably closed forcing). The latter relies on Moschovakis witnesses and reflection phenomena, while the former is a selection argument that holds for a narrow class of Σ_1 relations. Consider the special case of $L(K) = E(\omega_1)$ and $\rho = \omega_1$. Let $\mathscr{P}$ be a countable-chain-condition notion of forcing in $E(\omega_1)$. The relation on p, e and t given by

$$(1) \qquad (\mathrm{Er})(p \geq r \Vdash (E\sigma)[|\{e\}(t)| = \sigma])$$

is Σ_1 over $E(\omega_1)$. The proof of Theorem 4.1 shows how to compute an r (if there is one) that satisfies the matrix of (1) from p, t and $\mathscr{P}$. It follows that relation (1) is E-recursively enumerable on $E(\omega_1)$. Thus Theorem 4.1 is a selection theorem that shows certain Σ_1 relations, derived recursively from a chain condition, are E-recursively enumerable.

The absence of κ_r in the proof of Theorem 4.1 accounts for the difference between Exercise 2.6 and Corollary 4.4.

$\mathscr{P}$-genericity was defined in subsection 1.6. The point of Corollary 4.3 is at its sharpest when

$$L(\kappa) \models [\mathrm{gc}(\kappa) \text{ is singular}].$$

4.3 Corollary. *Suppose $L(\kappa)$ is E-closed and $\mathscr{P} \in L(\kappa)$ is a countable-chain-condition, set notion of forcing. Then every $\mathscr{P}$-generic extension of $L(\kappa)$ is E-closed.*

Proof. According to Exercise 4.6, Gandy selection implies $L(\kappa)$ obeys less than ω_1 selection. By Theorem 4.1 $\mathscr{P}$ satisfies effective bounding. Now apply Lemma 1.7. $\square$

4.4 Corollary (Slaman). *Let $L(\kappa)$ be E-closed and $\mathscr{P} \in L(\kappa)$ a set notion of forcing. Assume $L(\kappa) = E(\mathrm{gc}(\kappa))$, $\mathscr{P}$ satisfies the ρ-chain condition, and $L(\kappa)$ obeys less-than-ρ selection. Then*

$$\kappa_r^{G,a} \leq \kappa_r^a$$

for all $\mathscr{P}$-generic G and $a \in L(\kappa)$. (Parameters ρ, $\mathrm{gc}(\kappa)$ and $\mathscr{P}$ are suppressed.)

Proof. For simplicity suppress $\mathscr{P}$, ρ and various background parameters. It suffices to show

(1) $$\phi \Vdash^* [\kappa_0^{\mathscr{G},a} < \kappa_r^a].$$

To prove (1) a maximal antichain is constructed. r_γ and $\delta_\gamma (\gamma < \rho)$ are defined by recursion on γ. Suppose there are r and δ such that

(1a) r is incomparable with r_β for all $\beta < \gamma$, and

(1b) $(\mathrm{E}e)(r \Vdash [|\{e\}(\mathscr{G}, \underline{a})| = \underline{\delta}])$.

Note that (1a) is equivalent to $r \Vdash (\beta)_{\beta < \gamma} (\mathscr{G} \notin r_\beta)$. Let $\langle r_\gamma, \delta_\gamma \rangle$ be such an r and δ computed from

$$\{\langle r_\beta, \delta_\beta, \beta \rangle | \beta < \gamma\}, a$$

with aid of Theorem 4.1 and Gandy selection. If there are no such r and δ, then $\langle r_\gamma, \delta_\gamma \rangle$ is undefined.

The ρ-chain condition implies $\langle r_\gamma, \delta_\gamma \rangle$ is undefined for some $\gamma < \rho$; let γ_∞ be the least such. γ_∞ is the least solution of a predicate co-E-recursively enumerable in a. By Exercise 5.17.X, a corollary to Kechris's basis theorem (5.1.X), $\kappa_r^{a,\gamma_\infty} \leq \kappa_r^a$. Let

$$\delta_\infty = \sup\{\delta_\gamma | \gamma < \gamma_\infty\}.$$

Then $\delta_\infty \leq_E \gamma_\infty, a$, and so $\delta_\infty < \kappa_r^a$. Finally

$$\phi \Vdash^* \kappa_0^{\mathscr{G},a} \leq \delta_\infty. \quad \square$$

4.5–4.6 Exercises

4.5. Let $\mathscr{P}$ be Cohen forcing with finite conditions on a real. Suppose $L(\kappa)$ is E-closed and $G \subseteq \omega$ is $\mathscr{P}$-generic over $L(\kappa)$. Show

$$\kappa_0^{G,a} = \kappa_0^a$$

for all $a \in L(\kappa)$. (Note: do not assume $a \subseteq \kappa$.)

