APPLICATIONS OF THE RAYLEIGH RITZ METHOD TO VARIATIONAL PROBLEMS

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Introduction. Let R be a bounded either simply or multiply connected plane region with boundary Γ, consisting of a finite number of non-intersecting simply closed regular arcs of class c^k. A plane curve is a regular arc if the defining functions $x(t)$, $y(t)$, $a \leq t \leq b$ have continuous derivatives with $x'(t)^2 + y'(t)^2 \neq 0$ on $a \leq t \leq b$. A regular arc is of class c^k if the defining functions $x(s)$, $y(s)$, s being arc length, have continuous derivatives of order k. We shall say a function $h(x, y)$ defined on $\bar{R} = R + \Gamma$ is of class c^k if the partial derivatives of h of order r, $0 \leq r \leq k$ exist in R and have limits on Γ so as to define functions continuous on $\bar{R}$. Let $g(x, y)$ be a given function of class c^k on $\bar{R}$. The main problem considered is that of finding the function ψ_0 which yields minimum value to the functional

$$I[\psi] = \iint_R (a\psi_x^2 + b\psi_y^2 + c\psi^2 + 2f\psi)\,dx\,dy$$

defined over the admissible class of functions ψ which are of class c^k on $\bar{R}$ and assume the values of g on Γ.

We shall assume $a(x, y) > 0$, $b(x, y) > 0$, $c(x, y) \geq 0$ on $\bar{R}$; a, b, c bounded and integrable in $\bar{R}$; $f(x, y)$ integrable in $\bar{R}$. In the sequel, unless otherwise specified, integrations will be taken over R and the symbol R omitted.

Let $G(x, y)$ be of class c^k on $\bar{R}$, vanishing on Γ, positive in R, with normal derivative $\partial G/\partial\nu$ on Γ different from 0. We show that, if $k \geq 3$, every admissible function ψ has a uniformly convergent expansion on $\bar{R}$

$$\psi = g + \sum_{i=1}^{\infty} b_i f_i(x, y)$$

where f_i are obtained by a Gram-Schmidt process from the functions $\{G x^i y^j\}$ $i, j = 0, 1, 2, \cdots$ and b_i are generalized Fourier coefficients connected with the quadratic functional

$$D[\psi] = \iint (a\psi_x^2 + b\psi_y^2 + c\psi^2)\,dx\,dy\,.$$

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In fact, $b_i=D[\psi-g, f_i]$ where

$$D[\xi, \eta]=\iint(a\,\xi_x\,\eta_x+b\,\xi_y\,\eta_y+c\,\xi\eta)dx\,dy\,.$$

An estimate of the error obtained by using for ψ only the first n terms of the expansion is given in terms of n and k. Sufficient conditions are obtained for the convergence of

$$\nabla\left[g+\sum_{i=1}^{n}b_i f_i\right]$$

to $\nabla\psi$ and an estimate is given for the rate of convergence.

In particular, if ψ_0 is an admissible function minimizing $I[\psi]$, then the expansion

$$\psi_0=g+\sum_{i=1}^{\infty}a_i f_i$$

yields an explicit solution for ψ_0, since the coefficients a_i are given, in this case, by

$$a_i=-\iint f f_i\,dx\,dy-D[g, f_i]$$

which are independent of ψ_0.

The problem of minimizing the functional $I[\psi]$, with $g\equiv 0$, has been studied by Kryloff and Bogoliubov [4] and by Kantorovitch [2], both obtaining estimates for convergence to ψ_0 of functions obtainable by the Rayleigh Ritz method. The first paper deals with convex regions R, the second with regions R bounded by $x=0$, $x=1$, $y=g(x)$, $y=h(x)$; $h>g$ on $0\leq x\leq 1$. Neither obtains an explicit solution for ψ_0 nor studies the convergence of the derivatives.

In the final section of this paper, we assume the existence of a function ψ_0 yielding minimum value, for $p\geq 1$, to

$$D^p[\psi]=\iint_R(a\psi_x^2+b\psi_y^2+c\psi^2)^p\,dx\,dy\,,\qquad \psi=g \quad\text{on}\quad \Gamma$$

and obtain an estimate for the rate of convergence to ψ_0 of functions obtained by the Rayleigh Ritz method.

§ 1. **Preliminary Considerations.** A variation v shall mean a function of class c^k on $\bar{R}$ vanishing on Γ. Form the Hilbert space H by completing the linear manifold V of variations v using the positive definite quadratic form $D[v]$ as the square of the norm of a variation. If $h\in H$, we represent the norm of h by $\|h\|$. If ξ and η are variations, the inner product will be

$$(\xi, \eta)=D[\xi, \eta]\,.$$

Let f_i be any complete orthonormal set of variations in H. If ψ is admissible, then $\psi-g$ is a variation and thus expressible in H as

$$\psi-g=\sum_{i=1}^{\infty} b_i f_i$$

with $b_i=D[\psi-g,\ f_i]$.

If ψ_0 is an admissible function yielding a minimum value to $I[\psi]$, if λ is real, and v is a variation, then $\psi_0+\lambda v$ is admissible, and

$$I[\psi_0]\leq I[\psi_0+\lambda v]=I[\psi_0]+\lambda(2D[\psi_0,\ v]+\iint 2fv\,dx\,dy)+\lambda^2 D[v]\,.$$

This implies that the coefficient of λ must vanish so that

$$D[\psi_0,\ v]=-\iint fv\,dx\,dy$$

and

$$I[\psi_0+\lambda v]=I[\psi_0]+\lambda^2 D[v]$$

for every variation v.

The first relation shows that the Fourier coefficients of ψ_0-g,

$$a_i=D[\psi_0-g,\ f_i]=D[\psi_0,\ f_i]-D[g,\ f_i]=-\iint ff_i\,dx\,dy-D[g,\ f_i]$$

are independent of ψ_0.

The second relation implies that if ψ is admissible,

$$I[\psi]=I[\psi_0+\psi-\psi_0]=I[\psi_0]+D[\psi-\psi_0]\,.$$

Thus if

$$\phi_n=g+\sum_{i=1}^{n} a_i f_i\ ,$$

then

$$0=\lim_{n\to\infty}\left\|\psi_0-\left(g+\sum_{i=1}^{n} a_i f_i\right)\right\|^2=\lim_{n\to\infty} D[\psi_0-\phi_n]=\lim_{n\to\infty} I[\phi_n]-I[\psi_0]$$

so that ϕ_n is a minimizing sequence.

Moreover,

$$D\left[\psi_0-g-\sum_{i=1}^{n} c_i f_i\right]$$

is a minimum when $c_i=a_i$ implying that ϕ_n are chosen to yield minimum value to $I[\psi_n]-I[\psi_0]$ and hence to $I[\psi_n]$ in the class of functions

$$\psi_n=g+\sum_{i=1}^{n} c_i f_i\,.$$

Thus ϕ_n may be obtained by the Rayleigh Ritz process applied to the functional $I[\psi]$.

We will prove, in Theorem 1, that the class of functions $\{G P\}$ where P is a polynomial in x and y, is dense in H. This class is the linear manifold determined by the set $\{G x^i y^j\}$, a set linearly independent in H. For, if

$$v=\sum_{i=1}^{m}\sum_{i=1}^{n}\alpha_{ij}\, G\, x^i\, y^j,$$

then $D[v]=0$ implies $\alpha_{ij}=0$.

It follows that we can obtain an orthonormal set f_i complete in H by orthonormalizing the set $\{G x^i y^j\}$. Let

$$\begin{aligned} &v_1=G\, x^0\, y^0 \\ &v_2=G\, x^1\, y^0\ , \quad v_3=G\, x^0\, y^1 \\ &\vdots \\ &v_{\frac{k(k+1)}{2}+1}=G\, x^k\, y^0, \cdots,\ v_{\frac{k(k+1)}{2}+k+1}=G\, x^0\, y^k\ . \end{aligned}$$

Then

$$f_n=\frac{v_n-\sum_{j=1}^{n-1} f_j(v_n\ ,\ f_j)}{\left\| v_n-\sum_{j=1}^{n-1} f_j(v_n, f_j)\right\|}$$

$$=\begin{vmatrix} (v_1,\ v_1)\cdots(v_1,\ v_{n-1}) & v_1 \\ \vdots \qquad\qquad \vdots & \\ (v_n,\ v_1)\cdots(v_n,\ v_{n-1}) & v_n \end{vmatrix}\cdot\begin{vmatrix} (v_1,\ v_1) & & \\ & \ddots & \\ & & (v_{n-1}\ ,\ v_{n-1}) \end{vmatrix}^{-1/2}\cdot\begin{vmatrix} (v_1,\ v_1) & & \\ & \ddots & \\ & & (v_n\ ,\ v_n) \end{vmatrix}^{-1/2}.$$

The function f_n is of the form $G P_n$, where the degree of the polynomial P_n is that of v_n/G. If $v_n=G\, x^r\, y^s$ with $r+s=k$, then

$$\frac{k(k+1)}{2}+1\leq n\leq\frac{k(k+1)}{2}+k+1$$

so that $k^2<k(k+1)<2n-2$ and the degree k of P_n is less than $\sqrt{2n-2}$. Similarly k is greater than $\sqrt{2n}-2$.

§ 2. The Minimizing sequence. We shall use certain approximation theorems which can be derived by methods used by Mickelson [5][1]. To simplify the notation, let

$$x=(x_1,\cdots x_s)\ ,$$

$$x^{(1)}=(x_1^{(1)},\cdots x_s^{(1)})\ ,$$

[1] For detailed proofs of Lemmas 1, 2 see J. Indritz "Applications of the Rayleigh Ritz method to the solutions of partial differential equations" Ph. D. Thesis, U. of Minnesota, 1953.

$$f(x)=f(x_1,\cdots,x_s)\ ,$$

$$f_r(x)=\frac{\partial^{r_1+\cdots+r_s}}{\partial x_1^{r_1}\cdots\partial x_s^{r_s}}[f(x_1,\cdots,x_s)]\ ,$$

$$\|x^{(1)}-x^{(2)}\|=\sqrt{\sum_{i=1}^{s}(x_i^{(1)}-x_i^{(2)})^2}\ .$$

The modulus of continuity for a function f defined over a closed set A: $-1\leq x_i\leq 1$ $(i=1,\cdots,s)$ is

$$\Omega(\delta,\ f)=\sup\ |f(x^{(1)})-f(x^{(2)})|$$

for all points $x^{(1)}$, $x^{(2)}$ in A with $\|x^{(1)}-x^{(2)}\|\leq\delta$. The uniform modulus of continuity of a finite number of functions $f_1,\cdots,f_N$ is the largest of the moduli of each f_i for each δ.

LEMMA 1. *Let $F(\theta)$ be a continuous periodic function of period 2π in each θ_i and of class c^k. Let $\omega(\delta)$ be the uniform modulus of continuity of the partial derivatives of F of order 1 to k for $\delta\leq\pi\sqrt{s}$. Let $j\leq k$. Then, corresponding to every set $m_1,\cdots,m_s$ of positive integers, there is a trigonometric sum T^m of order at most m_i in θ_i such that*

$$|F_r(\theta)-T_r^m(\theta)|\leq K_1\Big(\sum_{i=1}^{s}\frac{1}{m_i}\Big)^{k-j}\sum_{i=1}^{s}\omega\Big(\frac{1}{m_i}\Big)\quad for\quad 0\leq r_1+\cdots+r_s\leq j$$

where K_1 is a constant independent of F, s, m_i.

If the partial derivatives of order 1 to k satisfy

$$|F_r(\theta^{(1)})-F_r(\theta^{(2)})|\leq L\Big(\sum_{i=1}^{s}|\theta_i^{(1)}-\theta_i^{(2)}|\Big)$$

then

$$|F_r(\theta)-T_r^m|\leq L\,K_2\Big(\sum_{i=1}^{s}\frac{1}{m_i}\Big)^{k-j+1}\quad 0\leq r_1+\cdots+r_s\leq j$$

where K_2 is also a constant independent of F, s, m_i.

If F is even in each θ_i separately, T contains only cosine terms.

