Part A

Introduction, Basic Theory and Examples

This part of the book provides a basic setting for the chapters that follow, by isolating examples and concepts that have emerged as central and by presenting some of the more basic methods and results. Chapter I discusses how the subject of model-theoretic logics got started, both the parts that have to do with extended logics, and the part having to do with abstract model theory. The chapter presupposes familiarity with only the most basic parts of first-order model theory, its syntax and semantics.

In Chapter II the basic concept of a logic is presented, with many examples, as well as the concepts of elementary and projective class and compactness, Löwenheim–Skolem and definability properties. The notion of one logic being stronger than another is introduced and studied. Examples discussed include higher-order logics, logics with cardinality and cofinality quantifiers, infinitary logics and other logics with generalized quantifiers and logical operations.

Given any particular logic $\mathscr{L}$ one central problem is that of understanding when two structures are $\mathscr{L}$-equivalent, that is, satisfy the same $\mathscr{L}$-sentences. Among the basic results of Chapter II is a characterization of $\mathscr{L}$-equivalence in terms of partial isomorphisms, for a wide range of $\mathscr{L}$. Here we have a good example of a method borrowed from first-order logic which really comes into its own only in the more general setting. Another important method presented in Chapter II is the use of projective classes (PC) for establishing countable compactness and recursive axiomatizability for a host of logics.

Chapter III begins with an exposition of Lindstrom's theorem, which shows that first-order logic is the strongest logic (of ordinary structures) which satisfies the compactness and Löwenheim–Skolem properties. First-order logic is also shown to be maximal with respect to other combinations of familiar properties. The methods used are those of partial isomorphisms and projective classes.

Lindstrom's theorem has become a paradigm for characterizing other logics. Among those discussed in Chapter III are certain infinitary logics and logics with added quantifiers. Chapter III ends with an abstract characterization theorem which covers Lindstrom's theorem as well as logics for other types of structures, like topological structures. This connects with work in Chapter XV.

These chapters are meant to be accessible to anyone with a knowledge of basic model theory for first-order logic. They provide the reader with the basic notions and viewpoint needed to appreciate what follows.

Chapter I
Model-Theoretic Logics: Background and Aims

by J. BARWISE

Two aspects to the study of model-theoretic logics are represented in this volume. First, there is the isolation and study of specific model-theoretic languages, or *logics* as they are called here, for the study of various mathematical properties. Second, there is the investigation into relations between these logics. These two parts of the subject are called *extended model theory* and *abstract model theory*, respectively, and are the two subjects of the two main sections of this chapter.

In writing this chapter I hope to give a perspective from which to view the study of model-theoretic logics. First (in Section 1.2) I will contrast the view of logic implicit in this endeavor with what I call the *first-order thesis*, a view of logic and mathematics which claims that logic *is* first-order logic. Then (in the rest of Sections 1 and 2) I will discuss some of the motivation, ideas, aims, and preconceptions of early workers in the subject. The first is needed to appreciate the most basic definitions. The second is needed to judge the progress made against those early hopes and preconceptions. Where it is natural, I will point ahead to later chapters, but more specific introductions to the chapters will be found at the beginning of each part of the book.

1. Logics Embodying Mathematical Concepts

In extended model theory one asks, "What is the logic of specific mathematical concepts?" More explicitly, given a particular mathematical property (like being a finite, infinite, countable, uncountable, or open set, or being a well-ordering or a continuous function, or having probability greater than some real number r), what is the logic implicit in the mathematician's use of the property? What sorts of mathematical structures isolate the property most naturally? What sorts of languages best mirror the mathematician's talk about the property? What forms of reasoning about it are legitimate? Which other properties are implicit in it or are presupposed by it?

1.1. Logic, Structures and Logics

A word of explanation is in order about the way we are using the words "logic", "structure" and "logics" here. For the person in the street, logic is the study of valid forms of reasoning, from the most mundane uses in our day-to-day lives to the most sophisticated uses in science and mathematics. If you and I are discussing some topic, like fixing the roof, a law of genetics, or the solution to some partial differential equation, and I say "The logic of that escapes me", what I mean is that I do not see how the conclusion you have come to follows from our shared assumptions and concepts, including the conception of the task at hand. How does it follow from the properties of roofs, or the laws of genetics that we both accept, or the concepts involved in differential equations? When I talk of *logic* as I have above, I am referring to this common sense, person-in-the-street notion.

On the common sense view of logic, all the concepts we use to cope with and organize our world have their own logic. As logicians, we are perfectly entitled to delve into their logic. However, as mathematical logicians, or metamathematicians, our interest is more specialized. What we seek to understand is the logic of precise mathematical concepts. Extended model theory makes a frontal attack on this problem by, where appropriate, building "logics" to get answers to some of the questions listed above.

We assume that the reader of this volume is familiar with first-order logic, its syntax, semantics and basic model theory, because first-order logic is the inspiration for extended model theory. The basic idea of model theory, first-order and beyond, is that one can profit by paying attention to the relationship between some *mathematical structures* and some collection of expressions of a language used to describe properties of such structures. The basic notion is that of *satisfaction*: $\mathfrak{M} \models \phi$ if the expression ϕ is *true of*, or satisfied by, the structure $\mathfrak{M}$. First-order logic considers mathematical structures of a particularly algebraic sort, domains of individuals with arbitrary sets and functions to serve as interpretations for various predicate and function symbols. It allows expressions that build in the concepts *and*, *or*, *not*, *every* and *some*, and concepts that can be expressed in terms of them, but nothing else.

First-order model theory is the study of the semantics of this language, and it has become a very sophisticated branch of mathematics, full of its own concepts and theorems, some of extraordinary beauty and complexity. These theorems give insight into and enrichment for those parts of mathematics that happen to fit the shoe of first-order logic. This includes a fairly extensive part of modern algebra. The book Chang and Keisler [1973] provides an excellent introduction to the model theory of first-order logic. In extended model theory, we take the basic idea and expand it in various ways, by allowing richer mathematical structures or richer expressive power in the language, or both.

As used in this book, then, *a logic* consists of a collection of mathematical structures, a collection of formal expressions, and a relation of satisfaction between the two. We are primarily interested in logics where the class of structures are those where some important mathematical property is built in, and where the language gives us a convenient way of formalizing the mathematician's talk about the property. We might say, then, that a logic is something we construct to study the logic of some part of mathematics.