4.6. Let ρ be ω_1 in the sense of some E-closed $L(\kappa)$. Show $L(\kappa)$ obeys less-than-ρ selection.

5. Normann Selection and Singular Cardinals

Several definitions are needed for the sake of an application of countable-chain-condition forcing over $E(\omega_{\omega_1})$. Let R be an arbitrary class of sets. The Normann schemes, (1)–(6) of Section 1.X, relativize readily to R. Simply replace $\{\ \}$ by $\{\ \}^R$ throughout (1)–(6), and add a new scheme,

$$(7) \qquad \{e\}^R(x_1, \ldots, x_n) = R \cap x_1 \quad \text{if} \quad e = \langle 7, n \rangle.$$

A partial function f from V into V is *partial E-recursive relative to* R if there exists an e such that $f(x) \simeq \{e\}^R(x)$ for all $x \in V$. Thus "partial E-recursive" is equivalent to "partial E-recursive relative to the empty set".

Let A be a transitive set. A is said to be E-closed relative to R if

$$\vec{y} \in A \quad \& \quad f(\vec{y})\downarrow \rightarrow f(\vec{y}) \in A$$

for every f partial E-recursive relative to R. The E-closure of an arbitrary set x relative to R, in symbols $E(x; R)$, is the least E-closed (relative to R) $y \subseteq TC(\{x\})$. Observe that

$$E(x; R) = E(x; R \cap E(x; R)).$$

Assume $R \subseteq E(2^\omega)$ and

$$E(2^\omega; R) \models [2^\omega \text{ is wellordered}].$$

It follows there is a greatest cardinal in the sense of $E(2^\omega; R)$; call it gc. Assume

$$(1) \qquad E(2^\omega; R) \models [\text{gc is regular}].$$

Statement (1) is Norman's regularity assumption. It is the principal hypothesis in Normann's selection theorem and its extensions discussed in the next chapter. One such extension has as its conclusion:

(2) If $\delta < gc$ and A is E-recursively enumerable relative to R, then $(A \cap \delta) \in E(2^\omega; R)$.

(2) is informally put as: every *small* R–RE class belongs to $E(2^\omega; R)$. (2) figured prominently in Normann's study of Post's problem. These matters will be discussed in greater detail in the next two chapters. The matter at hand is: is (1) necessary for (2)? The answer is no according to the next result.

5.1 Theorem. *There exists a model M of ZFC in which for some $R \subseteq \omega_{\omega_1} \times 2^\omega$:*

$$E(2^\omega; R) \models [\text{card}(2^\omega) = \omega_{\omega_1}],$$

and every small R–RE class belongs to $E(2^\omega; R)$.

Proof. Let $L(\alpha)$ be a model of ZFC, and $\mathscr{P}$ be Cohen forcing with finite conditions designed to produce a generic real. $\mathscr{P}_{\omega_{\omega_1}}$ is the result of iterating $\mathscr{P}$ with finite support ω_{ω_1} times. Let $L(\kappa)$ be $E(\omega_{\omega_1})$. $\mathscr{P}_{\omega_{\omega_1}} \in L(\kappa)$ and satisfies the countable chain condition in $L(\kappa)$. Let R be $\mathscr{P}$-generic over $L(\alpha)$, hence over $L(\kappa)$. Then $L(\kappa, R) = E(R)$ by Corollary 4.1. Standard set-theoretic arguments show $L(\alpha, R)$ is a model of ZFC, call it M, with the same cardinals as $L(\alpha)$. Note that in M, $L(\kappa, R) = E(2^\omega; R)$.

A typical small R–RE class is

$$(1) \qquad A = \{\gamma \mid \{e\}^R(\gamma)\downarrow \quad \& \quad \gamma < \delta\},$$

where $\delta < \omega_{\omega_1}$. To see that $A \in E(2^\omega; R)$, let

$$A_\gamma = \{p \mid p \Vdash (E\sigma)[|\{e\}^{\mathscr{R}}(\underline{\gamma})| = \sigma]\}$$

for each $\gamma < \delta$. In $L(\alpha)$: A_γ is equivalent to a countable antichain A_γ^0; $\{\langle \gamma, A_\gamma^0 \rangle \mid \gamma < \delta\}$ has the same complexity as a bounded subset of ω_{ω_1}, hence belongs to $L(\omega_{\omega_1})$. Then