LEMMA 2. *Let $f(x)$ be of class c^k in the set A: $-1\leq x_i\leq 1$ $(i=1,\cdots,s)$. Let M be the maximum of the absolute values of the derivatives of order 1 to k, and $\Omega(\delta)$ the uniform modulus of continuity of the derivatives of order k. Let B denote a closed set interior to A. Let $j\leq k$. Then, for every set of positive integers $m_1,\cdots,m_s$ with $m_i\geq k$ there is a polynomial P^m of order at most m_i in x_i such that*

$$|f_r(x)-P_r^m(x)|\leq K_3\Big(\sum_{i=1}^{s}\frac{1}{m_i}\Big)^{k-j}\sum_{i=1}^{s}\Omega\Big(\frac{1}{m_i}\Big)$$

for x in B and $0\leq r_1+\cdots+r_s\leq j$. Here K_3 is a constant independent of f and m_i.

If also, the k-th partial derivatives of $f(x)$ satisfy a Lipschitz condition with parameter λ, then, for x in B,

$$|f_r(x)-P_r^m(x)|\leq K_4\Big(\sum_{i=1}^{s}\frac{1}{m_i}\Big)^{k-j+1} \quad for \quad 0\leq r_1+\cdots+r_s\leq j,$$

and where K_4 is a constant independent of f and m_i.

To apply the lemmas to a function defined over the region R, we shall extend the domain of definition of the function. The question arises whether the differentiability properties of the function are maintained under the extension. The answer depends upon the properties of the boundary Γ of R. For example, Hirschfeld [1] has shown that even a cusp in the complementary region may prevent c^1 extension of a function of class c^∞ on a closed set through a continuous boundary arc. Whitney [6] has given a different definition for a function to be of class c^k in a closed set A. If f is of Whitney class c^k in A, then there exists an extension F to the whole plane E_2 which is of class c^k in the ordinary sense on E_2 and is analytic in E_2-A. The derivatives of F of order $\leq k$ coincide with those of f at any interior point of A. Moreover Whitney [7] has shown the following: Let (a) f be of class c^k on $R+\Gamma$, where R is a region, Γ its boundary, in the sense we have defined in the introduction, and (b) R have the property "P", that any two points P_1, P_2 in R, whose linear distance apart may be represented by $|P_1-P_2|$, can be joined by a rectifiable curve in R of length L, with $L/|P_1-P_2|$ bounded uniformly with respect to P_1 and P_2; then f is also of Whitney class c^k and thus can be extended to E_2 to be of class c^k on E_2.

For our purposes we assume R to be a bounded region with boundary Γ consisting of a finite number of non-intersecting simply closed regular arcs Γ_i and we will show R has property "P".

Choose, for each Γ_i, a $\delta>0$ such that no two tangents to Γ_i on any portion of arc length $<\delta$ make with each other an angle greater than 5°. We may choose δ independent of i and smaller than one-fourth the distance between any two Γ_i. Now fix i, and let P_1, P_2 be points on Γ_i on a subarc of length $<\delta$. There is a point Q on that subarc between P_1 and P_2 such that the tangent line at Q is parallel to the chord P_1P_2. Set up an (x, y) coordinate system at Q, using the tangent line as x-axis, the normal as y-axis, and note that the subarc considered has an equation $y=y(x)$ of class c^1 in view of the implicit function theorems. Let $P_1=(x_1, y_1)$, $P_2=(x_2, y_2)$, $\|P_1-P_2\|=$ distance between P_1 and P_2, $\|\widehat{P_1 P_2}\|=$length of the subarc joining P_1 to P_2. Then $\|P_1-P_2\|=|x_1-x_2|$ and $|y'(x)|\leq 1$ so that

$$\text{(1)}\qquad \begin{aligned}\|P_1-P_2\|\leq\|\widehat{P_1 P_2}\| &= \left|\int_{x_1}^{x_2}\sqrt{1+y'^2}\,dx\right| \\ &\leq\sqrt{2}\,|x_1-x_2|=\sqrt{2}\,\|P_1-P_2\|.\end{aligned}$$

Moreover, since $\tan 5°<1/10$, the mean value theorem shows that $\sup|y(x)|\leq\|P_1-P_2\|/10$.

We shall also use the well known property that if Γ_i is a regular arc, there is an $\omega_1>0$ such that for any subarc joining points P_3, P_4 on Γ_i, we have $\|\widehat{P_3 P_4}\|/\|P_3-P_4\|\leq\omega_1$. ω_1 can be chosen independent of i.

Now suppose S_1, S_2 are any two points interior to the region R. If the segment $S_1 S_2$ is interior to R, we of course have $\|\widehat{S_1 S_2}\|/\|S_1-S_2\|=1$ by using the segment as the arc. Otherwise, let Q_1 be the first intersection of the directed line $S_1 S_2$ with the boundary, say with Γ_1. Let Q_1^1 be a point on S_1Q_1 in R. Let Q_2 be the first point of intersection of the directed line $S_2 S_1$ with Γ_1 and Q_2^1 a point in R on S_2Q_2 such that the open interval $Q_2Q_2^1$ is also in R. Note that Q_1 and Q_2 may coincide. If $Q_2^1S_2$ is not in R, let Q_3 be the first point of intersection of the directed line $Q_2^1S_2$ with the boundary, say with Γ_2 and Q_3^1 in R and on $Q_2^1Q_3$. Let Q_4 be the first point of intersection of the directed line $S_2Q_2^1$ with Γ_2 and Q_4^1 a point in R, on Q_4S_2, with the open interval $Q_4Q_4^1$ in R. Continuing in this way, after at most n steps, we form a finite sequence of points $Q_0^1=S_1$, Q_1^1, $Q_2^1,\cdots,Q_{2m}^1$, $Q_{2m+1}^1=S_2$ such that Q_{2k-1} and Q_{2k} are on the same regular arc, and the lines joining Q_{2k}^1 to Q_{2k+1}^1, $k=0,\cdots,m$ are in R. If we can show there is an $\omega>0$, independent of the points, and arcs λ_1 in R joining consecutive points Q_j^1 to Q_{j+1}^1 such that $\|\widehat{Q_j^1 Q_{j+1}^1}\|\leq\omega\|Q_j^1-Q_{j+1}^1\|$, then we can attain the desired results by addition. It suffices to show that Q_1^1 and Q_2^1 and an arc λ joining Q_1^1 to Q_2^1 and in R may be chosen so that $\|\widehat{Q_1^1 Q_2^1}\|\leq\omega\|Q_1^1-Q_2^1\|$. Suppose first that Q_1 and Q_2 coincide. A sufficiently small circle with Q_1 as center will have one of the arcs cut off by S_1S_2 entirely in R and we may choose Q_1^1 and Q_2^1 as the intersections of S_1S_2 with this circle. In this case

$$\|\widehat{Q_1^1 Q_2^1}\|=\frac{\pi}{2}\|Q_1^1-Q_2^1\|.$$

Otherwise, let L be the length of an arc on Γ_1 joining Q_1 to Q_2.

Divide this arc into N equal segments of length $\beta = L/N$ where N is sufficiently large so that $\beta < \delta$. Draw circles of radius $r = \beta/\sqrt{2}$ about each of the division points and the end points. We first show that consecutive circles intersect. If R_1 and R_2 are two consecutive centers, (1) implies

$$\| R_1 - R_2 \| \leq \beta \leq \sqrt{2} \, \| R_1 - R_2 \|$$

so that

$$\frac{\| R_1 - R_2 \|}{2} \leq \frac{\| R_1 - R_2 \|}{\sqrt{2}} \leq \frac{\beta}{\sqrt{2}} \leq \| R_1 - R_2 \| ,$$

and the circles must intersect.

Moreover, since $r \geq \| R_1 - R_2 \| / \sqrt{2}$, the semi-length τ of the common chord is

$$\tau = \sqrt{r^2 - \frac{\| R_1 - R_2 \|^2}{4}} \geq \frac{\| R_1 - R_2 \|}{2} ,$$

whereas the arc joining R_1 to R_2 has distance $< \| R_1 - R_2 \| / 10$ from the chord. Hence the arc lies entirely within the circles.

Now let Q_1^1 be an intersection of $S_1 S_2$ with the circle whose center is Q_1 and Q_2^1 an intersection of $S_1 S_2$ with the circle whose center is Q_2, the points being chosen to lie in R and have the desired properties. Starting from Q_1^1 we may proceed to Q_2^1 via the circumferences of the circles. The total length of the curve thus formed will be less than

$$(N+1) 2\pi \frac{\beta}{\sqrt{2}} = \frac{N+1}{N} \frac{2\pi}{\sqrt{2}} L \leq \frac{4\pi}{\sqrt{2}} L$$

and

$$\frac{\| \widehat{Q_1^1 Q_2^1} \|}{\| Q_1^1 - Q_2^1 \|} \leq \frac{4\pi}{\sqrt{2}} \frac{L}{\| Q_1^1 - Q_2^1 \|} \leq \frac{4\pi}{\sqrt{2}} \frac{L}{\| Q_1 - Q_2 \|} \leq \frac{4\pi}{\sqrt{2}} \omega_1 .$$

This concludes the proof that R has property "P".

We will be particularly interested in extending a function of the form $v(x, y)/G(x, y)$ where $G(x, y) > 0$ in R, $\partial G / \partial \nu > 0$ on Γ, $G = v = 0$ on Γ and we seek differentiability conditions on v and G which insure that v/G is of class c^k on $R + \Gamma$. Here again the nature of the boundary is of importance. The next two lemmas deal with this problem. The letter P will refer to a point in R and Q to a point on Γ, the boundary of R. By a neighborhood $N(Q)$ in $R + \Gamma$ we will mean a set of points S in $R + \Gamma$ such that for some sufficiently small circle with center at Q, every point of the circle which lies in $R + \Gamma$ also lies in S.

LEMMA 3. a) *Let R be a region bounded by Γ, a finite number of closed Jordan curves, no two having a point in common. Let γ be a regular subarc of Γ, and Q_0 an interior point of γ. Let N be the normal to γ at Q_0. Then there is a neighborhood $N(Q_0)$ in $R+\Gamma$ such that through each point P in $RN(Q_0)$, the line parallel to N cuts $\gamma_0=\gamma N(Q_0)$ in one and only one point Q, PQ lies in $N(Q_0)$, and Q ranges over γ_0.*

b) *Let $\psi(x, y)$ be of class c^1 in $RN(Q_0)$ and suppose ψ, ψ_x, ψ_y have continuous limits on γ_0. Define $(\partial\psi/\partial s)(P)$ to be the derivative at $P\in RN(Q_0)$ in the direction of the tangent at the corresponding point Q on γ_0. The derivative $(\partial\psi/\partial s)(P)$ has continuous limits on γ_0 which we will denote by $(\partial\psi/\partial s)(Q)$.*

If $\psi=0$ on γ_0, then $(\partial\psi/\partial s)(Q)=0$ for Q on γ_0.

Proof. Let γ be given by $x(t)$, $y(t)$ and Q_0 defined by the parameter value t_0. Let (ξ, η) be rectangular axes along the tangent and normal at Q_0. In a suitable neighborhood of t_0, $t_1<t<t_2$, defining an arc λ_0 containing Q_0, γ admits a representation $\eta=\eta(\xi)$. We may assume λ_0 so small that no two tangents to it make with each other an angle greater than 5°. There is a positive distance d between $\Gamma-\lambda_0$ and the arc λ_1 defined by the parameter range $(t_1+t_0)/2<t<(t_0+t_2)/2$. Take

$$\delta<\min[d,\ |\xi(t_0)-\xi((t_0+t_2)/2)|,\ |\xi(t_0)-\xi((t_0+t_1)/2)|]$$

and draw a square T of side δ with sides parallel to the (ξ, η) axes and center at Q_0. Let $\gamma_0=\gamma T$, the projection of RT on γ by lines parallel to N, and let γ_h be the arcs formed by displacing γ_0 a distance h parallel to itself into R along N. For $h<h_1$ sufficiently small, $\gamma_h\subset T$. The regular arc γ_0 may be given a representation $x=x(s)$, $y=y(s)$, $0\leq s\leq L$, in terms of arc length s, where L is the length of γ_0. Then γ_h is given by

$$x=x(s)+h\cos\alpha, \qquad y=y(s)+h\cos\beta,$$

where $\cos\alpha$, $\cos\beta$ are the direction cosines of the line N directed inward into R. The neighborhood $N(Q_0)$ may be chosen as given by these equations with $0<s<L$. $0\leq h<h_1$.