1.2. The First-Order Thesis

If first-order logic is the inspiration for much of extended model theory, it is also its nemesis. The common sense, mathematician-in-the-street view of logic implicit in this subject is at variance with what we teach our students in basic logic courses. There we attempt to draw a line between "logical concepts", as embodied in the so-called "logical constants", and all the rest of the concepts of mathematics. In extended model theory we do not so much question the placement of this line, as question whether there is such a line, or whether all mathematical concepts have their own logic, something that can be investigated by the tools of mathematics.

To give ourselves a foil, let us call the view that attempts to define logic as the logic implicit in the "logical constants" the *first-order thesis.* (Among the numerous past and present adherents to this thesis there is a slight disagreement as to whether identity should be counted as a "logical constant".) Another way to state this view is to claim that logic is first-order logic, so that anything that cannot be defined in first-order logic is outside the domain of logic.

The reasons for the widespread, often uncritical, acceptance of the first-order thesis are numerous. Partly it grew out of interest in and hopes for Hilbert's program. Partly is was spawned by the great success in the formalization of parts of mathematics in first-order theories like Zermelo–Fraenkel set theory. And partly, it grew out of a pervasive nominalism in the philosophy of science in the mid-twentieth century, led by Quine, among others. As late as 1953, well after the Gödel incompleteness theorems, Quine wrote in his book *From a Logical Point of View*:

> The bulk of logical reasoning takes place on a level which does not presuppose abstract entities. Such reasoning proceeds mostly by quantification theory, the laws of which can be represented through schemata involving no quantification over class variables. Much of what is commonly formulated in terms of classes, relations, and even number, can easily be reformulated schematically within quantification theory plus perhaps identity theory. Quine [1953, p. 116].

As logicians we do our subject a disservice by convincing others that logic is first-order logic and then convincing them that almost none of the concepts of modern mathematics can really be captured in first-order logic. Paging through any modern mathematics book, one comes across concept after concept that cannot be expressed in first-order logic. Concepts from set-theory (like *infinite set*, *countable set*), from analysis (like *set of measure* 0 or *having the Baire property*), from topology (like *open set* and *continuous function*), and from probability theory (like *random variable* and *having probability greater than some real number r*), are central notions in mathematics which, on the mathematician-in-the-street view, have their own logic. Yet none of them fit within the domain of first-order logic. In some cases the basic presuppositions of first-order logic about the kinds of mathematical structures one is studying are inappropriate (as the examples from topology or analysis show). In other cases, the structures dealt with are of the sort studied in first-order logic, but the concepts themselves cannot be defined in terms of the "logical constants." For example, by the Löwenheim–Skolem theorem, any countable set of first-order sentences which is true in some structure is true in some countable structure. This shows that the complementary concepts of *countable* and *uncountable* cannot be defined in first-order logic. The compactness theorem, stated below, shows that the concepts of finite and infinite cannot be captured in first-order logic.

Extended model theory adds a new dimension and new tools to the study of the logic of mathematics. The first-order thesis, by contrast, confuses the subject matter of logic with one of its tools. First-order logic is just an artificial language constructed to help investigate logic, much as the telescope is a tool constructed to help study heavenly bodies. From the perspective of the mathematician in the street, the first-order thesis is like the claim that astronomy is the study of the telescope. Extended model theory attempts to take the experience gained in first-order model theory and apply it in ever broader contexts, by allowing richer structures and richer ways of building expressions. It attempts to build languages similar to the first-order predicate calculus to study concepts that are banned from logic by the first-order thesis.

It is not always straightforward to come up with the best language to capture a given concept. For example, the "best" one for studying the concepts of finite and infinite is not at all the one that first came to mind, as we shall see. Similarly, finding the "best" logic of topological structures was a process of successive approximations. In both cases the class of structures is clear: ordinary structures in the first case, topological structures in the second; but the choice of just the right language is difficult. In other cases, even finding just the right collection of structures has been problematic. Finding natural logics takes trial, error and experience. Part of the accumulated experience is discussed in the section on abstract model theory, below.

1.3. The Completeness Problem

Similarly, there is nothing straightforward about knowing the best questions to ask about a given logic. They will depend, in general, on the concepts it captures. But one question always suggests itself just by virtue of being a study of logic, the *completeness problem*: is there any kind of completeness theorem that goes with the logic, analogous to the completeness theorem for first-order logic? That is, given a logic $\mathscr{L}$, is there an effective list of axioms that are valid in all structures of the logic and a list of valid rules of inference that, together with the axioms, generate all valid theorems of the logic, i.e., the set of sentences that hold in all its structures?

Using the language of recursion theory, the completeness problem can be phrased quite abstractly (or crudely, depending on one's point of view). For if it has a positive solution, then the set of valid sentences is recursively enumerable. And, conversely, if the set of valid sentences is recursively enumerable, then in principle we can find such a completeness theorem. However, this does not give one a simple set of axioms and rules of inference which generate the valid sentences. Thus, up to aesthetic considerations, the first question about a logic $\mathscr{L}$ that we usually ask is: Is the set of valid sentences recursively ennumerable? This is sometimes called "abstract completeness."

The completeness problem ties up with the first-order thesis and an even older view of logic, where it was seen as the study of axioms and rules of inference. Of the logics studied here, some have a completeness theorem, some don't. If one thinks of logic as limited to the study of axioms and rules of inference, then logics without an abstract completeness theorem will not seem part of logic. But if you think of logic as the mathematician in the street, then the logic in a given concept is what it is, and if there is no set of rules which generate all the valid sentences, well, that is just a fact about the complexity of the concept that has to be lived with. It is this latter point of view that is implicit in the study of model-theoretic logics.

1.4. Compactness

A major theme in the early days of extended model theory was the search for compact logics, logics which satisfied the following (1) or (2), or some appropriate analogue of them where the concept of finite is replaced by a different notion of small.

(1) (Strong Compactness Property.) If T is any set of sentences of the logic, and if every finite subset of T has a model (i.e., is true in some structure of the logic) then T has a model.
(2) (Countable Compactness Property.) Same as (1), but only for countable sets T.

There are two reasons for interest in these results. One is closely related to the completeness problem. Usually a completeness theorem establishes that if ϕ is a logical consequence of some set (or perhaps countable set) T of assumptions, then it is derivable from some finite subset of T. In particular, if T is inconsistent and so has no models whatsoever, then some contradictory sentence is a consequence of T, in which case some finite subset of T will be inconsistent. That is, usually (1) or (2) fall out of a completeness theorem, if there is one.

