ON THE NUMBER OF INTEGERS IN THE SUM OF TWO SETS OF POSITIVE INTEGERS

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1. Introduction. Let $A, B, \cdots$ be sets of nonnegative integers. We define $A + B = \{a + b\}_{a \in A,\ b \in B}$. By $A^0, B^0, \cdots$ we shall denote the union of $A, B, \cdots$ and the number 0, by $A(n)$ the number of positive a's that do not exceed n. We further put

$$\text{g.l.b.}\ \frac{A(n)}{n} = \alpha \,, \tag{1}$$

$$\text{g.l.b.}\ \frac{A(n)}{n+1} = \alpha^* \,, \tag{2}$$

$$\liminf \frac{A(n)}{n} = \bar{\alpha} \,. \tag{3}$$

If $1, 2, \cdots, k-1 \in A$, $k \notin A$, we further put

$$\underset{n \geq k}{\text{g.l.b.}}\ \frac{A(n)}{n+1} = \alpha_1 \,. \tag{4}$$

The real number α is called the *density* of A, α_1 the *modified density*, and $\bar{\alpha}$ the *asymptotic density* of A. Densities of $A, B, C, \cdots$ will be denoted by the corresponding Greek letters $\alpha, \beta, \gamma, \cdots$.

Besicovitch [1] introduced α^*, and Erdös [2] α_1.

The author [3] proved: *If $C = A^0 + B$ for $B \ni 1$ and $A^0 + B^0$ otherwise, then for all $n \notin C$ we have*

$$C(n) \geq \alpha^* n + B(n) \,. \tag{5}$$

It was also shown [3] that in (5), α^* cannot be replaced by α.

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It is the purpose of the present note to improve (5) to the relation

$$C(n) \geq \alpha_1 n + B(n)\,. \tag{6}$$

The proof of (6) requires only a modification of the proof of (5), but will be given in full to make the present note self-sufficient.

The inequality (6) immediately yields

$$\bar{\gamma} \geq \alpha_1 + \bar{\beta} \tag{7}$$

if C has infinitely many gaps.

Now (7) is sometimes better and sometimes not as good as Erdös' [2] inequality

$$\bar{\gamma} \geq \bar{\alpha} + \bar{\beta}/2 \tag{8}$$

for the case $\alpha > \beta$, $B \ni 1$, $C = A^0 + B^0$. (To establish (8) it is really sufficient to assume that there is at least one b^0 such that $b^0 + 1 \in B$.) However (7) holds also for $C = A^0 + B$ if $B \ni 1$, and for $C = A^0 + B^0$ without any restriction on B.

2. Proof. We shall now give a proof of (6) for the case $C = A^0 + B$, $B \ni 1$, and then shall indicate the changes which have to be made if nothing is assumed about B but if $C = A^0 + B^0$. By $a, b, c, \cdots$ we shall denote unspecified integers in $A, B, C, \cdots$.

Let $n_1 < n_2 < \cdots$ be all the gaps in C. Put $n_r = n$, $n - n_i = d_i$ for $i < r$. If there is one $e \in B$ such that

$$a + e + d_i = n_j\,, \tag{9}$$

form all numbers $e + d_t$ for which

$$a + e + d_t = n_s\,, \qquad t < r\,, \quad s < r\,. \tag{10}$$

Let T be the set of indices occurring in (10). Put $B^* = \{e + d_s\}_{s \in T}$.

It is not difficult to prove the following propositions.

PROPOSITION 1. *The intersection $B \cap B^*$ is empty.*

PROPOSITION 2. *The integer n is not of the form $a + e + d_s$ for any s.*

Since (10) also implies

$$a + e + d_s = n_t\,, \tag{10'}$$

it follows that B^* contains as many numbers as there are gaps in C which precede n and which are not gaps in $A + B \cup B^*$. Hence we have the following result.

PROPOSITION 3. *If* $B \cup B^* = B_1$, $A + B_1 = C_1$, *then*

$$C_1(n) - C(n) = B_1(n) - B(n) . \tag{11}$$

Thus we have proved the following lemma.