It is clear that

$$\frac{\partial\psi}{\partial s}=\frac{\partial\psi}{\partial x}\cdot\frac{dx}{ds}+\frac{\partial\psi}{\partial y}\frac{dy}{ds}$$

has continuous limits on γ_0. Write

$$\frac{\partial\psi}{\partial s}(P)=\frac{\partial\psi}{\partial s}(x(s)+h\cos\alpha,\ y(s)+h\cos\beta)=F(s,\ h).$$

If λ is any closed subarc of γ_0, we have

$$\lim_{h\to 0} F(s,\ h) = \frac{\partial\psi}{\partial s}(Q)$$

uniformly in s.

Along γ_h we have

$$\psi(P_2) - \psi(P_1) = \int_{s_1}^{s_2} F(s,\ h)\,ds$$

where P_1 and P_2 are points on γ_h corresponding to points Q_1 and Q_2 on λ with parameter values s_1 and s_2. As h approaches 0, the limits on the integral remain fixed. Since $\psi=0$ on λ, we find, by letting $h\to 0$,

$$0 = \int_{s_1}^{s_2} \frac{\partial\psi}{\partial s}(Q)\,ds$$

for arbitrary s_1, s_2. Thus $(\partial\psi/\partial s)(Q)=0$ on λ and hence on γ_0.

LEMMA 4. *Let R, γ, Q_0, $N(Q_0)$, N, γ_0 be defined as in Lemma 3. Let $v(x,\ y)$ and $G(x,\ y)$ be of class c^0 on $N(Q_0)$ and of class c^1 on $N(Q_0)[R+Q_0]$. Let $v=G=0$ on γ_0, $G>0$ in $RN(Q_0)$, $(\partial G/\partial\nu)(Q_0)\neq 0$. Then there exists $\lim_{P\to Q_0} v(P)/G(P)$ for $P\in R$.*

If γ is of class c^{k+1} on $N(Q_0)$ and v, G are of class c^k in $N(Q_0)$ and of class c^{k+1} on $N(Q_0)[R+Q_0]$, then v/G is of class c^k on $N(Q_0)[R+Q_0]$.

Proof. Denote differentiation along a line parallel to N by $\partial/\partial h$. By the mean value theorem one finds that $(\partial G/\partial\nu)(Q_0)$ is the limiting value of $(\partial G/\partial h)(P)$ as $P\in RN(Q_0)$ approaches Q_0 along the normal at Q_0, and hence $(\partial G/\partial\nu)(Q_0)$ is the limiting value of $(\partial G/\partial h)(P)$ as P approaches Q_0 by any approach in $RN(Q_0)$. A similar statement is true for $(\partial v/\partial\nu)(Q_0)$.

Let P_n be any sequence of points in $RN(Q_0)$ converging to Q_0 and let Q_n be the points on γ_0 associated, by projection along N, with P_n. By the generalized mean value theorem,

$$\frac{v(P_n)}{G(P_n)} = \frac{v(P_n)-v(Q_n)}{G(P_n)-G(Q_n)} = \frac{(\partial v/\partial h)(P_n')}{(\partial G/\partial h)(P_n')}$$

where P_n' is interior to the line segment P_nQ_n.

Thus

$$\lim_{P_n\to Q_0} \frac{v(P_n)}{G(P_n)} = \frac{(\partial v/\partial\nu)(Q_0)}{(\partial G/\partial\nu)(Q_0)}\ .$$

It is clear from the construction of $N(Q_0)$ that the equations

$$x=X(s,\ h)=x(s)+h\cos\alpha\ ,\quad y=Y(s,\ h)=y(s)+h\cos\beta$$

yield a one to one transformation of $N(Q_0)$ into $N^*(Q_0):\ 0\leq h<h_1$, $0<s<L$ and γ_0 into γ_0^*: $h=0$, $0<s<L$ and Q_0 into Q_0^*: $h=0$, $s=s_0$. In fact, in view of the restriction on the slope of the tangent to γ_0, the Jacobian of the transformation is

$$J=x'(s)\cos\beta-y'(s)\cos\alpha\neq 0\ .$$

If $x(s)$, $y(s)$ are of class c^{k+1} on $0<s<L$ then so are $X(s,\ h)$, $Y(s,\ h)$ in $N^*(Q_0)-\gamma_0^*$. Any partial derivative of $X(s,\ h)$, $Y(s,\ h)$ of order $r\leq k+1$ converges, as $h\rightarrow 0$, uniformly on any closed subinterval of γ_0^* and thus this derivative has a continuous limit on γ_0^*. By the implicit function theorems, the inverse functions $s=S(x,\ y)$, $h=H(x,\ y)$ are of class c^{k+1} in $RN(Q_0)$. Moreover, the partial derivatives of S, H of order $r\leq k+1$ have continuous limits on γ_0, for the relationships

$$1=\frac{\partial X}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial X}{\partial h}\frac{\partial h}{\partial x}=x'(s)\frac{\partial s}{\partial x}+\cos\alpha\frac{\partial h}{\partial x}$$

$$0=\frac{\partial Y}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial Y}{\partial h}\frac{\partial h}{\partial x}=y'(s)\frac{\partial s}{\partial x}+\cos\beta\frac{\partial h}{\partial x}$$

can be solved for $\partial s/\partial x$, $\partial h/\partial x$, $\partial s/\partial y$, $\partial h/\partial y$ and the resulting equations indicate that these derivatives and their derivatives of order $\leq k$ have continuous limits on γ_0.

Under this transformation $v(x,\ y)$ becomes $v^*(s,\ h)$ and $G(x,\ y)$ becomes $G^*(s,\ h)$. It is sufficient to show v^*/G^* is of class c^k at Q_0^* since any partial derivative of order $r\leq k$ of $v(x,\ y)/G(x,\ y)$ is a polynomial in the derivatives of v^*/G^* and in the derivatives of s and h with respect to x and y of order $\leq r$.

By the hypothesis and comments above, $v^*(s,\ h)$ and $G^*(s,\ h)$ are of class c^k on $N^*(Q)$ and of class c^{k+1} on $(N^*(Q_0)-\gamma_0^*)+Q_0^*$. In view of the continuity of $\partial G/\partial h$ at Q_0. there is a neighborhood of Q_0 where $(\partial G/\partial h)(P)>\delta>0$, It is no loss of generality to assume $(\partial G/\partial h)>\delta>0$ in $N(Q_0)$ and we shall do so. By Lemma 3, $\partial v/\partial s$ and $\partial G/\partial s$ vanish on γ_0. By repeated application of Lemma 3, $\partial^r v/\partial s^r$ and $\partial^r G/\partial s^r$ $(0\leq r\leq k)$ vanish on γ_0.

The proof is greatly facilitated by an auxiliary transformation. Let $t=s$, $z=G^*(s,\ h)$ carrying Q_0^* into Q_0^{**}, γ_0^* into γ_0^{**}, $N^*(Q_0)$ into $N^{**}(Q_0)$. For each s, z is a monotone increasing function of h and the inverse function $h=H^*(t,\ z)$ is a monotone increasing function of z for each t. As above, we see that $v^*(s,\ h)=v^{**}(t,\ z)$ is of class c^k on

$N^{**}(Q_0)$ and of class c^{k+1} on $(N^{**}(Q_0)-\gamma_0^{**})+(Q_0^{**})$. Moreover, it suffices to prove that $v^{**}(t, z)/z$ is of class c^k at Q_0^{**}. For notational simplicity, let $w(t, z)=v^{**}(t, z)$. Note that $N^{**}(Q_0)$ is the set $0\leq z<G(t, h_1)$, $0<t<L$.

By induction, we verify

$$\frac{\partial^r}{\partial z^r}\left(\frac{w}{z}\right)=\frac{r!(-1)^r}{z^{r+1}}\left(w-z\frac{\partial w}{\partial z}+\frac{z^2}{2!}\frac{\partial^2 w}{\partial z^2}+\cdots+(-1)^r\frac{z^r}{r!}\frac{\partial^r w}{\partial z^r}\right)$$

for $0\leq r\leq k$ when $z>0$.

For t fixed, $w(t, z)$ has a Taylor expansion of the form

$$w(t, \zeta)=w(t, z)+\frac{\partial w}{\partial z}(t, z)(\zeta-z)+\cdots+\frac{\partial^r w}{\partial z^r}(t, z)\frac{(\zeta-z)^r}{r!}$$
$$+\frac{\partial^{r+1}w}{\partial z^{r+1}}(t, \xi)\frac{(\zeta-z)^{r+1}}{(r+1)!}$$

for $0\leq r\leq k$, where $0\leq\zeta<\xi(t, z, \zeta, r)<z$ so that, when $\zeta=0$,

$$0=w(t, 0)=w(t, z)-z\frac{\partial w}{\partial z}(t, z)+\cdots+\frac{(-1)^r}{r!}z^r\frac{\partial^r w}{\partial z^r}(t, z)$$
$$+\frac{(-1)^{r+1}z^{r+1}}{(r+1)!}\frac{\partial^{r+1}w}{\partial z^{r+1}}(t, \xi).$$

Hence

$$\frac{\partial^r}{\partial z^r}\left(\frac{w}{z}\right)=\frac{1}{r+1}\frac{\partial^{r+1}}{\partial z^{r+1}}w(t, \xi),$$

which has a limit as the point (t, z) approaches Q_0^{**}.

We have thus shown that the partial derivatives of w/z, with respect to z alone, of order $\leq k$ have limits at Q_0^{**}.

We next show that the partial derivatives of w/z with respect to t alone have limits at Q_0^{**}. First note that the derivatives of w with respect to t alone vanish at $z=0$. For, $w(t, z)=v^*(s, h)$ so that

$$\frac{\partial v}{\partial s}=\frac{\partial v^*}{\partial s}=\frac{\partial w}{\partial t}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}=\frac{\partial w}{\partial t}+\frac{\partial w}{\partial z}\frac{\partial G}{\partial s}$$

and, as we have seen, $\partial v/\partial s$ and $\partial G/\partial s$ vanish at $z=0$. Thus $\partial w/\partial t=0$ at $z=0$. Similarly, successive differentiation shows $\partial^r w/\partial t^r=0$ on γ_0^{**}, $0\leq r\leq k$.

We apply Taylor's theorem to obtain

$$\frac{\partial^r}{\partial t^r}\left(\frac{w}{z}\right)=\frac{1}{z}\frac{\partial^r w(t, z)}{\partial t^r}=\frac{1}{z}\left\{z\frac{\partial}{\partial z}\left[\frac{\partial^r w(t, \xi)}{\partial t^r}\right]\right\}=\frac{\partial}{\partial z}\frac{\partial^r w(t, \xi)}{\partial t^r},$$
$$0<\xi(z, r)<z$$

and conclude that $(\partial^r/\partial t^r)(w/z)$ has a limit at Q_0^{**} for $0 \leq r \leq k$,

Finally, any mixed derivative may be written as

$$\frac{\partial^{n+m}}{\partial z^n \partial t^m}\left(\frac{w}{z}\right), \qquad n+m=r\leq k$$

and this may be written as $\dfrac{\partial^n}{\partial z^n}\left\{\dfrac{1}{z}\dfrac{\partial^m w}{\partial t^m}\right\}$

where $\partial^m w/\partial t^m$ vanishes on γ_0^{**} and is of class c^{k-m} on $N^{**}(Q_0)$ and of class c^{k-m+1} on $(N^{**}(Q_0)-\gamma_0^{**})+Q_0^{**}$. By the first results for derivatives with respect to z, the mixed derivatives have the desired property.

THEOREM 1. *Let R be a bounded region whose boundary Γ consists of a finite number of non-intersecting simply closed regular arcs of class c^k, $(k \geq 2)$. Let $G(x, y)$ be a function of class c^k on $R+\Gamma$, vanishing on Γ, positive in R, with $\partial G/\partial \nu \geq \delta > 0$ on Γ.*

Let H be the Hilbert space formed by completing the linear vector space V of variations—functions of class c^k on $\bar{R}$ and vanishing on Γ—, using the functional

$$D[\xi]=\iint(a\xi_x^2+b\xi_y^2+c\xi^2)dx\,dy$$

for $\xi \in V$ as the square of the norm, where a, b, c are bounded and integrable, $a>0$, $b>0$, $c\geq 0$ in $R+\Gamma$.

Then the set of functions $G\tau$, where τ is a polynomial in x and y, is dense in H. The set $\{f_i\}$ obtained by orthonormalizing the set $\{Gx^i y^j\}$ is complete in H.