Secondly, in first-order model theory, the compactness theorem is a ubiquitous tool, applied at almost every turn. It was natural that it should have been deemed a crucial property for a logic to have, if one wanted to exploit experience gained in first-order model theory.

For some logics, like the infinitary logics discussed below, it was realized that finite was the wrong property, because proofs themselves could have infinitely many hypotheses, so various analogues of compactness were sought where *finite* was replaced by some other notion of small set. First attempts were in terms of cardinality. Later, and more successful attempts brought in notions of small from generalized recursion theory.

1.5. Mostowski's Proposal and Generalized Quantifiers

One of the first explicit proposals for studying extensions of first-order logic by the methods of model theory came in Mostowski [1957]. His idea was that since various concepts like finitely many and countably many are not definable in first-order logic but are important in modern mathematics, we should add quantifiers embodying such concepts directly. He suggested having a new syntactic rule:

if $\phi(x)$ is a formula, so is $Qx\phi(x)$,

where x is not free in the new formula. This formation rule is added in such a way that it can be iterated along with "and", "or", "not", "everything" and "something". The meaning of Q depends on a new semantic rule. In fact, given any cardinal number $\aleph_\alpha$ one has a logic $\mathscr{L}(Q_\alpha)$ defined by giving the semantics:

$$\mathscr{M} \models_\alpha Qx\phi(x) \quad \text{iff} \quad \begin{array}{l}\text{there are at least } \aleph_\alpha \text{ elements} \\ b \text{ such that } \mathscr{M} \models_\alpha \phi(b).\end{array}$$

In words, $Qx\phi(x)$ is true just in case there are at least $\aleph_\alpha$ elements b such that $\phi(b)$ is true. The logics $\mathscr{L}(Q_\alpha)$ all have the very same syntax but have different semantics assigning different meanings to the quantifier symbol Q.

The logic $\mathscr{L}(Q_0)$ builds in the finite/infinite distinction missing from first-order logic. It is a notion at the heart of much mathematics, especially in modern algebra. Using it one can define notions like torsion group, finitely generated group, finite-dimensional vector space, and one can define the natural numbers.

The logic $\mathscr{L}(Q_1)$ on the other hand, builds in the countable/uncountable distinction missing from first-order logic, but it does not include $\mathscr{L}(Q_0)$. Using it one can define notions like countably generated groups, uncountable structures, and the like.

One of the first surprises in extended model theory was the extent to which $\mathscr{L}(Q_1)$ is better behaved than the logic $\mathscr{L}(Q_0)$. For example, while there is no completeness theorem for $\mathscr{L}(Q_0)$ there is one for $\mathscr{L}(Q_1)$. Vaught [1964] proved a "two-cardinal theorem" of first-order model theory which had as a corollary an abstract completeness theorem for $\mathscr{L}(Q_1)$. The problem of finding a concrete completeness theorem for $\mathscr{L}(Q_1)$ was left open until a very elegant complete set of axioms and rules was found by Keisler [1970]. Similarly, Fuhrken [1965] used the proof of Vaught's two-cardinal theorem to show that $\mathscr{L}(Q_1)$ is countably compact; $\mathscr{L}(Q_0)$ is not. This is also an immediate consequence of Keisler's completeness theorem. To prove his result, Keisler had to develop much more refined techniques of building uncountable models than had been available before, techniques which have been incorporated into the heart of the subject. They are discussed in Kaufmann's chapter in Part B.

A great deal of effort has gone into studying the logics $\mathscr{L}(Q_\alpha)$ in general, and especially $\mathscr{L}(Q_1)$, as well as closely related logics. But cardinality is only one rather crude distinction between sets. Mostowski's idea of imposing various properties on definable sets has had a liberating effect on logic and has been extended in many different directions. Quantifiers based on measure theory, on probability and on other measures of size have been studied, for example. Lindstrom [1966a] proposed a very general definition of a quantifier, so that one could use practically any class K of structures to define a new quantifier Q_K that captures membership in that class. The notion of a Lindstrom quantifier is defined in Chapter II. Adding quantifiers to first-order logic is a central theme of extended model theory, and provides the focus of Part B of this book.

Most work in extended model theory assumes that one wants to study logics that are stronger than first-order logic, stronger in the sense of containing first-order logic. However, in investigating the logic of probability spaces, Keisler realized that to get the right logic, one wants to have all definable sets measurable, and that these measurability considerations dictate that the logic is strictly incomparable with first-order logic, since one cannot in general assume closure under the ordinary quantifiers "everything" and "something". Instead one has quantifiers of the form

$$(Px \geq r)\phi(x)$$

meaning that the probability of ϕ is at least r. But this logic has a rather weak expressive power unless one takes advantage of countable additivity by allowing infinitary propositional operations, as had already been studied in the more classical setting. (See the next subsection.) Besides the interesting applications, such logics give us a new kind of testing ground for our basic ideas about what a logic is and what, if anything, is so special about first-order logic.

1.6. Infinitary Logics

The logic $\mathscr{L}(Q_0)$ embodying the finite/infinite distinction turned out to have less than satisfactory properties. A number of logics more or less equivalent to $\mathscr{L}(Q_0)$ (e.g. weak second-order logic, that allows quantification over finite sets, and ω-logic, that allows quantification directly over the natural numbers) were worked on until they were gradually replaced by the study of logics with infinitely long formulas.

Actually, the investigation of such languages is older than that dealing with generalized quantifiers (see Zermelo [1931], Novikoff [1939, 1943], Bochvar [1940]), but had fallen on hard times until the late 1950's and early 1960's, when work of Tarski, Henkin, Karp, Scott, Lopez-Escobar, Hanf, and Keisler revitalized the subject. Part C of this book is devoted to infinitary languages and their applications.

Early work on infinitary logics dealt with certain languages $\mathscr{L}_{\kappa,\lambda}$ which were generated by allowing conjunctions and disjunctions of size less than κ and homogeneous strings of quantifiers of length less than λ. The early work looked for analogues of the compactness, completeness and Löwenheim–Skolem theorems. Initial results were discouraging, in that compactness was found to exist only under the rarest of circumstances. Indeed, work of Hanf [1964] showed that it required strong new set-theoretical assumptions to prove that there were any logics $\mathscr{L}_{\kappa,\lambda}$ that were compact in the hoped-for sense, of being κ-compact, where "finite" is replaced by "size less than κ" in the statement of compactness.

