ON A PROBLEM IN THE THEORY OF MECHANICAL QUADRATURES

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1. Introduction. In the present note we study a scheme of mechanical quadratures of the form

$$\int_{-1}^{+1} f(x)dx \sim \sum_{k=0}^{n} a_{nk} f(\lambda_{nk}) = Q_n(f) \, , \tag{1}$$

as applied to certain distinguished classes of analytic functions on $[-1, +1]$. The question of the convergence of $Q_n(f)$ to the integral in (1) has been solved completely by Pólya [4] when f is selected from the class of continuous functions. There seems to be less discussion of the problem when f is selected from the class of analytic functions on $[-1, +1]$ or from certain of its subclasses.

Let B designate a region in the complex $z=x+iy$ plane which we shall assume contains $[-1, +1]$ in its interior. By $L^2(B)$ we designate the class of functions which are analytic and single valued in B and are such that

$$\iint_B |f|^2 dxdy < \infty \, . \tag{2}$$

With

$$(f, g) = \iint_B f\bar{g} dxdy \tag{3}$$

as an inner product, and $\|f\|^2 = (f, f)$ as a norm, the space $L^2(B)$ becomes a well-known and very useful Hilbert space of functions, possessing a reproducing kernel $K_B(z, \bar{w})$ which is generally referred to as the Bergman kernel for B [1].

Let E be a bounded linear functional over $L^2(B)$; its norm (over the conjugate space of all linear functionals) may be obtained in the following way. Let $\varphi_n(z)$ $(n=0, 1, \cdots)$ be a complete orthonormal system for $L^2(B)$. Then it may be shown that

$$\|E\|^2 = \sum_{n=0}^{\infty} |E(\varphi_n)|^2 \, . \tag{4}$$

This may be expressed in the alternate but equivalent form,

$$\|E\|^2 = E_z E_{\bar{w}} K_B(z, \bar{w}) \, , \tag{5}$$

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where the subscripts on the E mean that the functional operation is to be carried out on the variable indicated. We have, then, for all $f \in L^2(B)$,

$$|E(f)| \leq ||E|| \, ||f|| , \tag{6}$$

with the equality sign being attained for some $f \in L^2(B)$. If now, the abscissas λ_{nk} lie in the interior of B, and the segment $[-1, +1]$ lies in the interior of B, then the linear functional

$$E_n(f) = \int_{-1}^{+1} f(x)dx - \sum_{k=0}^{n} a_{nk} f(\lambda_{nk}) \tag{7}$$

is bounded (cf. [2]) over $L^2(B)$, so that we have, for all $f \in L^2(B)$,

$$|E_n(f)| \leq ||E_n|| \, ||f|| . \tag{8}$$

2. Uniform convergence. We shall say that the quadrature scheme (1) converges *uniformly in* $L^2(B)$ if, having been given an $\varepsilon > 0$, there is an $n_0 = n_0(\varepsilon)$ such that, for all $f \in L^2(B)$ and $n \geq n_0$, we have

$$\left| \int_{-1}^{+1} f(x)dx - \sum_{k=0}^{n} a_{nk} f(\lambda_{nk}) \right| \leq \varepsilon \, ||f|| . \tag{9}$$

THEOREM 1. *A necessary and sufficient condition that the quadrature scheme* (1) *converges uniformly in* $L^2(B)$ *is that*

$$\lim_{n\to\infty} ||E_n||^2 = \lim_{n\to\infty} E_{nz} E_{n\bar{w}} K_B(z, \bar{w}) = 0 . \tag{10}$$

Proof. Suppose that (10) holds. Then given an $\varepsilon > 0$ we can find an $n_0(\varepsilon)$ such that $||E_n|| \leq \varepsilon$ for all $n \geq n_0(\varepsilon)$. Hence, by (6), the inequality (9) must hold. Conversely, suppose that (9) holds. For each n, it is possible to find a nontrivial function $f_n(z) \in L^2(B)$ such that

$$|E_n(f_n)| = ||E_n|| \, ||f_n|| . \tag{11}$$

By (9), given an $\varepsilon > 0$ we may find an $n = n_0(\varepsilon)$ such that for all $n \geq n_0(\varepsilon)$ and for all $f \in L^2(B)$ we have $|E_n(f)| \leq \varepsilon \, ||f||$. Hence, in particular, for the f_n of (11),

$$||E_n|| \, ||f_n|| = |E_n(f_n)| \leq \varepsilon \, ||f_n|| . \tag{12}$$

Therefore (10) must follow.

