ON THE L^p THEORY OF HANKEL TRANSFORMS

G. M. WING

1. Introduction. Under suitable restrictions on $f(x)$ and ν, the Hankel transform $g(t)$ of $f(x)$ is defined by the relation

(1) $$g(t) = \int_0^\infty (xt)^{1/2} J_\nu(xt) f(x)\, dx .$$

The inverse is then given formally by

(2) $$f(x) = \int_0^\infty (xt)^{1/2} J_\nu(xt) g(t)\, dt .$$

These integrals represent generalizations of the Fourier sine and cosine transforms to which they reduce when $\nu = \pm 1/2$. The L^p theory for the Fourier case has been studied in considerable detail. In this note we present some results concerning the inversion formula (2) in the L^p_x case.

It is clear that if $f(x) \in L$ and $\Re(\nu) \geq -1/2$ then the integral in (1) exists. It has been shown [3,6] that if $f(x) \in L^p$, $1 < p \leq 2$, then

(3) $$g_a(t) = \int_0^a (xt)^{1/2} J_\nu(xt) f(x)\, dx$$

converges strongly to a function $g(t)$ in $L^{p'}$. For this case Kober has obtained the inversion formula,

$$f(x) = x^{-1/2-\nu} \frac{d}{dx} \left\{ x^{\nu+1/2} \int_0^\infty \frac{(xt)^{1/2} J_{\nu+1}(xt)}{t}\, g(t)\, dt \right\} ,$$

which holds for almost all x. In her investigation of Watson transforms, Busbridge [1] has given analogous results for more general kernels. Except when p = 2 the question of the strong convergence of the inversion integral has apparently been considered only in the Fourier case [2]. We now investigate this problem

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for the Hankel transforms. We assume throughout that $\Re(\nu) \geq -1/2$.

2. Theorem. We shall establish the following result.

THEOREM 1. *Let $f(x) \in L^p$, $1 < p \leq 2$, and let $g(t)$ be the limit in mean of $g_a(t)$, $g(t) =$ l.i.m. $g_a(t)$, where $g_a(t)$ is defined by* (3). *If*

$$f_a(x) = \int_0^a (xt)^{1/2} J_\nu(xt) g(t)\, dt,$$

then

$$f_a(x) \in L^p \quad and \quad f(x) = \text{l.i.m. } f_a(x).$$

Proof. Write

$$f_a(x, b) = \int_0^a (xt)^{1/2} J_\nu(xt) g_b(t)\, dt$$

$$= \int_0^b (xu)^{1/2} f(u)\, du \int_0^a J_\nu(ut) J_\nu(xt) t\, dt.$$

Since $g_b(t)$ converges in the mean to $g(t)$ it follows that $\lim_{b\to\infty} f_a(x, b) = f_a(x)$. Hence

$$f_a(x) = \int_0^\infty (xu)^{1/2} K(x, u, a) f(u)\, du, \tag{4}$$

where [9]

$$K(x, u, a) = \int_0^a J_\nu(ut) J_\nu(xt) t\, dt \tag{5}$$

$$= a\{u J_{\nu+1}(ua) J_\nu(xa) - x J_{\nu+1}(xa) J_\nu(ua)\}/(u^2 - x^2).$$

An integral very similar to (4) has been studied in a previous paper [10]. The same methods may be used here to show that $\| f_a(x) \|_p < M_p \| f(x) \|_p$. Our theorem will now follow in the usual way if we can prove it for step functions which vanish outside a finite interval. Let $\phi(x)$ be a step function, $\phi(x) = 0$ for $x > A$, and let $\phi_a(x)$ correspond to it as in (4). Choose $\xi > 2A$, $a > A$, to get

$$\int_\xi^\infty |\phi_a(x) - \phi(x)|^p\, dx = \int_\xi^\infty dx \left| \int_0^A \phi(u) (xu)^{1/2} K(x, u, a)\, du \right|^p.$$