Completeness results were a little easier to come by. Building on work of Scott and Tarski [1958], Karp [1964] gave a completeness theorem for the logic $\mathscr{L}_{\omega_1\omega}$. Notice, though, that since the syntactic expressions are infinite, the recursion-theoretic formulation in terms of recursively ennumerable sets had to be abandoned—or better—generalized. What one wanted was a recursion theory over infinitary objects to capture the sense in which one notion of proof might be seen as appropriately effective, another not. Such generalized recursion theories were being developed at about this time (by Takeuti, Levy and Machover, Kripke, Kreisel and Sacks, and Platek) for independent reasons, but then led to a fruitful interaction with the work on infinitary logics.

One of the reasons for favoring an infinitary language over $\mathscr{L}(Q_0)$ had to do with the failure of the Craig interpolation theorem and its consequence, the Beth definability theorem. (The latter says that any notion that is implicitly definable in first-order logic is also explicitly definable in first-order logic.) Mostowski [1968] showed that there is a principled reason for the failure of these results in logics like $\mathscr{L}(Q_0)$, weak second-order logic, and ω-logic. What he showed was that any logic where the syntax is finite but where the notion of finite is definable has sets that are implicitly definable but not explicitly definable. Hence the obvious analogues of the Beth and Craig results fail. More to the point, though, his results show that such logics fail to capture all that is implicit in the logic of finiteness.

The moral is that if you want a logic where the notions of finite and infinite are expressible, and if you want it to be closed under implicit definability, then the syntax is going to have to be infinitary—in some sense. This is not the original motivation for the study of infinitely long formulas, but it is a sound one. The logic $\mathscr{L}_{\omega_1\omega}$ studied by Karp, Scott, Lopez-Escobar is a different way of building the notion of finite into a logic, one that *does* satisfy the obvious analogues of the Beth and Craig theorems, as shown in Lopez-Escobar [1965b]. It allows arbitrary countable conjunctions and disjunctions of formulas to be formulas. The logic $\mathscr{L}(Q_0)$ is a "sublogic", since "there exist infinitely many" can be defined by the following countable conjunction:

$$\bigwedge_{n\geq 0} \forall x_1 \ldots x_n \exists y(\phi(y) \wedge y \neq x_1 \wedge \cdots \wedge y \neq x_n).$$

Lopez-Escobar gave a completeness proof for a Gentzen-style system for $\mathscr{L}_{\omega_1\omega}$, from which he was able to derive an interpolation theorem, and an analogue of the Beth definability theorem.

One of the notions that has emerged as central to logic is that of an inductive definition, i.e., one of the form: the smallest relation R satisfying some closure condition. The notion comes up in the very definition of the syntax and semantics of specific logics, in recursion theory, and in various other branches of mathematics. It is only natural that logicians would look for logics where such implicit forms of definability were made explicit. Infinitely long formulas emerged again in this connection. Moschovakis [1972] showed that any inductive definition could be made explicit by using a formula with an infinite string of alternating quantifiers:

$$\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \ldots \bigwedge_n \phi_n(x_1, y_1, \ldots, x_n, y_n).$$

This generalized a theorem of Svenonius [1965] about PC-classes on countable models. Various suggestions for logics admitting such infinite alternating strings have been forthcoming. The most useful now appears to be the Vaught formulas built into the logic $\mathscr{L}$ studied in Kolaitis's chapter. Such infinite strings also have connections with work in game theory, higher recursion theory and descriptive set theory.

1.7. Second-Order Logic

Actually, there was another extension of first-order logic that was around for a long time before Mostowski's suggestion. Everyday mathematical experience shows us that the concepts of arbitrary set and function are important and powerful. Notions like finite, infinite, countable, uncountable, well-ordering, the natural and real numbers, are all definable in terms of these notions. Second-order logic is the extension of first-order logic where these concepts are built in by allowing quantifiers not just over individuals in the domain $\mathfrak{M}$, but also over subsets of that domain and over relations and functions on the domain.

Judged by the standards of first-order logic, the model theory of second-order logic was deemed unmanageable. None of the basic theorems of first-order logic extended to second order logic. There were no completeness, compactness, interpolation or Löwenheim–Skolem theorems. For many years the model-theory of second-order logic was thus largely ignored. In fact, in the early days of extended model theory, many of us saw ourselves as chipping away manageable fragments of second-order logic. However, the way we judged what it was to be a manageable theory was by comparing it to first-order model theory. In retrospect, this seems unimaginative, since there has turned out to be quite a rich model theory for second-order logic, once the right questions started being asked.

Second-order logic permits quantification over arbitrary functions on the domain of discourse as well as quantification over the elements in the domain of discourse. Since sets and relations can be represented by their characteristic functions, second-order logic embodies quantification over arbitrary sets and relations, as well. There is an obvious equivalence between functions and relations, but allowing quantification only over sets turns out to be weaker than full second-order logic. It is called "monadic" second-order logic, and it is much more expressive than first-order logic while being manageable enough to provide many interesting decidability and undecidability results. Some of these are discussed in Gurevich's chapter. For example, he discusses a classification of ordered abelian groups by means of properties definable in the monadic second-order logic of such groups. He also presents a proof of the famous result due to Rabin on the decidability of the monadic theory of the infinite binary tree.

Shelah [1973c] investigated what other types of restricted second-order quantifiers there were, besides the restriction to monadic quantification, but where the restrictions considered had to be first-order definable. He proved a striking and difficult result: there are only four first-order definable second-order quantifiers. Baldwin's chapter takes advantage of more recent work in model theory to give a simplified presentation of the result. The structural results implicit in the proof of the four definable second-order quantifiers theorem emphasize the importance of studying three theories in monadic logic: (i) the monadic theory of order, (ii) the monadic theory of the tree $\lambda^{\leq\omega}$, and (iii) the monadic theory of the tree $\lambda^{<\omega}$.

Both Baldwin's and Gurevich's chapters emphasize the importance for monadic logic of a basic result for first-order logic which does extend to this situation: the Feferman–Vaught theorem.

1.8. Applications to Mathematics

There are many kinds of applications of logic to mathematics. The most striking (at least the ones that strike most people) are those where some specific theorem or method from logic gives an outright solution to some open question in mathematics. Eklof surveys a number of applications of this sort, of infinitary logics within algebra. Keisler's chapter contains some applications of this sort to probability theory.

A second kind of application of logic is in the realm of independence results where it is shown that certain problems cannot be settled on the basis of the first-order axioms of set-theory. These results are really about the limitations of first-order logic, and so are outside the scope of this book, except to the extent that they have an impact on extended model theory itself. (See Section 2.6 below.)