$$\gamma \in A \leftrightarrow \gamma < \delta \quad \& \quad (\text{E}p)[p \in A_\gamma^0 \quad \& \quad R \text{ satisfies } p],$$

and so $A \in E(2^\omega; R)$. $\square$

An application of iterated, countable-chain-condition forcing, deeper than that of Theorem 5.1, has been made by Slaman [1983]. He shows: if ZFC is consistent, then ZFC and $2^\omega = \omega_{\omega_1}$ and the extended plus-one hypothesis is consistent. See subsection 3.7.XII for the missing definitions. Normann proved the extended plus-one hypothesis with the aid of his regularity assumption, formula (1) of Section 5. Slaman [1983] showed that regularity is not essential by an almost-disjoint forcing argument based on Harrington [1973]. In Slaman's model,

$$E(2^{\omega}) \models [\mathrm{card}\,(2^{\omega}) = \omega_{\omega_1}]$$

and the extended plus-one hypothesis holds. It is not known if the hypothesis fails in some model of ZFC in which

$$E(2^{\omega}) \models [2^{\omega} \text{ is wellordered}].$$

6. Further Forcing

Countably closed and countable chain condition forcings are special cases of Axiom A forcing, an invention of Baumgartner. Sacks [199?] shows that Axiom A forcing preserves E-closure, as does Shelah's proper forcing. Also given is an example of set forcing that preserves cardinals but not E-closure.

Some results on class forcing over E-closed structures are available. One such proves Theorem 6.1; the forcing conditions are E-pointed, perfect trees.

6.1 Theorem (Sacks & Slaman [1987]). *Let $L(\kappa)$ be countable, E-closed and not Σ_1 admissible with greatest cardinal* $\mathrm{gc}(\kappa)$. *If*

$$L(\kappa) \models [\mathrm{cf}\,(\mathrm{gc}(\kappa)) > \omega],$$

then for some $G \subseteq \omega_1^{L(\kappa)}$, $L(\kappa, G) = E(G)$.

A partial converse to Theorem 6.1 is proved in the next chapter, Corollary 5.2.XII. It states: if x is a set of ordinals and

$$E(x) \models [\text{cofinality of the greatest cardinal is } \omega],$$

then $E(x)$ is Σ_1 admissible.

Another use of class forcing over an E-closed structure, inspired by Steel forcing, has been found by Slaman. At one swoop he settles many questions.

6.2 Theorem (Slaman [1985]). *Let $L(\kappa)$ be countable and E-closed. There exists an $x \subseteq 2^{\omega}$ such that:*

(i) $L(\kappa, x) = E(x)$;
(ii) $L(\kappa, x)$ *does not admit Moschovakis witnesses;*
(iii) if $L(\kappa)$ is Σ_1 admissible but not the E-closure of any $y \in L(\kappa)$, then $L(\kappa, x)$ is Σ_1 admissible, but Σ_1 over $L(\kappa, x)$ differs from E-recursively enumerable on $L(\kappa, x)$;
(iv) if $\kappa_r^a < \kappa$ for all $a \in L(\kappa)$, then $\kappa_r^a < \kappa$ for all $a \in L(\kappa, x)$;
(v) there is no Σ_1 formula $\mathscr{F}(x)$ such that for all $a \in L(\kappa, x)$,

$$L(\kappa_r^a + 1, x) \models \mathscr{F}(\underline{a}) \quad \textit{but} \quad L(\kappa_r^a, x) \models \sim \mathscr{F}(\underline{a}).$$

The content of 6.2 is considerable. (i) says each countable E-closed ordinal is the least such relative to some set of reals. (ii) says the E-closure of a set of reals can be inadmissible yet not admit Moschovakis witnesses. Thus the assumption that x is a set of ordinals, in Thoerem 5.7.X, was needed. (v) says there is no uniform (in a) failure of Σ_1 reflection at $\kappa_r^a + 1$ in $L(\kappa, x)$ for some $x \subseteq 2^\omega$. It is tempting to think there is a uniform failure having to do with existence of Moschovakis witnesses. That idea succeeds in $E(2^\omega)$, as pointed out by Harrington [1973]. For each $a \subseteq \omega$, $\kappa_r^{a, 2^\omega}$ is the least ordinal that suffices to construct witnesses for all e such that $\{e\}(a, 2^\omega)\uparrow$. The idea also succeeds in $E(z)$ when z is a set of ordinals and $E(z)$ is inadmissible.

A question not answered by Theorem 6.2 is: is there a set x of ordinals and a $y \in E(x)$ such that $E(x)$ is inadmissible and $\kappa_r^y \notin E(x)$?