From the relations

$$(6)\qquad x^{1/2} J_\nu(x) = (2/\pi)^{1/2} \{\cos(x+\delta_\nu) + x^{-1}A_\nu \sin(x+\delta_\nu)\} + O(x^{-2}) \qquad (x \longrightarrow \infty),$$

where

$$A_\nu = (1-4\nu^2)/8, \qquad \delta_\nu = -(2\nu+1)\pi/4,$$

and

$$(7)\qquad J_\nu(x) = O(x^{\nu_1}) \qquad (x \longrightarrow 0),$$

where $\nu_1 = \Re(\nu)$, it is easy to see that

$$(xu)^{1/2} |K(x,u,a)| < M/|u-x| ,$$

so that we have

$$\int_\xi^\infty |\phi_a(x) - \phi(x)|^p\, dx < M \int_\xi^\infty \frac{dx}{|x-A|^p} \int_0^A |\phi(u)|^p\, du < \epsilon$$

for ξ sufficiently large. Now

$$\|\phi_a(x) - \phi(x)\|_p^p = \int_0^\xi + \int_\xi^\infty |\phi_a(x) - \phi(x)|^p\, dx$$

$$\leq M \left\{ \int_0^\xi |\phi_a(x) - \phi(x)|^2\, dx \right\}^{p/2} + \epsilon .$$

As $a \longrightarrow \infty$ the integral goes to zero by the L^2 theory for Hankel transforms (see [7, Chapter 8]). This completes the proof.

3. The case $p = 1$. Theorem 1 fails to hold in the case $p = 1$. The proof, similar to that given by Hille and Tamarkin in the Fourier case [2] , will only be sketched.

THEOREM 2. *There exists a function $h(t)$, the Hankel transform of a function $\psi(x) \in L$, such that if*

$$(8)\qquad \psi_a(x) = \int_0^a (xt)^{1/2} J_\nu(xt) h(t)\, dt$$

then l.i.m. $\psi_a(x)$ *fails to exist.*

Proof. Let $h(t) = t^{1/2} J_\nu(t)/\log(t+2)$. Two integrations of (8) by parts and use of formulas (5), (6), and (7) yield

$$\psi_a(x) = \frac{ax^{3/2} J_\nu(a) J_{\nu+1}(ax)}{(x^2-1)\log(a+2)} + O(x^{-2}) \tag{9}$$

for large x.

Now define $\psi(x) = \lim_{a\to\infty} \psi_a(x)$. It is evident from (8) that $\psi(x)$ is continuous except perhaps at $x = 1$, while (9) shows that $\psi(x) = O(x^{-2})$. To show that $\psi(x) \in L$ it suffices to consider the neighborhood of $x = 1$. Formula (6) yields, after some calculation,

$$\psi(x) = \int_0^\infty \frac{\cos(1-x)t}{\log(t+2)}\,dt + \alpha(x),$$

where $\alpha(x)$ is continuous near $x = 1$. Thus

$$\int_{1+\epsilon}^2 \{\psi(x) - \alpha(x)\}\,dx = -\int_0^\infty \frac{\sin t}{t\log(2+t/\epsilon)}\,dt + \int_0^\infty \frac{\sin t}{t\log(2+t)}\,dt.$$

The first integral on the right tends to zero as $\epsilon \to 0^+$. Since $\psi(x) - \alpha(x)$ is positive (see [2]) it follows that $\psi(x) - \alpha(x)$ is integrable over $(1,2)$ [8, p. 342]. The interval $(0,1)$ may be handled similarly. Hence $\psi(x) \in L$.

That $h(t)$ is indeed the Hankel transform of $\psi(x)$ is a consequence of a result of P. M. Owen [5, p. 310]. But it may be seen from (9) that $\psi_a(x)$ is not in L, so that l.i.m. $\psi_a(x)$ surely fails to exist.