Most important in the long run, it seems, is where logic contributes to mathematics by leading to the formation of concepts that allow the right questions to be asked and answered. A simple example of this sort stems from "back and forth arguments" and leads to the concept of partially isomorphic structures, which plays such an important role in extended model theory. For example, there is a classical theorem of Erdos, Gillman and Henriksen; two real-closed fields of order type η_1 and cardinality $\aleph_1$ are isomorphic. However, this way of stating the theorem makes it vacuous unless the continuum hypothesis is true, since without this hypothesis there are no fields which satisfy both hypotheses. But if one looks at the proof, there is obviously something going on that is quite independent of the size of the continuum, something that needs a new concept to express. This concept has emerged in the study of logic, first in the work of Ehrenfeucht and Fraissé in first-order logic, and then coming into its own with the study of infinitary logic. And so in his chapter Dickmann shows that the theorem can be reformulated using partial isomorphisms as: Any two real-closed fields of order-type η_1, of any cardinality whatsoever, are strongly partially isomorphic. There are similar results in the theory of abelian torsion groups which place Ulm's theorem in its natural setting.

Notice the shift of perspective here. While we started with the idea of taking concepts that were already explicit in mathematics and studying their logic, we now see the possibility of exploring concepts that are only implicit in existing mathematics, making them explicit, and using them to go back and re-examine and enrich mathematics itself. Isolating the notions of inductive definability implicit in so much of mathematics is another example mentioned above. The results mentioned from Keisler's and Gurevich's chapters are also of this nature, bringing in new concepts with which the right questions can be asked and answered. Similarly, much of Shelah's work in extended model theory can be seen in this light, taking some important construction from mathematics or logic and building the construction into a new logic. Extended model theory provides a framework within which to understand existing mathematics and push it forward with new concepts and tools.

2. *Abstract Model Theory*

Once there are lots of similar structures around one begins to study the relationships that exist between them. And so it is with extended model theory. Once there are lots of logics around, one begins to study their interrelationships. This part of the subject is known as *abstract model theory.*

2.1. Lindstrom's Theorem

One of the first equations that must be settled is, just what makes a logic natural? What are the guiding principles which help one find interesting and useful logics?

Here the experience built up with many examples suggests three principles:

(1) build into the semantics natural and important notions from some particular domain of mathematical activity;
(2) keep the semantics constrained so that it embodies just those notions one intends to study, and notions implicit in them; and
(3) find a syntax in which the basic notions of the logic find natural expression.

It was obvious from the start that there is a trade-off in the construction of logics. You can't build in some concept that goes beyond first-order logic without paying the piper. For example, if some particular theorem about first-order model theory shows that adding a new quantifier is a genuine strengthening of first-order logic, then the obvious analogue of that theorem will fail for the new logic. For example, the countable compactness of first-order logic has as an easy corollary that the quantifier "there exists at most finitely many" is not definable therein. It follows from the proof that $\mathscr{L}(Q_0)$ and $\mathscr{L}_{\omega_1\omega}$ are not countably compact. Similarly, the Löwenheim–Skolem theorem (if a countable set of sentences has a model, it has one that is at most countable) has as a corollary that "there exist uncountably many" is not definable in first-order logic. Hence the analogous statement will fail for the logic $\mathscr{L}(Q_1)$.

There is an important theorem lurking here, one discovered by Lindstrom [1969]; it is a result that opened up a new aspect to the study of logic. What Lindstrom showed is that what we have just observed in these two cases is in fact quite general. Any attempt to build a logic that is more expressive than first-order logic will fail to satisfy the obvious analogue of either the countable compactness theorem or the Löwenheim–Skolem theorem. Or, to state it more positively, first-order logic can be characterized as the strongest logic satisfying the following two properties:

(1) (Countable Compactness Property.) If a countable set of sentences has no model then some finite subset has no model; and
(2) (Löwenheim Property.) If a sentence has an infinite model, it has a countable model.

$\mathscr{L}(Q_1)$ is countably compact; $\mathscr{L}_{\omega_1\omega}$ satisfies the Löwenheim property. This striking result has led to much important research after lying largely unnoticed for several years. It was the rediscovery of the result and its widespread circulation in Friedman [1970a] that in many ways woke logicians to the potential in abstract model theory. A proof of Lindstrom's theorem is contained in Chapter III.

Characterizing a given logic $\mathscr{L}$ as the strongest logic with some property presupposes an understanding of just what a logic is. What kinds of syntactic and semantic closure conditions does one build into the notion of a logic? Obviously the more one builds in, the fewer logics there are and so the weaker a characterization theorem becomes. On the other hand, for the other aims of extended model theory, one wants a notion that captures the important examples and systematizes the common assumptions.

Lindstrom and Friedman managed to side-step this problem. To get around the difficulties of saying just what a logic is, they dealt entirely with classes of structures and closure conditions on these classes, thinking of the classes definable in some logic. That is, they avoided the problem of formulating a notion of a logic in terms of syntax, semantics, and satisfaction, and dealt purely with their semantic side. From the point of view of logic, this is at best a stop-gap measure, to be replaced by an analysis of just what makes up a logic. But the task of coming up with a general definition of just what constitutes a logic has been a large one, one that may still be not entirely settled. The one given in this book has emerged as fairly stable over time, and most useful for a variety of investigations.

2.2. *Characterization Theorems*

The compactness and Löwenheim–Skolem theorems are two of the most striking results in first-order model theory—and probably the most frequently used tools of the first-order model-theorist. This made Lindstrom's characterization theorem of first-order logic somewhat disheartening, initially at least, since it says that the model-theorist interested in extensions of first-order logic is going to have to give up at least one of his most cherished tools. Luckily, however, there had already been enough success in the model theory of $\mathscr{L}_1$, $\mathscr{L}(Q_1)$, $\mathscr{L}_{\omega_1\omega}$, and some other logics to whet the appetites of those interested in extensions of first-order logic and to convince them that there was room to maneuver around the failures of these results. And there was enough intrinsic interest in these logics that workers attempted to find Lindstrom-style characterization theorems for them.

There have been some successes finding such characterizations, but they have been few and far between. What there are can be found in the chapters by Flum and Väänänen. But there are still no satisfactory characterizations of $\mathscr{L}_{\omega_1\omega}$ or $\mathscr{L}(Q_1)$. Indeed, search for such results has led to the study of even stronger logics that are based on the same sorts of mathematical concepts, but there is no satisfactory characterization of these stronger logics either.