LEMMA. *If there is at least one equation of the form* $a + b + d_i = n_j$, *then there exists a* $B_1 \supset B$ *such that* $C_1 = A + B_1$ *does not contain n, and such that*

$$C_1(n) - C(n) = B_1(n) - B(n) > 0 . \tag{12}$$

Now let $C = A^0 + B$, $B \ni 1$. Clearly, $n_1 > 1$. The numbers smaller than n_1 are either in B, or of the form $n_1 - a$, or of neither of these two sorts. Also $n_1 \notin B$, since $C \supset B$. Hence we have

$$C(n_1) = n_1 - 1 \geq A(n_1 - 1) + B(n_1) . \tag{13}$$

Since $B \ni 1$, we must have $n_1 - 1 \notin A$, $(n_1 - 1) \geq k$. Thus, we obtain

$$C(n_1) \geq \alpha_1 n_1 + B(n_1) . \tag{14}$$

We proceed by induction and assume (6) proved, when n is the jth gap, $j < r$. We distinguish two cases.

Case 1: $d_{r-1} < n_1$. Then

$$C \ni n_1 - d_{r-1} = a + b .$$

We now apply the lemma. Let n be the jth gap in C_1. Then $j < r$, and we have, by induction,

$$C_1(n) \geq \alpha_1 n + B_1(n) , \tag{15}$$

and, by the lemma,

$$C_1(n) - C(n) = B_1(n) - B(n) . \tag{16}$$

Subtracting (16) from (15), we obtain (6).

Case 2: $d_{r-1} \geq n_1$. Now

$$n - n_{r-1} - 1 \geq n_1 - 1 \notin A .$$

Hence we have

$$A(n-n_{r-1}-1)\geq \alpha_1(n-n_{r-1})\,.$$

The numbers between n_{r-1} and n are either of the form $n - a$, or in B, or of neither of these two sorts. But $n \notin B$; hence,

$$(17)\qquad n-n_{r-1}-1\geq A(n-n_{r-1}-1)+B(n)-B(n_{r-1})$$
$$\geq \alpha_1(n-n_{r-1})+B(n)-B(n_{r-1})\,.$$

By induction we have

$$(18)\qquad C(n_{r-1})=n_{r-1}-(r-1)\geq \alpha_1 n_{r-1}+B(n_{r-1})\,.$$

Adding (17) and (18), we obtain (6).

From the proof it is evident that we may obtain the even stronger inequality

$$(6')\qquad C(n)\geq \alpha_1 n+B(n)+\min_{n_i\leq n}\left[\frac{A(n_i-1)}{n_i}-\alpha_1\right]n_i\,.$$

To establish (6) for $C = A^0 + B^0$ without the restriction $B \ni 1$, we first remark that in (13) the term $A(n_1 - 1)$ can be replaced by $A(n_1)$. The cases to be distinguished are $d_{r-1} \leq n_1$ and $d_{r-1} > n_1$. The proof of Case 1 is then word by word the same when we replace B by B^0 and B_1 by B_1^0. In Case 2 we have

$$n-n_{r-1}-1\geq n_1\geq k\,,$$

so that $A(n - n_{r-1} - 1) \geq \alpha_1(n - n_{r-1})$; the remainder of the argument remains unchanged. For $C = A^0 + B^0$, we can obtain the even stronger inequality

$$(6'')\qquad C(n)\geq \alpha_1 n+B(n)+\min_{n_i\leq n}\left[\frac{A(n_i)}{n_i}-\alpha_1\right]n_i\,,$$

which again implies the even stronger result

$$C(n)\geq \max\left\{\alpha_1 n+B(n)+\left[\frac{A(n_1)}{n_1}-\alpha_1\right]n_1\,,\right.$$
$$\left. A(n)+\beta_1 n+\min_{n_i\leq n}\left[\frac{B(n_i)}{n_i}-\beta_1\right]n_i\right.\,.$$

To establish (7), it is sufficient to show that for any set S we have

$$\frac{S(m)}{m} > \frac{S(n)}{n}$$

if $m > n$, $n \notin S$, $S(m) - S(n) = m - n$. However, this can easily be verified. Thus if S has infinitely many gaps, then

$$\bar{\sigma} = \liminf \frac{S(m)}{m} = \liminf_{n \notin S} \frac{S(n)}{n} .$$

It thus appears that in (7) we may replace $\bar{\beta}$ by

$$\liminf_{n \notin C} \frac{B(n)}{n} \geq \bar{\beta} .$$

If $C = A^0 + B^0$, we may of course write

$$\bar{\gamma} \geq \max \left(\alpha_1 + \bar{\beta}, \quad \bar{\alpha} + \beta_1\right) .$$

References

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2. P. Erdös, *On the asymptotic density of the sum of two sequences*, Ann. of Math. 43 (1942), 65-68.

3. H. B. Mann, *A proof of the fundamental theorem on the density of sums of sets of positive integers*, Ann. of Math. 43 (1942), 523-527.

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