If $g(x, y)$ is a function of class c^k on $\bar{R}$ and Ψ is the set of functions ψ of class c^k on $\bar{R}$, assuming the values of $g(x, y)$ on Γ, and if for any $\psi \in \Psi$ we define $b_i=D[\psi-g, f_i]$, then

$$\left\|\psi-g-\sum_{i=1}^n b_i f_i\right\|^2=D\left[\psi-g-\sum_{i=1}^n b_i f_i\right]\leq\frac{\theta(n)}{n^{k-2}}$$

where $\lim_{n\to\infty}\theta(n)=0$, θ depending on $\psi-g$.

In particular, if f is integrable,

$$I[\psi]=\iint(a\psi_x^2+b\psi_y^2+c\psi^2+2f\psi)dx\,dy\,,$$

and there exists an admissible function ψ_0 which minimizes $I[\psi]$ for $\psi \in \Psi$, and we define

$$a_i = -\iint f f_i \, dx\, dy - D[g, f_i], \quad \phi_n = g + \sum_{i=1}^{n} a_i f_i ,$$

then

$$\| \psi_0 - \phi_n \|^2 = D[\psi_0 - \phi_n] \leq \frac{\theta(n)}{n^{k-2}}$$

where $\lim_{n\to\infty} \theta(n) = 0$.

Proof. If v is a variation, we show there is a sequence Q_j of polynomials such that

$$\lim_{j\to\infty} \| v - G\,Q_j \|^2 = \lim_{j\to\infty} \iint [a(v - G\,Q_j)_x^2 + b(v - G\,Q_j)_y^2 + c(v - G\,Q_j)^2]\, dx\, dy = 0.$$

In view of Lemma 4, v/G is of class c^{k-1} on $\bar{R}$ and it is thus possible to extend the definition of v/G over the entire plane so that it is of class c^{k-1} over the entire plane. Let $\Omega(\delta)$ be the uniform modulus of continuity of the $(k-1)$st partial derivatives of v/G over a rectangle with sides parallel to the axes containing R in its interior.

By Lemma 2, with $s=2$, $j=1$, $m_1 = m_2 = j$ there is a sequence Q_j of polynomials of degiee $2j$ in x and y such that, for (x, y) in $\bar{R}$,

$$\left| \frac{v}{G} - Q_j \right| , \quad \left| \left(\frac{v}{G}\right)_x - Q_{j_x} \right| , \text{ and } \left| \left(\frac{v}{G}\right)_y - Q_{j_y} \right| \text{ are all } O\left(\frac{1}{j^{k-2}}\, \Omega\left(\frac{1}{j}\right)\right).$$

Hence

$$(v - G\,Q_j)_x^2 = \left[\left(\frac{v}{G} - Q_j\right) G\right]_x^2 = \left[\left(\frac{v}{G} - Q_j\right)_x G + \left(\frac{v}{G} - Q_j\right) G_x\right]^2$$

$$\leq \left(\frac{v}{G} - Q_j\right)_x^2 G^2 + G_x^2 \left(\frac{v}{G} - Q_j\right)^2 + 2G|G_x| \left|\left(\frac{v}{G} - Q_j\right)_x\right| \left|\frac{v}{G} - Q_j\right|$$

$$= O\left(\frac{1}{j^{2(k-2)}} \left[\Omega\left(\frac{1}{j}\right)\right]^2 \right) .$$

A similar result is true for $(v - G\,Q_j)_y^2$ and $(v - G\,Q_j)^2$. Thus $\lim_{j\to\infty} D[v - G\,Q_j] = 0$ for $k \geq 2$.

It has thus been proved that the linear manifold formed by $\{G x^i y^j\}$ is dense in V and thus in H. By the previous discussion the set $\{f_i\}$ is complete in H.

Now let v in the above be the particular variation $\psi - g$ and let $[N]$ represent the largest integer $\leq N$. For fixed n, let $j = [(\sqrt{n}/2) - 1]$ and $\tau_n(x, y) = Q_j(x, y)$. Thus there is a sequence τ_n of degree at most

$$2j \leq \left[2\left(\frac{\sqrt{n}}{2} - 1 \right) \right] = [\sqrt{2n} - 2]$$

such that

$$D[\psi-g-G\tau_n]=O\left(\frac{1}{n^{k-2}}\Omega^2\left(\frac{8}{\sqrt{n}}\right)\right).$$

Now $\sum_{i=1}^{n} b_i f_i = G\mu_n$ where μ_n is a polynomial of degree greater than $\sqrt{2n}-2$, and it is known that

$$\left\|\psi-g-\sum_{i=1}^{n} c_i f_i\right\|$$

is a minimum when $c_i=(\psi-g, f_i)=b_i$. Thus

$$D\left[\psi-g-\sum_{i=1}^{n} b_i f_i\right]=O\left(\frac{1}{n^{k-2}}\theta\right), \quad \lim_{n\to\infty}\theta=0.$$

In particular, if ψ_0 minimizes $I[\psi]$, then we have seen that

$$D[\psi_0-g, f_i]=-\iint f f_i\,dx\,dy-D[g, f_i].$$

Thus, in this case, the Fourier coefficients depend only on known quantities.

COROLLARY
$$b_n=O\left(\sqrt{\frac{\theta(n)}{n^{k-2}}}\right).$$

Proof.
$$\begin{aligned} b_n^2=D[b_nf_n]&=D\left[\left(\psi-g-\sum_{i=1}^{n-1} b_i f_i\right)-\left(\psi-g-\sum_{i=1}^{n} b_i f_i\right)\right]\\ &\leq D\left[\psi-g-\sum_{i=1}^{n-1} b_i f_i\right]+2\left(D\left[\psi-g-\sum_{i=1}^{n-1} b_i f_i\right]\right.\\ &\quad\left.\cdot D\left[\psi-g-\sum_{i=1}^{n} b_i f_i\right]\right)^{1/2}+D\left[\psi-g-\sum_{i=1}^{n} b_i f_i\right]=O\left(\frac{\theta(n)}{n^{k-2}}\right). \end{aligned}$$

§ 3. Expansion Theorems.

We use the notations in Theorem 1 and seek conditions which insure that convergence in H yields uniform convergence in $\bar{R}$.

THEOREM 2. *Let R be a bounded region with boundary Γ. Let ψ, ψ_n be continuous on $\bar{R}$, absolutely continuous on each line in $\bar{R}$ and all taking on the same values on Γ. Let $D[\psi]<\infty$, $D[\psi_n]<\infty$. If $\lim_{n\to\infty} D[\psi-\psi_n]=0$, then a necessary and sufficient condition that $\lim_{n\to\infty}\psi_n=\psi$ uniformly on $\bar{R}$ is that ψ_n be equicontinuous on $\bar{R}$. If $\lim_{n,m\to\infty} D[\psi_n-\psi_m]=0$ then a necessary and sufficient condition that $\lim_{n\to\infty}\psi_n$ exists uniformly on $\bar{R}$ is that ψ_n be equicontinuous on $\bar{R}$.*

Proof. The necessity is clear since a sequence of continuous functions which converge uniformly are equicontinuous.

Let $u(x, y)$ be a function with the continuity properties of $\psi(x, y)$ and vanishing on Γ. Let P_0 be a point interior to R. Place polar coordinates at P_0. If a ray from P_0 meets the circle S_ρ of radius $\rho \leq d$, d being the diameter of R, with P_0 as center, before it meets Γ, label P_1 the first intersection point with S_ρ and Q the first intersection with Γ. Otherwise both P_1 and Q will refer to the first intersection point of the ray and Γ.

$$\frac{1}{2\pi}\left\{\int_0^{2\pi}|u(P_1)|\,d\theta\right\}^2 \leq \int_0^{2\pi} u^2(P_1)d\theta = \int_0^{2\pi}\left[\int_{P_1}^{Q}\frac{\partial u}{\partial r}dr\right]^2 d\theta$$

$$= \int_0^{2\pi}\left[\int_{P_1}^{Q}\frac{1}{\sqrt{r}}\sqrt{r}\frac{\partial u}{\partial r}dr\right]^2 d\theta \leq \int_0^{2\pi}\log\frac{d}{\rho}\int_{P_1}^{Q} r\left(\frac{\partial u}{\partial r}\right)^2 dr\,d\theta$$

$$\leq \log\frac{d}{\rho}\iint(u_x^2+u_y^2)dx\,dy \leq \alpha\log\frac{d}{\rho}D[u]$$

where $\alpha = 1/\min_{R+\Gamma}(a, b)$, since

$$\iint(au_x^2+bu_y^2+cu^2)dx\,dy \geq \min(a, b)\iint(u_x^2+u_y^2)dx\,dy\,.$$

Apply this result to the functions $u_n = \psi - \psi_n$ (or to $u_{nm} = \psi_n - \psi_m$) which are equicontinuous on $R+\Gamma$ and thus have a uniform modulus of continuity $\omega(\delta)$, which approaches 0 with δ.

Since P_1 is on or interior to the circle of radius ρ, we have $|u_n(P_1) - u_n(P_0)| \leq \omega(\rho)$, whence $|u_n(P_1)| \geq |u_n(P_0)| - \omega(\rho)$ and

$$2\pi[|u_n(P_0)| - \omega(\rho)] \leq \sqrt{2\pi\alpha\log d/\rho\, D[u_n]}\,.$$

Thus

$$|u_n(P_0)| \leq \sqrt{\frac{\alpha}{2\pi}D[u_n]\log\frac{d}{\rho}} + \omega(\rho)\,,$$

which is true even if P_0 is on Γ.

Now, for $\varepsilon > 0$, choose $\rho = \rho_1$ so small that $\omega(\rho_1) < \varepsilon/2$ and then choose N so large that

$$\frac{\alpha}{2\pi}D[u_n]\log\frac{d}{\rho_1} < \frac{\varepsilon^2}{4}$$

for $n > N$. Hence

$$\varepsilon > 0 \cdot\supset\cdot \exists\, N(\varepsilon) \ni n > N \cdot\supset\cdot |\psi(P_0) - \psi_n(P_0)| < \varepsilon\,.$$

LEMMA 5. *Let R be a bounded region with boundary Γ and diameter d.*

Let $u(x, y)$ be continuous on $R+\Gamma$, absolutely continuous on each line in $R+\Gamma$, and vanish on Γ, and let $0<D[u]<\infty$. Let $\alpha=1/\min_{R+\Gamma}(a, b)$. Let $P_0 \in R+\Gamma$. If there exists $\delta>0$, $K\geq 0$ and

$$|u(P)-u(P_0)|\leq K\,\|P-P_0\|^{\delta}$$

for all points P such that the ray P_0P is in $R+\Gamma$, then

$$|u(P_0)|\leq\sqrt{\frac{\alpha D[u]}{2\pi\delta}\log^{+}\frac{d^{\delta}K}{\Delta D[u]}}+\Delta D[u]$$

where Δ is any number >0, and

$$\log^{+}x=\begin{cases}\log x & \text{if } x>1\\ 0 & \text{if } x\leq 1\,.\end{cases}$$

Proof. If P_0 is interior to R, and $\rho\leq d$, then as in Theorem 2

$$\frac{1}{2\pi}\left\{\int_0^{2\pi}|u(P_1)|\,d\theta\right\}^2\leq \alpha \log d/\rho\, D[u],$$

where P_1 is a point which is the first intersection of a ray from P_0 with either Γ or the circle of radius $\rho\leq d$ about P_0 as center.

Since P_1 is on or interior to the circle of radius ρ, we have

$$|u(P_1)-u(P_0)|\leq K\rho^{\delta},\quad |u(P_1)|\geq|u(P_0)|-K\rho^{\delta},$$

$$2\pi[u(P_0)-K\rho^{\delta}]\leq\sqrt{2\pi\alpha \log d/\rho\, D[u]},$$

$$|u(P_0)|\leq\sqrt{\frac{\alpha}{2\pi}D[u]\log\frac{d}{\rho}}+K\rho^{\delta},$$

which holds even if P_0 is on Γ.

Let $\Delta>0$. If

$$\left(\frac{\Delta D[u]}{K}\right)^{1/\delta}<d\,,$$

choose

$$\rho=\left(\frac{\Delta D[u]}{K}\right)^{1/\delta}$$

to obtain

$$|u(P_0)|\leq\sqrt{\frac{\alpha}{2\pi\delta}D[u]\log\frac{d^{\delta}K}{\Delta\, D[u]}}+\Delta\, D[u]\,.$$

Otherwise,

$$\left(\frac{\Delta\, D[u]}{K}\right)^{1/\delta}\geq d\,,\qquad K\leq\frac{\Delta\, D[u]}{d^{\delta}}$$

and we may replace K to obtain

$$|u(P_0)| \leq \sqrt{\frac{\alpha}{2\pi} D[u] \log \frac{d}{\rho}} + \frac{\varDelta\, D[u]}{d^\delta} \rho^\delta .$$

Choose $\rho = d$ to obtain $|u(P_0)| \leq \varDelta\, D[u]$.