2.3. *Uses of Abstract Model Theory*

Abstract model theory has turned out to have more to say about the *relations between* various properties of logics than about the characterization of logics by their properties. In general, abstraction can serve many different masters. It can be used to systematize a body of examples, notions and results, and in this organization, help us to understand more explicitly what we already know. This usually leads to the emergence of new concepts for unifying properties of the material, concepts which are overlooked in specific cases. And new problems and theorems that can be formulated in terms of the new concepts that emerge.

Studying only the model theory of first-order logic would be analogous to the study of real analysis never knowing of any but the polynomial functions: core concepts like continuity, differentiability, analyticity, and their relations would remain at best vaguely perceived. It is only in the study of more general functions that one sees the importance of these notions, and their different roles, even for the simple case.

One of the aims of abstract model theory is develop an analogous classification of logics by means of their most important properties. This entails understanding the relationships between these properties. Properties of logics that are co-extensive in the first-order case often have quite different extensions in the general setting. For example, in first-order logic, the interpolation theorem and the Robinson consistency theorem appear to be equivalent results. However, in general, the latter is much more powerful than the former. $\mathscr{L}_{\omega_1\omega}$, for example, has the interpolation property but not the Robinson consistency property. So too, the difference between strong compactness and countable compactness is not too noticeable in first-order logic, because of the Löwenheim–Skolem theorem. In general, however, countable compactness is much weaker.

Like properties of logics, so too methods of proof that seem more or less equivalent in the context of first-order model theory often split and come into their own in abstract model theory. For example, the Ehrenfeucht–Fraissé partial isomorphism method has come to the fore in two ways. First, it generalizes in different ways to a host of model-theoretic logics. Second, it is used as a means of classifying logics, into those that have and those that do not have the "Karp property". In the next subsections, we discuss three particularly important links that come up repeatedly in extended and abstract model theory, the Δ-closure of a logic, and the least ordinal pinned down by a bounded logic, and the Hanf number of a logic. In each we have a property of first-order logic that is largely overlooked until put in the context of the more general theory.

2.4. *The Interpolation Theorem and the Δ-Closure*

The interpolation theorem illustrates a number of the issues discussed above. The Craig interpolation theorem (stated below) shows that first-order logic is closed under a very general form of implicit definability, so that the concepts embodied in first-order logic are all given explicitly. Closure under implicit definability is obviously a highly desirable result from the perspective of defining logics that embody a given mathematical notion. Craig's result was discovered about the same time as the Robinson consistency theorem, and they were widely perceived to be more or less the same result, one that implied the Beth definability theorem.

As mentioned above, the Robinson consistency property turns out to be a much stronger property of logics than the Craig interpolation property in the context of extended model theory. In fact, as long as the number of symbols in any single sentence is finite, or at all reasonable in size, one can say that a logic has the Robinson consistency property just in case it satisfies *both* the compactness property and the Craig interpolation property (see Chapter XVIII).

Neither $\mathscr{L}(Q_0)$ nor $\mathscr{L}(Q_1)$ satisfy the Craig interpolation theorem. But whereas Mostowski found a principled reason for the failure of interpolation for $\mathscr{L}(Q_0)$, there is no such explanation known for $\mathscr{L}(Q_1)$. (Keep in mind that $\mathscr{L}(Q_1)$ is not in any sense an extension of $\mathscr{L}(Q_0)$. The logic $\mathscr{L}(Q_1)$ satisfies the countable compactness property so "finite" is not definable in this logic.) Rather, the counter-examples that were found to the Craig and Beth theorems for $\mathscr{L}(Q_1)$ and related logics have repeatedly suggested additional concepts that were in the constellation of notions around countability but that were not definable in $\mathscr{L}(Q_1)$. That is, the counter-examples all suggested that we just did not yet have the right logic, rather than that there was an essential obstacle. This is presumably part of the reason there is no convincing characterization theorem for any of these logics.

The problem of finding a countably-compact logic extending $\mathscr{L}(Q_1)$ with the interpolation property has become known as Feferman's problem. It has led to the study of many interesting and useful extensions of $\mathscr{L}(Q_1)$—extensions that remedy various deficiencies in $\mathscr{L}(Q_1)$ by building in other notions that seem still in the spirit of the countable/uncountable distinction. Some of these extensions are discussed in Kaufmann's chapter. Nevertheless, there is still no conclusive solution to Feferman's problem either positively, or negatively by a result that shows, under some reasonable assumption, that an essential obstacle exists.

Feferman's motivation in stating the problem goes back to the issue of completeness. For first-order logic, there are both model-theoretic and proof-theoretic proofs of the interpolation theorem, the latter deriving the theorem from the completeness of Gentzen's cut-free set of axioms and rules. (Gentzen's rule of "cut" is the analogue of modus ponens for his system. He showed that this rule is redundant in his system.) For $\mathscr{L}_{\omega_1\omega}$, it was this latter proof that Lopez-Escobar managed to generalize. It was harder to find a purely model-theoretic proof. The basic idea of the proof-theoretic proof is that if you are able to prove $\psi(R, T)$ from $\phi(R, S)$, where R, S and T are relation symbols, and if the proof does not use "cut", then there should be a proof that only uses the common symbol R in an essential way, in that you should be able to isolate a sentence $\theta(R)$ so that both $\phi(R, S) \to \theta(R)$ and $\theta(R) \to \psi(R, T)$ are provable.

One can use the interpolation property as a yardstick for measuring whether there is a good proof theory. In the case of $\mathscr{L}(Q_1)$, knowing that interpolation fails shows that one is not going to have a good Gentzen style proof theory for $\mathscr{L}(Q_1)$. What Feferman was after was a richer logic that had a better completeness theorem in this sense, and he was using the interpolation property as a model-theoretic test for such a better theorem.