4. A summability method. It is natural to try to include the case $p = 1$ into the theory by introducing a suitable summability method. Our interest will be confined to the Cesàro method. If $f(x) \in L$ and $g(t)$ is its Hankel transform then we shall define

$$f_a(x) = \int_0^a (1 - t/a)^k (xt)^{1/2} J_\nu(xt) g(t)\,dt \tag{10}$$

$$= \int_0^\infty f(y) C_k(x,y,a)\,dy,$$

where

$$C_k(x, y, a) = \int_0^a (xy)^{1/2} u J_\nu(xu) J_\nu(yu)(1 - u/a)^k \, du. \tag{11}$$

Offord [4] has studied the local convergence properties of $f_a(x)$ for $k = 1$. We are able to extend his results to the case $k > 0$, but the estimates required are too long and tedious for presentation here. Instead we investigate the strong convergence.

THEOREM 3. *Let $f(x) \in L$, $k > 0$. If $f_a(x)$ is defined by* (10), *then $f_a(x)$ converges strongly to $f(x)$.*

Proof. We shall first prove that $C_k(x, y, a) \in L$ and $\|C_k(x, y, a)\| < M$, where the norm is taken with respect to x and the bound M is independent of y and a. An integration by parts and a change of variable in (11) give

$$C_k(x, y, a) = -\frac{ka}{2} \int_0^1 (1 - s)^{k-1} s (xy)^{1/2} Q \, ds \tag{12}$$

where

$$Q = \frac{J_{\nu+1}(ays)J_\nu(axs) - J_\nu(ays)J_{\nu+1}(axs)}{y - x} + \frac{J_{\nu+1}(ays)J_\nu(axs) + J_\nu(ays)J_{\nu+1}(axs)}{y + x}.$$

Consider

$$I = \int_{|y-x|>1/a} \frac{dx}{|y-x|} \left| \int_0^1 (1-s)^{k-1} (ays)^{1/2} J_{\nu+1}(ays)(axs)^{1/2} J_\nu(axs)\, ds \right|$$

$$= \int_{|ay-z|>1} \frac{dz}{|ay - z|} \left| \int_0^\infty G(a, y, s)(zs)^{1/2} J_\nu(zs)\, ds \right|,$$

where

$$G(a, y, s) = \begin{cases} (1 - s)^{k-1}(ays)^{1/2} J_{\nu+1}(ays) & (0 \le s < 1), \\ 0 & (s \ge 1). \end{cases}$$

Now, as a function of s, $G(a, y, s) \in L^p$ for some $p > 1$ so that

$$F(a, y, z) = \int_0^\infty G(a, y, s)(sz)^{1/2} J_\nu(sz)\, ds$$

is in $L^{p'}$ as a function of z [3]. Also

$$\left\{\int_0^\infty |F(a, y, z)|^{p'} dz\right\}^{1/p'} \leq A_p \left\{\int_0^\infty |G(a, y, s)|^p ds\right\}^{1/p} < M,$$

where M is a constant independent of a and y. Thus

$$I \leq \left\{\int_{|ay-z|>1} \frac{dz}{|ay - z|^p}\right\}^{1/p} \left\{\int_0^\infty |F(a, y, z)|^{p'} dz\right\}^{1/p'} < M.$$

The other parts of (12) may be cared for similarly, so that we have

$$\int_{|y-x|>1/a} |C_k(x, y, a)|\, dx < M.$$

The range $|y - x| \leq 1/a$ is easily handled since, by (11), for this range we have $|C_k(x,y,a)| < Ma$. Hence $\|C_k(x,y,a)\| < M$. We see at once from (10) that

$$\int_0^\infty |f_a(x)|\, dx = \int_0^\infty dx \left|\int_0^\infty f(y)\, C_k(x, y, a)\, dy\right|$$

$$\leq \int_0^\infty |f(y)|\, dy \int_0^\infty |C_k(x, y, a)|\, dx,$$

so $\|f_a(x)\| < M\|f(x)\|$. The proof may now be completed by the methods of Theorem 1.

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UNIVERSITY OF CALIFORNIA, LOS ANGELES