COROLLARY 1. *A sufficient condition that a sequence u_n, absolutely continuous on each line in $\bar{R}$, vanishing on Γ, continuous on $\bar{R}$, and having $\lim_{n\to\infty} D[u_n]=0$, converge to 0 at P_0 is that $\exists\, \delta>0$ and a sequence K_n, with $\lim_{n\to\infty} D[u_n] \log K_n = 0$ such that*

$$|u_n(p) - u_n(P_0)| \leq K_n \| P - P_0 \|^\delta$$

for all P with ray P_0P in $\bar{R}$. If δ, K_n are independent of P_0, the convergence is uniform. In any case,

$$|u_n(P_0)| \leq \sqrt{\frac{\alpha}{2\pi\delta} D[u_n] \log^+ \frac{d^\delta K_n}{\varDelta_n D[u_n]}} + \varDelta_n D[u_n]$$

for any $\varDelta_n > 0$.

LEMMA 6. *Let R be a bounded domain with boundary Γ. Let $P_0 \in \bar{R}$ and suppose there is a circle of radius ε lying in $\bar{R}$ and containing P_0. Place polar coordinates (r, θ) at P_0. Let $u(x, y)$ be of class c^1 in $\bar{R}$ and suppose that there exist $\lambda>0$, $\sigma \geq 0$ such that*

$$|u_r(P) - u_r(P_0)| \leq \sigma \| P - P_0 \|^\lambda$$

for all points P such that the ray P_0P is in $\bar{R}$.

Then

$$|\nabla u(P_0)|^2 \leq (\sigma^2 D[u]^\lambda)^{1/(\lambda+1)} \left(\frac{\alpha}{\lambda\pi}\right)^{\lambda/(\lambda+1)} 2^{(5\lambda+3)/(\lambda+1)} (\lambda+1) + \frac{8\alpha\, D[u]}{\pi\varepsilon^2} \left(\frac{\lambda+1}{\lambda}\right) .$$

Proof. $$|u_r(P_0)| \leq |u_r(P)| + \sigma r^\lambda$$

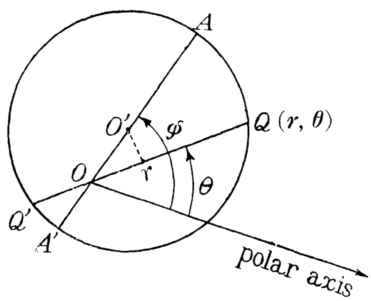
Integrating over a circle S_ρ of radius $\rho \leq \varepsilon$ which contains P_0, $S_\rho \subset S_\varepsilon$, we obtain

$$\iint_{S_\rho} |u_r(P_0)|^2 r\, dr\, d\theta \leq 2 \iint_{S_\rho} u_r(P)^2 r\, dr\, d\theta + 2 \iint_{S_\rho} \sigma^2 r^{2\lambda} r\, dr\, d\theta .$$

We may assume that the polar axis lies in the direction of $\nabla u(P_0)$. Hence $u_r(P_0) = |\nabla u(P_0)| \cos\theta$ and

$$\iint_{S_\rho} |\nabla u(P_0)|^2 (\cos^2\theta) r\, dr\, d\theta < 2\alpha\, D[u] + 2\sigma^2 (2\rho)^{2\lambda} \cdot \pi\rho^2 .$$

We will show that the minimum value of $\iint_{S_\rho}(\cos^2\theta)r\,dr\,d\theta$ is $\pi\rho^2/4$. Suppose first that the pole O is interior to S_ρ. Let $r(\theta)$ be the equation of the circle relative to the pole O. Let Q be the point $(r(\theta), \theta)$ and Q' the point $(r(\theta+\pi), \theta+\pi)$. Q and Q' are thus the intersections of a ray through O with the circle. Let O' be the center of the circle and suppose the coordinates of O' relative to O are (c, ϕ). Then the angle between OQ and OO' is $\phi-\theta$. Drop a perpendicular from O' to QOQ' hitting the latter at T, the length of OT being $|c\cos(\phi-\theta)|$. Thus one of the lengths $\|OQ\|$, $\|OQ'\|$ is $m+|c\cos(\phi-\theta)|$ and the other is $m-|c\cos(\phi-\theta)|$ where $2m$ is the length of QQ', and the product $\|OQ'\|\cdot\|OQ\|=m^2-c^2\cos^2(\phi-\theta)$. Also, if OO' meets the circle in points A, A' it is easily seen that $\|OA'\|\,\|OA\|=\|OQ\|\,\|OQ'\|$ so that $(\rho+c)(\rho-c)=m^2-c^2\cos^2(\phi-\theta)$ and $m^2=\rho^2-c^2+c^2\cos^2(\phi-\theta)$. Hence



$$\|OQ\|^2+\|OQ'\|^2=[m+|c\cos(\phi-\theta)|]^2+[m-|c\cos(\phi-\theta)|]^2$$
$$=2m^2+2c^2\cos^2(\phi-\theta)=2\rho^2-2c^2+4c^2\cos^2(\phi-\theta).$$

We note that

$$\iint_{S_\rho}(\cos^2\theta)r\,dr\,d\theta=\frac{1}{2}\int_0^{2\pi}r^2(\theta)\cos^2\theta\,d\theta=\frac{1}{2}\int_0^{\pi}(\|OQ\|^2+\|OQ'\|^2)\cos^2\theta\,d\theta$$
$$=\frac{1}{2}\int_0^{\pi}[2\rho^2-2c^2+4c^2\cos^2(\phi-\theta)]\cos^2\theta\,d\theta\,.$$

Moreover this formula holds even if O is a point on the circumference for in this case

$$\iint_{S_\rho}(\cos^2\theta)\,r\,dr\,d\theta=1/2\int_\gamma^{\gamma+\pi}r^2(\theta)\cos^2\theta\,d\theta$$

where γ is the angle between the polar axis and the tangent to the circle at O in that direction which has the area to the left of the tangent line. Here $r^2=[2\rho\cos(\phi-\theta)]^2$ and since the square of the cosine has period π, the integral reduces to

$$\frac{1}{2}\int_0^{\pi}4\rho^2\cos^2(\phi-\theta)\cos^2\theta\,d\theta\,.$$

Thus, in any case,

$$\iint_{S_\rho} (\cos^2\theta) r\,dr\,d\theta = \frac{1}{2}\int_0^\pi [2\rho^2 - 2c^2 + 4c^2\cos^2(\phi-\theta)]\cos^2\theta\,d\theta$$
$$= \frac{1}{2}\pi\rho^2 - \frac{1}{2}\pi c^2 + \frac{\pi c^2}{4}[1 + 2\cos^2\phi]\,.$$

For fixed c, the minimum is obtained when $\phi = \frac{\pi}{2}$, and is $\frac{\pi\rho^2}{2} - \frac{1}{4}\pi c^2$. The absolute minimum is obtained when $c=\rho$ and is $\pi\rho^2/4$.

It follows from this result, that

$$\frac{\pi\rho^2}{4}|\nabla u(P_0)|^2 \le 2\alpha D[u] + 2\sigma^2(2\rho)^{2\lambda}\pi\rho^2\,,$$
$$|\nabla u(P_0)|^2 \le \frac{8\alpha\,D[u]}{\pi\rho^2} + 2^{2\lambda+3}\sigma^2\rho^{2\lambda}\,.$$

Consider the function $y = A/\rho^2 + B\rho^{2\lambda}$ where $A = 8\alpha\,D[u]/\pi$, $B = 2^{2\lambda+3}\sigma^2$. The minimum value is

$$y_{\min} = A^{\lambda/(\lambda+1)}B^{1/(\lambda+1)}(\lambda+1)\lambda^{-\lambda/(\lambda+1)}$$
$$= (\sigma^2 D[u]^\lambda)^{1/(\lambda+1)}\left(\frac{\alpha}{\lambda\pi}\right)^{\lambda/(\lambda+1)}(\lambda+1)\,2^{(5\lambda+3)/(\lambda+1)}$$

obtained when

$$\rho = \left(\frac{A}{B\lambda}\right)^{1/(2\lambda+2)} = \left(\frac{\alpha\,D[u]}{\lambda\pi\sigma^2 2^{2\lambda}}\right)^{1/(2\lambda+2)}.$$

If

$$\left(\frac{\alpha D[u]}{\lambda\pi\sigma^2 2^{2\lambda}}\right)^{1/(2\lambda+2)} \le \varepsilon\,.$$

choose

$$\rho = \left(\frac{\alpha\,D[u]}{\lambda\pi\sigma^2 2^{2\lambda}}\right)^{1/(2\lambda+2)}$$

and have

$$|\nabla u(P_0)|^2 \le (\sigma^2\,D[u]^\lambda)^{1/(\lambda+1)}\left(\frac{\alpha}{\pi\lambda}\right)^{\lambda/(\lambda+1)} 2^{(5\lambda+3)/(\lambda+1)}(\lambda+1).$$

However, if

$$\left(\frac{\alpha\,D[u]}{\lambda\pi\sigma^2 2^{2\lambda}}\right)^{1/(2\lambda+2)} > \varepsilon\,,$$

we have

$$\sigma^2 \le \frac{\alpha\,D[u]}{2^{2\lambda}\lambda\pi\varepsilon^{2\lambda+2}}$$

and integrating over S_ε, as in the beginning of this proof, we find that

$$\iint_{S_\varepsilon} |u_r(P_0)|^2 r\,dr\,d\theta \leq 2\iint_{S_\varepsilon} u_r(P)^2 r\,dr\,d\theta + 2\frac{\alpha\,D[u]}{2^{2\lambda}\lambda\pi\varepsilon^{2\lambda+2}}(2\varepsilon)^{2\lambda}\pi\varepsilon^2,$$

$$\frac{\pi\varepsilon^2}{4}|\nabla u(P_0)|^2 \leq 2\alpha D[u] + \frac{2\alpha\,D[u]}{\pi\lambda\varepsilon^2}\cdot\pi\varepsilon^2,$$

$$|\nabla u(P_0)|^2 \leq \frac{8\alpha\,D[u]}{\pi\varepsilon^2} + \frac{8\alpha\,D[u]}{\lambda\pi\varepsilon^2} = \left(1+\frac{1}{\lambda}\right)\frac{8\alpha\,D[u]}{\pi\varepsilon^2}.$$

Thus, in any case,

$$|\nabla u(P_0)|^2 \leq (\sigma^2 D[u]^\lambda)^{1/(\lambda+1)}\left(\frac{\alpha}{\lambda\pi}\right)^{\lambda/(\lambda+1)} 2^{(5\lambda+3)/(\lambda+1)}(\lambda+1) + \frac{8\alpha\,D[u]}{\pi\varepsilon^2}\left(\frac{\lambda+1}{\lambda}\right).$$

LEMMA 7. *Let R be a bounded region with boundary Γ and diameter d and let R have the property that there exists an $\varepsilon>0$ such that every point of $R+\Gamma$ is within some circle of radius ε lying in $R+\Gamma$.*

Let $u(x, y)=G\tau+H$ where τ is a polynomial of degree m, G and H are of class c^1 on $R+\Gamma$ and vanish on Γ, $G>0$ in R, $|\nabla G|\geq\delta>0$ on Γ. Let $|G|\leq G_1$, $|H|\leq H_1$, $|\nabla G|\leq G_2$, $|\nabla H|\leq H_2$ for constants G_1, G_2, H_1, H_2.

Suppose also that

$$|G_x(P)-G_x(P_0)|\leq G_0\,\|P-P_0\|, \quad |G_y(P)-G_y(P_0)|\leq G_0\,\|P-P_0\|,$$

$|H_x(P)-H_x(P_0)|\leq H_0\,\|P-P_0\|$, $|H_y(P)-H_y(P_0)|\leq H_0\,\|P-P_0\|$ *for constants G_0, H_0, whenever P, P_0 are points in $\bar{R}$ such that the line P_0P is in $\bar{R}$. Let A be an upper bound for $D[u]$ and $D[u]\log m$.*

Then there exists a constant B, depending only on α, A, G_0, G_1, G_2, H_0, H_1, H_2, δ, ε, d, G but not on m or τ, such that for $P_0\in\bar{R}$.

$$|u(P_0)|\leq\sqrt{\frac{2\alpha}{\pi}D[u]\log^+\frac{m}{\Delta\,D[u]}} + B(\Delta\,D[u])^4(D[u])^{1/4}$$

for any $\Delta>0$. (m to be replaced by 1 if it is 0).