The proof theory of strong logics has not kept pace with their model theory, partially due to the interests of the people working in the field, partially due to the fact that proof theory is not seen as being particularly central to the subject since many of the logics do not have an r.e. set of valid sentences. And from a model-theoretic point of view, it has turned out that interpolation is not a particularly important or natural property for a logic to have. Interpolation is a much stronger property than is needed for a logic to be closed under implicit definability. The notion that has turned out to be more important in this respect is that of a *Δ-closed logic.*

A class K of structures is called PC (or Σ_1^1) in a logic $\mathscr{L}$ if there is a class K' of structures that is definable in $\mathscr{L}$ so that $\mathfrak{M} \in K$ if and only if some expansion

$\mathfrak{M}'$ of $\mathfrak{M}$ is in K'. The interpolation theorem can be restated as: If K_0 and K_1 are disjoint PC classes then there is a definable class K containing one and disjoint from the other. An obvious consequence is that *if a class K is both PC and co-PC (that is, its complement is PC) in $\mathscr{L}$ then K is definable in $\mathscr{L}$*. A logic with this property is called Δ-*closed*. Any logic satisfying the interpolation property is automatically Δ-closed, but not conversely. And whereas there is no known way to start with a logic $\mathscr{L}$ where interpolation fails and find a smallest extension where it holds, there is a way to define a smallest logic $\Delta(\mathscr{L})$ containing $\mathscr{L}$ and Δ-closed, called the Δ-*closure* of $\mathscr{L}$. This operation on logics preserves many of the nice properties of the original logic.

The Δ-closure is completely overlooked in first-order logic because we have so much more. And Δ-closure, rather than the stronger interpolation property, is really what shows us that we have a well-rounded logic.

A frequent use of the Δ-closure is to show that two logics $\mathscr{L}$ and $\mathscr{L}'$ are really the same up to implicit definability by showing that $\Delta(\mathscr{L}) = \Delta(\mathscr{L}')$. Several such results appear in Chapters VI and XVII. For example, the various logics $\mathscr{L}(Q_0)$, weak second-order logic (where one quantifies over finite sets) and ω-logic are the same up to implicit definability. Makowsky [1975a] and I (Barwise [1974a]) independently noticed that Mostowski's result, that no logic with finitary syntax that can define finite and infinite has the interpolation property, could be turned into a characterization of the common Δ-closure of these logics as a certain infinitary logic, the "hyperarithmetic" fragment of $\mathscr{L}_{\omega_1\omega}$ (see Chapter XVII for a proof of this result).

2.5. *Pinning Down Ordinals*

Another property of first-order logic that goes all but unnoticed in that setting, but assumes a central place in the general theory, is the undefinability of well-orderings. The distinction between logics where well-ordering is undefinable and those where it is definable turns out to be an important one.

A logic $\mathscr{L}$ is said to be *bounded* by an ordinal α if α is greater than all ordinals that can be "characterized" in the logic. Second-order-like logics are those where the notion of well-ordering is definable and so are unbounded.

First-order logic is bounded by ω, the first infinite ordinal, as the (countable) compactness theorem shows. Indeed, any extension of first-order logic that is countably compact will be bounded by ω. For example, $\mathscr{L}(Q_1)$ is bounded by ω. $\mathscr{L}(Q_0)$, by contrast, is bounded not by ω but by a certain countable ordinal ω_1^c, the least non-recursive ordinal. $\mathscr{L}_{\omega_1\omega}$ is bounded by ω_1, the least uncountable ordinal. On the other hand, second-order logics $\mathscr{L}_{\omega_1\omega_1}$, and logic with the game quantifier are not bounded.

For some applications, the failure of the compactness theorem can be circumvented in applications by knowing that the logic is bounded. For example, first-order logic can be characterized in terms of the Löwenheim–Skolem theorem and the assumption that the logic is bounded by ω. Similarly, for many "Hanf number" calculations (see the next subsection) one needs to know a bound for the logic.

In first-order logic, the fact that the logic is bounded by ω is such a simple consequence of compactness, that we do not even notice that the property is important. In more general logics, this notion assumes its rightful place in the web of properties of logics.

2.6. Hanf Numbers

In elementary textbooks on logic one often finds the Löwenheim–Skolem theorem for first-order logic stated as: If a theory has an infinite model, then it has models of all infinite cardinalities. The proof, however, when given, always breaks into two parts. There is a "downward" half, that allows one to get smaller models from bigger, and an "upward" half that allows one to get bigger from smaller. The downward version uses some form of submodel argument, the upward a compactness argument. Not surprisingly, these two arguments generalize quite differently, to different logics.

Many logics have some form of downward Löwenheim–Skolem theorem, with a proof analogous to the usual one, with the difference being just how small the submodel can be. But almost no logics have a simple analogue of the upward version. In $\mathscr{L}(Q_0)$, for example, one can define theories with model of quite large infinite cardinalities, but without arbitrarily large models. Hanf observed, however, that as long as the expressions of a logic $\mathscr{L}$ form a set, as opposed to a proper class, that one can show quite easily, though very non-constructively, that there must be *some* cardinal κ such that if a sentence ϕ of $\mathscr{L}$ has a model of size at least κ, then it has arbitrarily large models. The least such cardinal has come to be known as the *Hanf number* $h(\mathscr{L})$ of $\mathscr{L}$.

A fair amount of work has gone into calculating the Hanf number of various logics. The reader can find a number of such calculations for infinitary logics in Chapter IX. For bounded logics, the Hanf number is often related to the least ordinal that cannot be pinned down in the following manner. Define

$$\beth_0 = \aleph_0,$$

$$\beth_{\alpha+1} = 2^{\beth_\alpha},$$

and, for limit ordinals λ,

$$\beth_\lambda = \sup_{\alpha<\lambda} \beth_\alpha.$$

Then for many logics $\mathscr{L}$, like $\mathscr{L}(Q_0)$, $\mathscr{L}_{\omega_1\omega}$, $\mathscr{L}_{\kappa,\omega}$, one has $h(\mathscr{L}) = \beth_\lambda$, where λ is the least ordinal that cannot be pinned down by the logic. For logics that are not bounded, there is very little that can be said about the size of the Hanf number.

Shelah has suggested a structural explanation for the relation between the ease of computing the Hanf number and the boundedness of the logic. The situation is clearer if we consider the Hanf number $h(T, L)$ of a countable theory T in a logic L, the least κ such that for any L-sentence ϕ if $T \cup \{\phi\}$ has a model of power κ then $T \cup \{\phi\}$ has arbitrarily large models. (Setting T as the "empty" theory we specialize to $h(L)$.) Similarly, we can define T to be bounded or unbounded in the logic L.

The important structural distinction can be expressed by considering the class of models of T. Each model of T can be decomposed as a "product" of countable models if and only if T is bounded if and only if the Hanf number of T can be easily computed. The proof of this result for logics with definable second-order quantifiers, a characterization of theories according to this classification, and an account of the ensuing computation of Hanf numbers occurs in Chapter XII. Shelah has identified a similar dichotomy between superstable theories with and without the dimensional order property. The resulting structure theory also analyzes a model of power λ in terms of countable models and subtrees of $\lambda^{\leq \omega}$.