We note that, in view of (4), the condition (10) can, in principle, be converted into a necessary and sufficient condition on the weights a_{nk} and abscissas λ_{nk}.

The following special case is of considerable interest. Let $\mathscr{E}_\rho$, $\rho>1$, designate an ellipse with foci at $(-1, 0)$ and $(1, 0)$ and with semimajor and semiminor axes a and b respectively, and where ρ is given by

$$\rho=(a+b)^2, \quad a=(\rho+1)/2\rho^{1/2}, \quad b=(\rho-1)/2\rho^{1/2}. \tag{13}$$

Observe that as $\rho\to1$, $\mathscr{E}_\rho$ collapses to $[-1, +1]$. If $U_n(z)$ $(n=0, 1, \cdots)$ designates the Tschebysheff polynomials of the second kind defined by

$$U_n(z)=(1-z^2)^{-1/2}\sin((n+1)\operatorname{arc}\cos z), \tag{14}$$

then it is well known that the system of polynomials

$$\varphi_n(z)=2\sqrt{\frac{n+1}{\pi}}(\rho^{n+1}-\rho^{-n-1})^{-1/2}U_n(z) \quad (n=0, 1, 2, \cdots) \tag{15}$$

will be complete and orthonormal over $L^2(\mathscr{E}_\rho)$. Thus we have:

THEOREM 2. *A necessary and sufficient condition in order that the quadrature scheme* (1) *converge uniformly in* $L^2(\mathscr{E}_\rho)$ *is that*

$$\operatorname*{limit}_{n\to\infty}\frac{4}{\pi}\sum_{k=0}^{\infty}(k+1)\frac{|E_n(U_k)|^2}{\rho^{k+1}-\rho^{-k-1}}=0. \tag{16}$$

3. **Interpolatory quadrature.** An important class of quadrature schemes is formed by those which are of interpolatory type. For such quadratures we have

$$Q_n(f)=\int_{-1}^{+1}f(x)dx, \tag{17}$$

whenever f is a polynomial of degree not larger than n. If the scheme is of interpolatory type then (16) becomes

$$\lim_{n\to\infty}\frac{4}{\pi}\sum_{k=n+1}^{\infty}(k+1)\frac{|E_n(U_k)|^2}{\rho^{k+1}-\rho^{-k-1}}=0. \tag{18}$$

In view of the inequalities

$$\rho^{-1}\cdot\rho^{-k}\leq(\rho^{k+1}-\rho^{-k-1})^{-1}\leq(\rho-\rho^{-1})^{-1}\rho^{-k}, \qquad (\rho>1), \tag{19}$$

condition (18) is equivalent to

$$\lim_{n\to\infty}\sum_{k=n+1}^{\infty}(k+1)\frac{|E_n(U_k)|^2}{\rho^k}=0. \tag{20}$$

If we now define

$$\sigma_k = \begin{cases} 0 & (k \text{ odd}) , \\ \dfrac{2}{k+1} & (k \text{ even}) , \end{cases} \tag{21}$$

then (20) becomes

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} (k+1)\Big(\sigma_k - \sum_{j=0}^{n} a_{nj} U_k(\lambda_{nj})\Big)^2 \rho^{-k} = 0 . \tag{22}$$

The following sufficient condition for the uniform convergence in $L^2(\mathscr{E}_\rho)$ of an interpolatory quadrature scheme can now be obtained. Set

$$M_n = \sum_{j=0}^{n} |a_{nj}| , \tag{23}$$

and observe that for real absissas λ in $[-1, +1]$ we have

$$|U_k(\lambda)| \leq k+1 . \tag{24}$$

Then, using (21) and (23), for fixed $\rho > 1$ we get

$$\begin{aligned} \sum_{k=n+1}^{\infty} (k+1)\Big(\sigma_k - \sum_{j=0}^{\infty} a_{nj} U_k(\lambda_{nj})\Big)^2 \rho^{-k} &\leq \sum_{k=n+1}^{\infty} (k+1)(\sigma_k + (k+1)M_n)^2 \rho^{-k} \\ &\leq 4 \sum_{k=n+1}^{\infty} ((k+1)\rho^k)^{-1} + 4M_n \sum_{k=n+1}^{\infty} (k+1)\rho^{-k} + M_n^2 \sum_{k=n+1}^{\infty} (k+1)^3 \rho^{-k} \\ &< o(1) + C_1 M_n n \rho^{-n} + C_2 M_n^2 n^3 \rho^{-n}, \qquad (n \to \infty) , \end{aligned} \tag{25}$$

where C_1 and C_2 are two positive constants which may depend upon ρ but are independent of n. Thus, we have the following result.