Proof. We may assume $D[u]>0$ for otherwise $u=0$ in $\bar{R}$.

Let $L=\max_{\bar{R}}|\tau|$. By a theorem of Kellogg [3], $|\nabla\tau(P)|\leq Lm^2/\varepsilon$ for $P\in\bar{R}$.

If P and P_0 are on a straight line in $\bar{R}$, then

$$|\tau(P)-\tau(P_0)| = \left|\int_{P_0}^{P}\frac{\partial\tau}{\partial r}dr\right| \leq \frac{Lm^2}{\varepsilon}\|P-P_0\|.$$

$$|H(P)-H(P_0)|\leq H_2\,\|P-P_0\|, \quad |G(P)-G(P_0)|\leq G_2\,\|P-P_0\|,$$

$$|u(P)-u(P_0)|\leq|G(P)\tau(P)-G(P)\tau(P_0)|+|G(P)\tau(P_0)-G(P_0)\tau(P_0)+|H(P)-H(P_0)|$$

$$\leq\left(G_1\frac{Lm^2}{\varepsilon}+LG_2+H_2\right)\|P-P_0\|=K\|P-P_0\|\,.$$

By Lemma 5, with $\Delta=D[u]^{-1/2}$,

$$(2)\qquad |u(P_0)|\leq\sqrt{\frac{\alpha}{2\pi}D[u]\log^+\frac{dK}{\sqrt{D[u]}}}+\sqrt{D[u]}\,.$$

Also, τ_x, τ_y are polynomials of degree m and absolute value less than or equal to Lm^2/ε, so that $|\nabla\tau_x|\leq(Lm^2/\varepsilon)(m^2/\varepsilon,)$ and

$$|\tau_x(P)-\tau_x(P_0)|\leq\int_{P_0}^{P}|\nabla\tau_x|dr\leq\frac{Lm^4}{\varepsilon^2}\|P-P_0\|\,.$$

Thus

$$|\nabla\tau(P)-\nabla\tau(P_0)|\leq\frac{2Lm^4}{\varepsilon^2}\|P-P_0\|\,.$$

Then

$$\begin{aligned}
&|\nabla u(P)-\nabla u(P_0)|\\
&\quad\leq|G(P)\nabla\tau(P)-G(P_0)\nabla\tau(P_0)|+|\tau(P)\nabla G(P)-\tau(P_0)\nabla G(P_0)|+|\nabla H(P)-\nabla H(P_0)|\\
&\quad\leq|G(P)\nabla\tau(P)-G(P)\nabla\tau(P_0)|+|G(P)\nabla\tau(P_0)-G(P_0)\nabla\tau(P_0)|\\
&\qquad+|\tau(P)\nabla G(P)-\tau(P)\nabla G(P_0)|+|\tau(P)\nabla G(P_0)-\tau(P_0)\nabla G(P_0)|\\
&\qquad+|H_x(P)-H_x(P_0)|+|H_y(P)-H_y(P_0)|\\
&\quad\leq\left(G_1\frac{2\,Lm^4}{\varepsilon^2}+\frac{Lm^2}{\varepsilon}G_2+L\,2G_0+G_2\frac{Lm^2}{\varepsilon}+2H_0\right)\|P-P_0\|=\sigma\|P-P_0\|\,.
\end{aligned}$$

Whence Lemma 6 yields

$$(3)\qquad |\nabla u(P_0)|\leq\sqrt{(\sigma^2D[u])^{1/2}\left(\frac{\alpha}{\pi}\right)^{1/2}32+\frac{16\alpha\,D[u]}{\pi\varepsilon^2}}\,.$$

By use of inequalities (2) and (3) we now find a bound for L.

Either $L\leq1$ or else there exist constants c_1, c_2 such that $K<c_1Lm^2$, $\sigma<c_2Lm^4$ where the factor m is to be omitted if it is zero, and c_1, c_2 depend only on ε, G_1, G_2, H_2, H_0, G_0.

Assume $L>1$. Since $|\nabla G|\neq0$ on Γ, there exists a continuous curve (or curves) γ dividing $\bar{R}$ into two closed sets $\bar{R}_1$ and $\bar{R}_2$, such that $\bar{R}_1\bar{R}_2=\gamma$, $\bar{R}_1$ being a boundary set where $|\nabla G|\geq\delta/2>0$, and $\bar{R}_2$ the set separated from Γ by γ. There is a constant c_3 such that $G(P)\geq c_3>0$ for $P\in\bar{R}_2$.

Suppose first that $|\tau|$ assumes its maximum L at a point $P_0\in\bar{R}_2$. Then, by (2),

$$|G(P_0)\tau(P_0)+H(P_0)|\leq\sqrt{\frac{\alpha}{2\pi}D[u]\log^+\frac{d\,c_1\,Lm^2}{\sqrt{D[u]}}}+\sqrt{D[u]}$$

or,

$$(4) \qquad L \leq \frac{1}{c_3}\left[H_1 + \sqrt{\frac{\alpha}{2\pi} D[u] \log^+ \frac{d\, c_1 L m^2}{\sqrt{D[u]}}} + \sqrt{D[u]}\right].$$

Since $D[u] \log m$ and $D[u]$ are bounded by A, equation (4) implies the existence of a constant c_4 depending on c_3, c_1, d, A, α, H_1 such that $L < c_4$.

On the other hand, if $|\tau|$ assumes its maximum L at a point $P_0 \in \bar{R}_1$, write

$$\nabla u = G\nabla\tau + \tau\nabla G + \nabla H \quad \tau^2\nabla G = \tau\nabla u - u\nabla\tau + H\nabla\tau - \tau\nabla H,$$

$$\tau^2|\nabla G| \leq |\tau|\,|\nabla u| + |u|\,|\nabla\tau| + |H|\,|\nabla\tau| + |\tau|\,|\nabla H|$$

$$\leq L\sqrt{(c_2^2 L^2 m^8 D[u])^{1/2}\left(\frac{\alpha}{\pi}\right)^{1/2} 32 + \frac{16\alpha\, D[u]}{\pi\varepsilon^2}}$$

$$+ \frac{Lm^2}{\varepsilon}\left(\sqrt{\frac{\alpha}{2\pi} D[u] \log^+ \frac{d\, c_1 L m^2}{\sqrt{D[u]}}} + \sqrt{D[u]}\right) + H_1 \frac{Lm^2}{\varepsilon} + LH_2.$$

Therefore,

$$(5) \quad L \leq \frac{2}{\delta}\left\{\sqrt{32\,(c_2^2 L^2 m^8 D[u])^{1/2}\left(\frac{\alpha}{\pi}\right)^{1/2} + \frac{16\alpha\, D[u]}{\pi\varepsilon^2}}\right.$$

$$\left. + \frac{m^2}{\varepsilon}\left(\sqrt{\frac{\alpha\, D[u]}{2\pi} \log^+ \frac{d\, c_1 L m^2}{\sqrt{D[u]}}} + \sqrt{D[u]}\right) + \frac{H_1 m^2}{\varepsilon} + H_2\right\}.$$

This inequality, which is of the form

$$L \leq K_1 + K_2 m^2 + K_3 m^2 \sqrt{\log L} + K_4 m^2 \sqrt{L},$$

shows that

$$\sqrt{L} \leq \frac{K_1}{\sqrt{L}} + \frac{K_2 m^2}{\sqrt{L}} + K_3 m^2 \sqrt{\frac{\log L}{L}} + K_4 m^2 \leq K_1 + K_2 m^2 + K_3 m^2 + K_4 m^2,$$

since $L > 1$, whence $L \leq c_5 m^4$ for some constant c_5.

Thus, in any case, there is a constant c_6 such that $L < c_6 m^4$, where the factor m is to be omitted if it is zero. From this one can conclude that $K < c_1 c_6 m^6$. However, we may obtain a better estimate by noticing that K merely serves as a number such that $|u(P) - u(P_0)| \leq K\|P - P_0\|$ whenever P and P_0 are on a straight line in $\bar{R}$. Hence K may be replaced by $\sup_{\bar{R}} |\nabla u|$.

The inequality $\sigma < c_2 L m^4 < c_2 c_6 m^8$ and formula (3) yield

$$|\nabla u(P_0)| \leq \sqrt{c_2 c_6 m^8 D[u]^{1/2}\left(\frac{\alpha}{\pi}\right)^{1/2} \cdot 32 + \frac{16\alpha D[u]^{1/2} D[u]^{1/2}}{\pi\varepsilon^2}} = c_7 D[u]^{1/4} m^4,$$

since $D[u] \leq A$. Thus we may replace K by $c_7 D[u]^{1/4} m^4$ and substitute in Lemma 5 to obtain

$$|u(P_0)| \leq \sqrt{\frac{\alpha}{2\pi} D[u] \log^+ \frac{dc_7 D[u]^{1/4} m^4}{\Delta D[u]}} + \Delta D[u] \,.$$

Let $\Delta = dc_7 \Delta_1^4 D[u]^{13/4}$ and $B = dc_7$ to obtain the conclusion.

LEMMA 8. *Let R, G, have the properties in Lemma 7 and let $u = G\tau$ where τ is a polynomial of degree m.*

Then $D[u] \geq c_{12} |u(P_0)|^2 / \log m$ where $c_{12} > 0$ is a constant depending only on G_0, G_1, G_2, d, ε, α, δ, G. The factor $\log m$ is to be omitted if $m = 0$ or 1.

Proof. Whether $L \leq 1$ or not, the formulas for K, σ show that $K < c_1 L m^2$, $\sigma < c_2 L m^4$. Moreover, either formula (4) or (5) holds, with $H_1 = 0$, $H_2 = 0$. If (4) holds, we have

$$\frac{L}{\sqrt{D[u]}} \leq \frac{1}{c_3} \left[\sqrt{\frac{\alpha}{2\pi} \log^+ \frac{dc_1 L m^2}{\sqrt{D[u]}}} + 1 \right] .$$

Let $w = L / \sqrt{D[u]}$. The above inequality is then of the form $w \leq K_1 \sqrt{\log wm^2} + K_2$ whence $L / \sqrt{D[u]} \leq c_8 \log m$ for some constant c_8, depending on α, c_3, d, c_1. Here the factor $\log m$ is to be omitted if $m = 0$ or 1. On the other hand, if (5) holds, we have

$$\frac{L}{\sqrt{D[u]}} \leq \frac{2}{\delta} \left\{ \sqrt{c_2 m^4 \frac{L}{\sqrt{D[u]}}} \left(\frac{\alpha}{\pi} \right)^{1/2} \cdot 32 + \sqrt{\frac{16\alpha}{\pi \varepsilon^2}} \right.$$
$$\left. + \frac{m^2}{\varepsilon} \left(\sqrt{\frac{\alpha}{2\pi} \log^+ dc_1 m^2 \frac{L}{\sqrt{D[u]}}} + 1 \right) \right\} ,$$

from which we conclude $L / \sqrt{D[u]} \leq c_9 m^4 \log m$ (m and $\log m$ to be omitted if $m = 0$ or 1).

Thus, in any case, there is a constant c_{10} such that $L / \sqrt{D[u]} \leq c_{10} m^5$. Therefore

$$\frac{K}{\sqrt{D[u]}} \leq \frac{c_1 L m^2}{\sqrt{D[u]}} \leq c_1 c_{10} m^7 .$$

Substituting in equation (2), we have

$$|u(P_0)| \leq \sqrt{\frac{\alpha}{2\pi} D[u] \log^+ dc_1 c_{10} m^7} + \sqrt{D[u]} \leq c_{11} \sqrt{\log m} \sqrt{D[u]} ,$$

m to be omitted if it is 0 or 1.