2.7. *Strong Logics and First-Order Set Theory*

There is an older approach to the study of the relationship between logic and concepts that lie outside of first-order logic, one subscribed to by those who accept the first-order thesis. One gives a first-order approximation to one's meta-theory T, something like Zermelo–Fraenkel set theory (ZF) in which all the notions in question can be defined *relative to* the notion of set, or perhaps a weaker or stronger metatheory. To the extent that one can view some branch of mathematics as consequences of this theory, one has an account of that part of mathematics.

This has become something like the orthodox position of remaining mathematical formalists, those who see mathematics as the working out of consequences of some formal first-order theory by means of the axioms and rules of first-order logic. In particular, one can step back and look at extended model theory itself from this perspective. We can define many of the logics discussed here relative to the notion of set in ZF set theory. Hence, we can examine the relationship between the properties of logics and their definitions in set theory. This is an approach which I initiated in Barwise [1972a], motivated by an acceptance of the first-order thesis. While it now seems to me that my motivations were misguided, the approach has led to some very interesting work on the relationship between strong logics and set theory, work that is discussed in Chapter XVII.

From the early days of infinitary logic there has been a close interplay between strong logics and set-theoretic principles that go beyond ZF set-theory in various ways, especially so called "large cardinal" assumptions. These are assumptions that are not justified by clear-cut intuitions about sets, at least not by intuitions shared by the silent mathematical majority. Weakly and strongly compact cardinals κ are defined in terms of the associated infinitary logic $\mathscr{L}_{\kappa,\kappa}$ satisfying an analogue of the countable or full compactness property, for example. The assumption that there are such cardinals goes beyond the intuitions about sets built into ZF. Measurable cardinals come up in the discussion of the Robinson consistency property. It, too, is a strong assumption that goes beyond ZF. An even stronger assumption, Vopenka's principle, is equivalent to the statement that every finitely generated logic has a strong compactness cardinal, that is, has a cardinal κ so that any inconsistent theory T of the logic has a subset of size less than κ which is inconsistent. These and related results are discussed in Part F.

It is not clear what to make of results like these. Luckily, most of them have to do with very abstract logics, or with abstract logic itself, not with the concrete logics that arise from natural mathematical concepts.

2.8. Other Types of Structures

Lindstrom's theorem poses a dilemma: Give up either compactness or Löwenheim–Skolem. However, there is an escape from the horns of the dilemma mentioned earlier. Implicit in the discussion in this section has been the assumption that we were discussing logics that have the same basic sort of syntax and semantics as first-order logic. There is always the possibility of violating one or both of these assumptions by studying logics that have different sorts of structures, or have syntactic rules that are stronger in some ways than first-order logic but weaker in others.

Part E of the book is devoted to the study of some of the logics that have been developed for different kinds of mathematical structures. The most extensively studied class of structures is the class of topological models, models where there is an underlying topology. In this setting there has been a great deal of effort that has gone into discovering the analogue of first-order logic.

Harvey Friedman initiated the study of logic on the real numbers incorporating the notions of measure and category, a topic pursued in Chapter XVI. Keisler, on the other hand, initiated the investigation into the logic of probability spaces. These logics are interesting not just for what they say about the logic of the reals and the logic of probability, but also because they force us to examine additional assumptions that are usually implicit in extended model theory, assumptions that do not hold in these settings.

2.9. Unnatural Logics

We should give a word of warning about some of the logics one will meet in this book. Recall that the aim of extended model theory is to discover natural logics that embody important mathematical notions. This leads to abstract model-theory and the study of the relationships between properties of logics. There are a number of logics that have arisen simply as counterexamples to show that some one property of logics does not imply some other, not with the real goals of extended model-theory in mind at all. And, too, some of the logics that seemed superficially natural turned out not to be. $\mathscr{L}(Q_0)$ is one such. Time will tell which logics are truly significant. There is no more point in getting bogged down in the study of purely artificial and unnatural logics than there is in the study of hemi-demi-semi-groups with chain conditions.

3. Conclusion

The reader of this volume will find many topics that have not been discussed above, for the book, like the subject, is a large one. Even so, there are topics in the field of extended model theory and abstract logics that could not be included in this volume, for one reason or another. Beyond that, there are many topics that fit under the general heading described by the title of this book, "model-theoretic logics," but which are not usually considered part of extended model theory since they do not fit so well under the general framework that has been developed in abstract model theory. Consequently, we have not attempted to include this work here.

The most glaring omission of this sort is work on the semantics and logic of computer languages. This is a rich domain of research that would need a volume of at least equal size to treat adequately. In the long run, it seems that a unified view of logic and semantics will require us to come up with a framework that encompasses both fields, but we are far from such a conception at present.

The semantics of computer languages, and the differences that emerge in that work from more traditional model theory, points to a shortcoming in the latter, namely its failure to come to grips with activity, as opposed to objects and static relations between them. This same shortcoming causes problems with traditional attempts to apply model theory to human languages, another topic not treated here.

Traditional model theory focuses on truth (and satisfaction) of sentences, and so leaves out the use of language to affect change. This is a shortcoming that has been emphasized by Austin and other writers on natural language in the tradition of "speech act" theory. This power of language to effect change (e.g., in so-called "side effects") is one of the things that makes the semantics of computer languages strikingly different.

Another area where work on computer and human languages makes the traditional work in logic appear too static is in the treatment of inference. Inference, whether by man or machine, is an activity, a process of extracting information, whereas the tradition attempts to reduce inference to objects (proofs, strings of symbols). In another paper I have discussed the need to place the study of logic within a setting where traditional inference is seen as just one form of information preserving activity. I think such an approach has much to contribute to the understanding of mathematical activity, and hence to mathematical logic, but the development of these ideas will have to take place elsewhere. Even the traditional approach to inference in logic has not made great inroads in extended model theory. There are few genuine completeness theorems and even fewer extensions of proof theory.

Mathematicians often lose patience with logic simply because so many notions from mathematics lie outside the scope of first-order logic, and they have been told that that *is* logic. The study of model-theoretic logics should change that, by getting at the logic of the concepts mathematicians actually use, by finding applications, and by the isolation of still new concepts that enrich mathematics and logic. I do not know just how much of the work presented in this volume will find a permanent place in mathematics, because it is, after all, a young and vigorous subject. But whatever the fate of the particulars, one thing is certain. There is no going back to the view that logic is first-order logic.