THEOREM 3. *Let*

$$\lim_{n\to\infty} M_n n^{3/2} \rho^{-n/2} = 0 . \tag{26}$$

Then an interpolatory quadrature scheme converges uniformly in $L^2(\mathscr{E}_\rho)$

Pólya [4, p. 285] has remarked that if

$$\lim_{n\to\infty} (M_n)^{1/n} = 1 \tag{27}$$

then an interpolatory quadrature scheme converges for all functions which are analytic in the closed basic interval. Under hypothesis (27), we have

$$M_n = (1+\varepsilon_n)^n , \quad \varepsilon_n \to 0,$$

so that (26) holds with all $\rho > 1$. Thus, under Pólya's hypothesis, we see that the convergence is also uniform in every $L^2(\mathscr{E}_\rho)$, $\rho > 1$.

4. Newton-Cotes quadrature. We turn now to a specific quadrature scheme on $[-1, +1]$, namely, the Newton-Cotes scheme. In this scheme, we have

$$Q_n(f)=a_{n0}f(-1)+a_{n1}f(-1+2/n)+a_{n2}f(-1+4/n)+\cdots+a_{nn}f(1) \tag{28}$$
$$(n=1, 2, \cdots),$$

where the Cotes numbers a_{nk} have been determined so that

$$Q_n(f)=\int_{-1}^{+1} f\,dx$$

holds for an arbitrary polynomial of degree $\leq n$. We have now the following estimate due to J. Ouspensky [3] (Ouspensky's basic interval is $[0, 1]$):

$$a_{nk}=-\frac{2}{n(\log n)^2}\binom{n}{k}\left[\frac{(-1)^k}{k}+\frac{(-1)^{n-k}}{n-k}\right](1+\eta_{nk})\ , \tag{29}$$

where $\eta_{n,k}\to 0$ as $n\to\infty$ uniformly for $k=1, 2, \cdots, n-1$, while

$$a_{n0}=a_{nn}=\frac{2}{n\log n}(1+\varepsilon_n)\ , \qquad \varepsilon_n\to 0\ . \tag{30}$$

Thus,

$$M_n=\sum_{j=0}^{\infty}|a_{nj}|\leq\frac{4(1+\delta_n)}{n(\log n)^2}\sum_{k=1}^{n-1}\binom{n}{k}+\frac{4}{n\log n}(1+\varepsilon_n)\ , \tag{31}$$

where we have written $\eta_{nk}<\delta_n$ $(k=1, 2, \cdots, n-1)$, $\delta_n\to 0$. Hence,

$$M_n\leq\frac{4(1+\delta_n)2^n}{n(\log n)^2}+\frac{4}{n\log n}(1+\varepsilon_n)\ . \tag{32}$$

Condition (26) now holds with $\rho^{1/2}>2$. We have therefore arrived at the following result:

THEOREM 4. *The Newton-Cotes quadrature scheme converges uniformly in $L^2(\mathscr{E}_\rho)$ whenever $\rho>4$.*

Investigation of the convergence of the Newton-Cotes quadrature scheme has an interesting history which is worth retelling here. T. Stieltjes in 1884 first proved the convergence of the Gauss mechanical quadrature for the class of Riemann integrable functions, and in a letter to Hermite raised the question of the convergence of the Newton-Cotes scheme. In 1925 J. Ouspensky [3] arrived at the asymptotic result (29), and from the growth of Cotes numbers concluded only that the Newton-Cotes scheme is devoid of practical value. In 1933

G. Pólya [4] showed that this scheme is not valid for all continuous functions, and, indeed, is not valid for the class of analytic functions. Pólya's counterexample, referred to the interval $[-1, +1]$ is

$$f(w) = -\sum_{k=4}^{\infty} a^{k!} \frac{\sin k!\{(w+1)/2\}}{\cos \pi\{(w+1)/2\}} \qquad (1/2 < a < 1), \tag{33}$$

for which the Newton-Cotes scheme diverges. The functions $f(w)$ is regular in the strip

$$|\mathscr{I}(w)| < \frac{-2 \log a}{\pi} \tag{34}$$

and has a natural boundary along the sides of the strip. The widest such strip must be less than

$$|\mathscr{I}(w)| < \frac{2 \log 2}{\pi} = 0.4412 .$$

The function (33) cannot, therefore, be continued analytically to $\mathscr{E}_\rho = 4$, for which the semiminor axis is $b = .7500$. Theorem 4, therefore, rehabilitates the Newton-Cotes quadrature scheme for functions which are regular over a sufficiently large portion of the complex plane.

References

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