THEOREM 3. *Let R be a bounded region whose boundary Γ consists of a finite number of simply closed regular arcs of class c^k, $k\geq 3$. Let $G(x, y)$ be a function of class c^k on $R+\Gamma$, vanishing on Γ, positive in R, with $\partial G/\partial\nu\geq\delta>0$ on Γ. Let f_i be the set obtained by orthonormalizing the set $\{Gx^iy^j\}$ using the functional*

$$D[\xi]=\iint_R (a\xi_x^2+b\xi_y^2+c\xi^2)dx\,dy$$

as the square of the norm, where a, b, c are bounded and integrable, $a>0$, $b>0$, $c\geq 0$ on $R+\Gamma$. Let $g(x, y)$ be any function of class c^k on $R+\Gamma$. Let $\psi(x, y)$ be any function of class c^k on $R+\Gamma$ assuming the values of $g(x, y)$ on Γ. Define $b_i=D[\psi-g, f_i]$.

Then

$$\left|\psi-g-\sum_{i=1}^n b_i f_i\right|=O\left(\sqrt{\frac{\sqrt{\theta(n)}}{n^{k-2}}}\log\frac{n}{(\log n)^N}\right),$$

where

$$D\left[\psi-g-\sum_{i=1}^n b_i f_i\right]\leq\frac{\theta(n)}{n^{k-2}},$$

with $\lim_{n\to\infty}\theta(n)=0$, θ depending on $\psi-g$, and where N is any fixed constant >0. Moreover, if $k\geq 10$, then

$$\left|\nabla\psi-\nabla\left(g+\sum_{i=1}^n b_i f_i\right)\right|=O\left(\left[\frac{\theta(n)}{n^{k-10}}\right]^{1/4}\right).$$

Finally, if S is any closed domain in R, $k\geq 7$, then for points P in S,

$$\left|\nabla\psi-\nabla\left(g+\sum_{i=1}^n b_i f_i\right)\right|=O\left(\left[\frac{\theta(n)\log n}{n^{k-6}}\right]^{1/4}\right).$$

Proof. Let $u_n=\psi-g-\sum_{i=1}^n b_i f_i$. Then u_n is of the form $G\tau_n+H$ where the degree m_n of τ_n is less than $\sqrt{2n}-2$ and greater than $\sqrt{2n}-2$. By Theorem 1, $D[u_n]\leq\theta(n)/n^{k-2}$, $k\geq 3$, where $\lim_{n\to\infty}\theta(n)=0$ so that $D[u_n]\log m_n\leq A$ for some constant A independent of n. By Lemma 7,

$$|u_n(P_0)|\leq\sqrt{\frac{2\alpha}{\pi}D[u_n]\log^+\frac{m_n}{\Delta_n D[u_n]}}+B(\Delta_n D[u_n])^4 D[u_n]^{1/4}$$

for any $\Delta_n>0$.

$$|u_n|\leq\sqrt{\frac{2\alpha}{\pi}D[u_n]\log^+\frac{\sqrt{2n}}{\Delta_n D[u_n]}}+B(\Delta_n D[u_n])^4 D[u_n]^{1/4}.$$

There is a constant E, depending on N, such that $1/n < E/(\log n)^N e$, $n \geq 3$. Then

$$D[u_n] \leq \frac{\theta(n)}{n^{k-2}} \leq \frac{\theta(n)E^{k-2}}{e^{k-2}(\log n)^{N(k-2)}} = \frac{\sqrt{2n}}{e^{k-2}\Delta_n} \leq \frac{\sqrt{2n}}{e\Delta_n}$$

if

$$\Delta_n = \frac{\sqrt{2n}(\log n)^{N(k-2)}}{\theta(n)E^{k-2}} .$$

The function $x \log(\sqrt{2n}/\Delta x)$ is monotone increasing for $0 \leq x \leq \sqrt{2n}/e\Delta$ so that we may replace $D[u_n]$ by $\theta(n)/n^{k-2}$ to obtain

$$\begin{aligned}|u_n| &\leq \sqrt{\frac{2\alpha}{\pi}\frac{\theta(n)}{n^{k-2}}\log^+\frac{\theta(n)E^{k-2}}{(\log n)^{N(k-2)}\theta(n)/n^{k-2}}} \\ &\quad + B\left(\frac{\sqrt{2n}(\log n)^{N(k-2)}}{\theta(n)E^{k-2}}\frac{\theta(n)}{n^{k-2}}\right)^4\left(\frac{\theta(n)}{n^{k-2}}\right)^{1/4} \\ &= O\left(\sqrt{\frac{\sqrt{\theta(n)}}{n^{k-2}}\log\frac{n}{(\log n)^N}}\right) .\end{aligned}$$

In the proof of Lemma 7, we saw that $L < c_6 m^4$ and $\sigma < c_2 L M^4 < c_2 c_6 m^8$. Hence by equation (3) of Lemma 7,

$$|\nabla u_n| \leq \sqrt{c_2 c_6 m_n^8 \left(\frac{\theta(n)}{n^{k-2}}\right)^{1/2}\left(\frac{\alpha}{\pi}\right)^{1/2} \cdot 32 + \frac{16\alpha}{\pi\varepsilon^2}\frac{\theta(n)}{n^{k-2}}} .$$

Since $m_n < \sqrt{2n}$, we obtain the statement of the theorem regarding uniform convergence in $\bar{R}$ of $|\nabla u_n|$ for $k \geq 10$.

Next, let S be any closed domain in R. We may suppose the boundary Γ' of S is sufficiently smooth so that a circle of radius ε may be rolled around Γ' while lying in S. Let $L_n' = \sup_S |\tau_n|$ and $P_0^{(n)}$ be the point in S where $L_n' = |\tau_n(P_0^{(n)})|$. As in the proof of Lemma 7,

$$|\nabla u_n(P_0)| \leq \sqrt{(\bar{\sigma}_n^2 D[u_n])^{1/2}\left(\frac{\alpha}{\pi}\right)^{1/2} \cdot 32 + \frac{16\alpha D[u_n]}{\pi\varepsilon^2}} \qquad \text{for } P_0 \in S$$

where

$$\bar{\sigma}_n = G_1 \frac{2L_n' m_n^4}{\varepsilon^2} + \frac{L_n' m_n^2}{\varepsilon} G_2 + L_n' 2G_0 + \frac{G_2 L_n' m_n^2}{\varepsilon} + 2H_0 .$$

Using $G\tau_n$ as the function u of Lemma 8 defined over $\bar{R}$ and remembering that $G\tau_n = -\sum_{i=1}^n b_i f_i$, we obtain

$$|G(P_0^{(n)})L_n'| \leq \sqrt{D\left[\sum_{i=1}^{n} b_i f_i\right] \log m_n} \Big/ c_{12} \; .$$

In S, $|G(P)| \geq c_{13} > 0$. Also

$$D\left[\sum_{i=1}^{n} b_i f_i\right] = \sum_{i=1}^{n} b_i^2 \leq \sum_{i=1}^{\infty} b_i^2 = D[\psi - g] \; .$$

Therefore $L_n' < c_{14}\sqrt{\log n}$, $\bar{\sigma}_n < c_{15} n^2 \sqrt{\log n}$, and

$$|\nabla u_n(P_0)| \leq \sqrt{c_{15} n^2 \sqrt{\log n}\left(\frac{\theta(n)}{n^{k-2}}\right)^{1/2}\left(\frac{\alpha}{\pi}\right)^{1/2} \cdot 32 + \frac{16\alpha\theta(n)}{\pi\varepsilon^2 n^{k-2}}}$$

$$= O\left(\left[\frac{\theta(n) \log n}{n^{k-6}}\right]^{1/4}\right) .$$

THEOREM 4. *Let R, Γ, G, f_i be defined as in Theorem 3. Then there is a constant c_{17} such that whenever $P_0 \in \bar{R}$, then*

$$\sum_{K=1}^{n} f_K^2(P_0) \leq c_{17} \log n \; .$$

The theorem is true if P_0 is a point where $f_1, \cdots, f_n$ all vanish, in particular on Γ. Let P_0 be a point in R where not all f_K, $K=1, \cdots, n$ vanish. Consider the problem of minimizing $D[u]$, where u is of the form $u = \sum_{K=1}^{n} c_K f_K$, under the condition $u(P_0) = T \neq 0$. Now

$$D[u] = D\left[\sum_{K=1}^{n} c_K f_K\right] = \sum_{K=1}^{n} c_K^2 \; ,$$

so that we must minimize $\sum_{K=1}^{n} c_K^2$ under the condition $\sum_{K=1}^{n} c_K f_K(P_0) = T$. By Lagrange multipliers we find a necessary condition for a minimum to be

$$c_K = \bar{c}_K = \frac{T f_K(P_0)}{\sum_{j=1}^{n} f_j^2(P_0)} \; ,$$

and the function $\bar{u} = \sum_{K=1}^{n} \bar{c}_K f_K$ satisfies

$$D[\bar{u}] = T^2 \Big/ \sum_{K=1}^{n} f_K^2(P_0) \; .$$

This is actually a minimum value, for, if $u = \sum_{K=1}^{n} c_K f_K$, then

$$T^2 = \left[\sum_{K=1}^{n} c_K f_K(P_0)\right]^2 \leq \sum_{K=1}^{n} c_K^2 \sum_{K=1}^{n} f_K^2(P_0)$$

so

$$\frac{T^2}{\sum_{K=1}^{n} f_K^2(P_0)} \leq \sum_{K=1}^{n} c_K^2 = D[u] .$$

Now $\bar{u}$ is of the form $G\ \tau_n$ where τ_n has the degree of f_n and this degree is less than $\sqrt{2n-2}$.

By Lemma 8, we have $D[\bar{u}] \geq c_{12} T^2 / \log \sqrt{2n-2}$.
Hence

$$\frac{T^2}{\sum_{K=1}^{n} f_K^2(P_0)} = D[\bar{u}] \geq \frac{c_{12} T^2}{\log \sqrt{2n-2}} ,$$

$$\sum_{K=1}^{n} f_K^2(P_0) \leq \frac{1}{c_{12}} \log \sqrt{2n-2} .$$

4. An Associated Problem. As in the previous sections, let R be a bounded region whose boundary Γ consists of a finite number of simply closed regular arcs of class c^k, $k \geq 3$; $G(x, y)$ be a function of class c^k on $R+\Gamma$, vanishing on Γ, positive in R, with $\partial G/\partial \nu \geq \delta > 0$ on Γ; $g(x, y)$ be any function of class c^k on $R+\Gamma$; a variation be a function of class c^k on $R+\Gamma$ vanishing on Γ.

Let

$$D^p[\xi] = \iint_R (a\xi_x^2 + b\xi_y^2 + c\xi^2)^p dx\, dy ,$$

where $a>0$, $b>0$, $c \geq 0$ on $R+\Gamma$; a, b, and c are bounded and integrable on $\bar{R}$; p is a real number greater than or equal to 1.

Assuming the existence of a function ψ_0, yielding minimum value to $D^p[\psi]$ in the set of admissible functions of class c^k on $R+\Gamma$, which take the value of g on Γ, can we obtain ψ_0 by the Rayleigh Ritz method? This question is answered in the affirmative and an estimate is obtained for the rate of convergence.

Let $\|\xi\| = (D^p[\xi])^{1/2p}$, for ξ in the set of functions of class c^k on $R+\Gamma$. This functional has the properties $\|\xi\| \geq 0$, $\|a\xi\| = |a|\,\|\xi\|$ for real a, $\|\xi+\eta\| \leq \|\xi\| + \|\eta\|$.

The functional $\|\xi\|$ is a true norm in the linear space V of variations. Let H be the Banach space formed by completing V with respect to this norm. As in the proof of Theorem 1, we see that the set of functions $G\tau$, where τ is a polynomial in x and y, is dense in H. Moreover, if ψ is admissible, there exists a sequence of polynomials Q_j of degree at most j such that

$$\|\psi-g-GQ_j\|=D^p[\psi-g-GQ_j]^{1/2p}=O\left(\frac{\theta(j)}{j^{k-2}}\right),$$

where θ depends on $\psi-g$ and $\lim_{j\to\infty}\theta(j)=0$.

There exists $\inf D^p[\psi]\geq 0$ for admissible ψ. Let τ_j be a polynominal of degree at most j which makes $D^p[g+G\tau_j]\leq D^p[g+GQ_j]$ for all polynomials Q_j of degree at most j.

That such a polynomial τ_j exists can be seen as follows. The class of all functions GQ_j where Q_j is a polynominal of degree at most j is also the linear manifold determined by $f_i=GT_i$, the orthonormal sequence of Theorem 1, whose polynomial factor T_i is of degree at most j. As stated in the introduction, $1\leq i\leq j\binom{j+1}{2}+j+1=\sigma$ so that we may write $GQ_j=\sum_{i=1}^{\sigma}c_if_i$. Now let Q'_j be any fixed Q_j. We may restrict ourselves to those Q_j such that $D^p[g+GQ_j]\leq D^p[G+GQ'_j]$. For such Q_j we have

$$\|g\|+\|g+GQ'_j\|\geq\|g\|+\|g+GQ_j\|\geq\|GQ_j\|.$$

Since $D[\xi]\leq D^p[\xi]^{1/p}|R|^{1/q}$ where $(1/p)+(1/q)=1$, $|R|=$area of R, we find that

$$|R|^{1/q}[\|g\|+\|g+GQ'_j\|]^2\geq D[GQ_j]=D\left[\sum_{i=1}^{\sigma}c_if_i\right]=\sum_{i=1}^{\sigma}c_i^2.$$

Thus the permissible c_i lie in a bounded closed set S in σ-dimensional space. Since

$$D^p[g+GQ_j]=D^p\left[g+\sum_{i=1}^{\sigma}c_if_i\right]$$

is a continuous function of c_i in S, it attains its minimum in S.

Since $D^p[g+G\tau_j]$ is a decreasing function of j, we have

$$\lim_{j\to\infty}\|g+G\tau_j\|\leq\liminf_{j\to\infty}\|g+GQ_j\|.$$

Let ψ be admissible and choose Q_j so that $\lim_{j\to\infty}D^p[\psi-g-GQ_j]=0$. Then $\|g+GQ_j\|\leq\|\psi\|+\|\psi-g-GQ_j\|$ implies that $\liminf_{j\to\infty}\|g+GQ_j\|\leq\|\psi\|$. It follows that $\lim_{j\to\infty}\|g+G\tau_j\|\leq\|\psi\|$ for every admissible ψ and thus $g+G\tau_j$ is a minimizing sequence.

If $c>0$ in a set of positive measure in R, the functional $\|\xi\|$ is a true norm in the linear space (c^k) of functions of class c^k on $R+\Gamma$. If $c=0$, a.e. in R, this is still true provided we identify functions differing by a constant. In either case we will complete the space (c^k) to form a Banach space B.

A set S in a normed linear space is uniformly convex if there exists a continuous monotone increasing function $g(\varepsilon)$, $0\leq\varepsilon<1$, with $\lim_{\varepsilon\to\infty} g(\varepsilon)=0$, such that whenever ξ, η are in S and $\|\xi\|=\|\eta\|=1$, $\|(\xi+\eta)/2\|\geq 1-\varepsilon$, then $\|\xi-\eta\|\leq g(\varepsilon)$.

We shall show (c^k) is uniformly convex. It is easily verified that if α, β are ≥ 0, and $p\geq 1$ then

$$3\alpha^p+\beta^p\leq 2|\alpha+\beta|^p+|\alpha-\beta|^p.$$

Apply the inequality to the integrand below, where we assume ϕ and ψ are in (c^k), $\|\phi\|=\|\psi\|=1$, $\|(\phi+\psi)/2\|\geq 1-\varepsilon$.

$$\iint 3\left[a\left(\frac{\phi+\psi}{2}\right)_x^2+b\left(\frac{\phi+\psi}{2}\right)_y^2+c\left(\frac{\phi+\psi}{2}\right)^2\right]^p$$

$$+\left[a\left(\frac{\phi-\psi}{2}\right)_x^2+b\left(\frac{\phi-\psi}{2}\right)_y^2+c\left(\frac{\phi-\psi}{2}\right)^2\right]^p dx\,dy$$

$$\leq\frac{2}{2^p}\iint[(a\phi_x^2+b\phi_y^2+c\phi^2)+(a\psi_x^2+b\psi_y^2+c\psi^2)]^p dx\,dy$$

$$+\iint|a\phi_x\psi_x+b_y\phi_y\psi_y+c\phi\psi|^p dx\,dy$$

$$\leq\frac{2}{2^p}\left\{\left(\iint(a\phi_x^2+b\phi_y^2+c\phi^2)^p dx dy\right)^{1/p}+\left(\iint_R(a\psi_x^2+b\psi_y^2+\psi c^2)^p\,dx\,dy\right)^{1/p}\right\}^p$$

$$+\iint(\sqrt{a\phi_x^2+b\phi_y^2+c\phi^2}\sqrt{a\psi_x^2+b\psi_y^2+c\psi^2}\,)^p dx\,dy$$

$$\leq 2+\sqrt{\iint(a\phi_x^2+b\phi_y^2+c\phi^2)^p dx\,dy}\sqrt{\iint(a\psi_x^2+b\psi_y^2+c\psi^2)^p\,dx\,dy}=3$$

Hence

$$\iint\left(a\left(\frac{\phi-\psi}{2}\right)_x^2+b\left(\frac{\phi-\psi}{2}\right)_y^2+c\left(\frac{\phi-\psi}{2}\right)^2\right)^p dx\,dy\leq 3-3(1-\varepsilon)^{2p},$$

and

$$\|\phi-\psi\|=\left[\iint\left(a(\phi-\psi)_x^2+b(\phi-\psi)_y+c(\phi-\psi)^2\right)^p dx\,dy\right]^{1/2p}$$
$$\leq 2(3[1-(1-\varepsilon)^{2p}])^{1/2p}\leq 2(3^{1/2p})(2p\varepsilon)^{1/2p}=g(\varepsilon)$$

for $\varepsilon<1$, since the function $y=[1-(1-x)^{2p}]-2px$ vanishes at 0 and is a decreasing function of x for $0\leq x<1$.

LEMMA 9. *Let B be a Banach space, Y a set in B with the property that if y_1, y_2 are in Y, then so is $(y_1+y_2)/2$. Let the linear manifold spanned by Y be a uniformly convex set in B. Let*

$$\rho = \inf_{y \in Y} \| y \| > 0,$$

let y_n be a sequence in Y with

$$\lim_{n\to\infty} \| y_n \| = \rho, \quad \rho_n = \| y_n \| .$$

Then there exists a unique x in B such that $\| x \| = \rho$ and we have

$$\| x - y_n \| \leq \rho g\left(\frac{\rho_n - \rho}{2\rho}\right) + \rho_n - \rho ,$$

where $g(\varepsilon)$ is the function in the definition of uniform convexity. If $\rho = \inf_{y \in Y} \| y \| = 0$, and $\lim_{n\to\infty} \| y_n \| = \rho$, then there is a unique x in B such that $\| x \| = \rho$, and we have $\| x - y_n \| = \rho_n - \rho$.

Proof. Let $z_n = y_n / \rho_n$ so that $\| z_n \| = 1$. Write

$$\frac{z_n + z_m}{2} - \frac{1}{\rho}\left(\frac{y_n + y_m}{2}\right) = \frac{y_n}{2}\left(\frac{1}{\rho_n} - \frac{1}{\rho}\right) + \frac{y_m}{2}\left(\frac{1}{\rho_m} - \frac{1}{\rho}\right).$$

$$\left\| \frac{z_n + z_m}{2} \right\| \geq \frac{1}{\rho}\left\| \frac{y_n + y_m}{2} \right\| - \frac{\|y_n\|}{2}\left(\frac{1}{\rho} - \frac{1}{\rho_n}\right) - \frac{\|y_m\|}{2}\left(\frac{1}{\rho} - \frac{1}{\rho_m}\right)$$

$$\geq \frac{1}{\rho} \cdot \rho - \frac{\rho_n}{2}\left(\frac{1}{\rho} - \frac{1}{\rho_n}\right) - \frac{\rho_m}{2}\left(\frac{1}{\rho} - \frac{1}{\rho_m}\right) = 1 - \frac{(\rho_n - \rho) + (\rho_m - \rho)}{2\rho} .$$

Hence

$$\| z_n - z_m \| \leq g\left(\frac{(\rho_n - \rho) + (\rho_m - \rho)}{2\rho}\right) \quad \text{for} \quad \frac{\rho_n - \rho + \rho_m - \rho}{2\rho} < 1 .$$

Thus there exists $z = \lim_{n\to\infty} z_n$ in B. Let $x = \rho z = \lim_{n\to\infty} \rho z_n = \lim_{n\to\infty} \rho_n z_n = \lim_{n\to\infty} y_n$. Then $\| x \| = \lim_{n\to\infty} \| y_n \| = \rho$. Also $\| z_n - z \| \leq g((\rho_n - \rho)/2\rho)$ implies

$$\| x - y_n \| = \| \rho z - \rho_n z_n \| \leq \| \rho z - \rho z_n \| + \| \rho z_n - \rho_n z_n \| \leq \rho g\left(\frac{\rho_n - \rho}{2\rho}\right) + \rho_n - \rho .$$

To show x is unique, suppose also $y'_n \in Y$, $\lim_{n\to\infty} \| y'_n \| = \rho$, $x' \in B$, $\| x' \| = \rho$, $x' = \lim_{n\to\infty} y'_n$. Then form the sequence $\{y''_n\} = y_1, y'_1, y_2, y'_2$, etc. of elements of Y with $\lim_{n\to\infty} y''_n = \rho$. As above, $\exists x'' \in B$ with $x'' = \lim_{n\to\infty} y''_n = \lim_{n\to\infty} y'_n = \lim_{n\to\infty} y_n$. The last part of the lemma is obvious, since only $\| 0 \| = 0$.

To apply the lemma, let B be the completion of (c^k), Y the set of admissible functions,

$$y_n = g + G\tau_n, \quad \rho = \inf D^p[\psi]^{1/2p} ,$$

for admissible ψ. By the lemma, there is a unique x such that $\| x \| =$

ρ. Assuming that $x=\psi_0$ is in Y, we can choose polynomials Q_j of degree at most j such that

$$\|\psi_0-g-GQ_j\|=O\left(\frac{\theta(j)}{j^{k-2}}\right).$$

Then

$$\begin{aligned}\rho_j-\rho&=\|g+G\tau_j\|-\|\psi_0\|\leq\|g+GQ_j\|-\|\psi_0\|\\&=\|g+GQ_j-\psi_0+\psi_0\|-\|\psi_0\|\leq\|\psi_0-g-GQ_j\|=O\left(\frac{\theta(j)}{j^{k-2}}\right).\end{aligned}$$

By the lemma,

$$\|\psi_0-g-G\tau_j\|\leq 2\rho(6p)^{1/2p}\left(\frac{\rho_j-\rho}{2\rho}\right)^{1/2p}+\rho_j-\rho=O\left(\frac{\theta(j)}{j^{k-2}}\right)^{1/2p},$$

a better result, $O(\theta(j)/j^{k-2})$, is obtained in the case $\rho=0$.

Since

$$D[u]<(D^p[u])^{1/p}|R|^{1/q},$$

where $|R|$ is the area of R and $(1/p)+(1/q)=1$, we find

$$D[u_j]\leq\left(\frac{\theta(j)}{j^{k-2}}\right)^{1/p},$$

where $\lim_{j\to\infty}\theta(j)=0$, when we take $u_j=\psi_0-g-G\tau_j$. A proof similar to that of Theorem 3 can now be constructed for the following result.

THEOREM 5. *Let R be a bounded region whose boundary Γ consists of a finite number of simply closed regular arcs of class c^k, $k\geq 3$. Let $G(x, y)$ be a function of class c^k on $R+\Gamma$, vanishing on Γ, positive in R,* with *$\partial G/\partial\nu\geq\delta>0$ on Γ. Let a, b, c be bounded and integrable on $\bar{R}$, and $a>0$, $b>0$, $c\geq 0$ on $\bar{R}$. Let $g(x, y)$ be any function of class c^k on $R+\Gamma$. Choose polynomials τ_j minimizing $D^p[g+GQ_j]$ in the set of all polynomials Q_j of degree at most j. Then, if ψ_0 yields minimum value to $D^p[\psi]$ for ψ in the set of functions of class c^k on $R+\Gamma$ assuming the values of g on Γ, we have*

$$|\psi_0-g-G\tau_j|=O\left(\sqrt{\left(\frac{\sqrt{\theta(j)}}{j^{k-2}}\right)^{1/p}\log\frac{j}{(\log j)^N}}\right),$$

where N is any fixed positive constant, $\theta(j)$ depends on ψ_0-g and $\lim_{j\to\infty}\theta(j)=0$.

If $k \geq 16p+2$, *then*

$$|\nabla\psi_0 - \nabla(g+G\tau_j)| = O\left(\left[\frac{\theta(j)}{j^{k-2-16p}}\right]^{1/4p}\right).$$

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