ALTERNATING METHOD ON ARBITRARY RIEMANN SURFACES

LEO SARIO

1. Introduction. Schwarz gave the first rigorous construction of harmonic functions with given singularities on closed Riemann surfaces, by means of his alternating method for domains with annular intersection [16]. The method also is directly applicable to open Riemann surfaces of finite genus, since these can always be continued so as to form closed surfaces [7; 8]. For surfaces of infinite genus, this continuation is no longer possible. But if the surface is of parabolic type, Schwarz's method can still be used, a "null boundary" having no effect on the behaviour of the alternating functions [5; 11]. In the general case, there are two obstacles which prevent using Schwarz's method as such. First, if the surface has a large (ideal) boundary, the alternating functions are not determined by their values on the relative boundaries. Second, Schwarz's convergence proof fails, since the Poisson integral is inapplicable on arbitrary Riemann domains. We are going to show that, by certain changes of Schwarz's original method, these difficulties can be overcome.

This paper is a detailed exposition of a reasoning outlined in preliminary notes [9-11]. The manuscript of the paper was communicated (in French) to the Helsinki University in December, 1949. In the meanwhile, the author published a linear operator method [13], which also can be used to establish the results of these notes. A presentation of the classical alternating method for arbitrary Riemann surfaces seems, however, to have independent interest from a methodological viewpoint; such a presentation is the purpose of this paper.

The alternating method on Riemann surfaces, as sketched in [9-11], was referred to also in the recent papers of Kuramochi [1], Kuroda [2], Mori [3], and Ohtsuka [6]. A historical note on the method was given in [15].

2. Functions with vanishing conjugate a_0-periods. We start with two lemmas, which are basic for the alternating procedure.

Let R be an arbitrary Riemann surface, and G a subdomain, compact or not. The relative boundary a_0 of G, that is, the set of boundary points of G, interior

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to R, is assumed to consist of a finite number of closed analytic Jordan curves. On a_0, let f be a real single-valued function, harmonic in an open set containing a_0.

LEMMA 1. *There exists always a harmonic single-valued function u in G with the following properties:*

1°. *u takes on the values f on a_0.*

2°. *u is bounded in G and satisfies*

$$\min_{a_0} f \leq u \leq \max_{a_0} f. \tag{1}$$

3°. *u has a finite Dirichlet integral over G,*

$$D(u) = \iint_G |\operatorname{grad} u|^2 \, dx\, dy < \infty. \tag{2}$$

Here $z = x + iy$ is a local uniformizer of R.

4°. *The period along a_0 of the harmonic function v, conjugate to u, vanishes,*

$$\int_{a_0} dv = 0. \tag{3}$$

Proof. If G is compact, the lemma is evident. Suppose now G is noncompact. We form an exhaustion $G_1 \subset G_2 \subset G_3 \subset \cdots$ of G, such that the boundary of G_n consists of a_0 and a set a_n of closed analytic Jordan curves tending, for $n \longrightarrow \infty$, to the ideal boundary of G. Let u_n be a harmonic function in G_n which coincides with f on a_0 and assumes on a_n a constant value c_n. By Schwarz's reflexion principle, it is easy to see that u_n is harmonic still on $a_0 + a_n$.

We fix the constant c_n as follows. We observe that u_n depends continuously on c_n. The same is also true for the normal derivative $\partial u_n/\partial n$ on a_0 and, consequently, for

$$\int_{a_0} dv_n = \int_{a_0} \frac{\partial v_n}{\partial s}\, ds = \int_{a_0} \frac{\partial u_n}{\partial n}\, ds, \tag{4}$$

where the meaning of v_n and ds is evident. If we choose

$$c_n = \min_{a_0} f,$$

then obviously $\partial u_n/\partial n \geq 0$, if n denotes the interior normal of a_n with respect to

G_n. Hence, the period (4) is nonpositive in this case. If we had chosen

$$c_n = \max_{a_0} f,$$

we would have found by the same reasoning that the integral (4) is nonnegative. Thus, there must exist a value c_n such that

$$\min f \leq c_n \leq \max f,$$

and such that the integral (4) vanishes. In the sequel we suppose that the constant c_n has been selected according to this condition. We have then in G_n the uniform estimate

$$\min f \leq u_n \leq \max f. \tag{5}$$

In the sequence of the uniformly bounded functions u_n, there is a subsequence which converges uniformly in every closed subdomain of G to a single-valued function u, harmonic on $G + a_0$.

In order to see that u is uniquely determined, we shall prove that the sequence $\{u_n\}$ itself, not only a subsequence, converges. Let x_n be the harmonic function in G_n with $x_n = 0$ on a_0, $x_n = 1$ on a_n. The sequence $\{x_n\}$ decreases monotonically, converging to a harmonic function x on G with $x = 0$ on a_0. If $x \equiv 0$, u is necessarily unique, since the difference $u' - u''$ of two functions u would assume, by

$$\int d(v' - v'') = 0,$$

both positive and negative values, and would be dominated by a multiple of x. Hence we can confine our attention to the case $x \not\equiv 0$.

By Green's formula

$$\int_{-a_0 + a_n} x_n \, d\bar{v}_n - u_n \, d\bar{y}_n = 0,$$

where y_n is the harmonic conjugate of x_n, we have

$$c_n = -\frac{\displaystyle\int_{a_0} f \, d\bar{y}_n}{\displaystyle\int_{a_0} d\bar{y}_n}. \tag{6}$$

Hence c_n converges to a unique constant c. Now let z be an arbitrary (fixed) point of G. For sufficiently large n, z is an interior point of G_n. Let g_n be the Green's function of G_n with the logarithmic pole at z. Draw a small circle C about z. The Green's formula

$$\int_{-a_0+a_n+C} u_n \, d\bar{h}_n - g_n \, d\bar{v}_n = 0,$$

where h_n is the harmonic conjugate of g_n, yields, if we let C shrink to the point z,

$$(7) \qquad u_n(z) = \frac{1}{2\pi}\left[\int_{a_0} f \, dh_n + c_n\left(2\pi - \int_{a_0} d\bar{h}_n\right)\right].$$

This shows the convergence of $u_n(z)$ and thus uniqueness of u.

In order to prove that the function u satisfies the conditions $1° - 3°$, we note that the u_n converge uniformly even on the closure of G_n. In fact, for $\epsilon > 0$ and for m, q sufficiently large, we have $(u_m - u_q) < \epsilon$ on a_n; and on a_0 this difference vanishes. By the maximum principle, the Cauchy criterion is fulfilled on the closure of G_n. In view of the harmonic boundary values and Schwarz's reflexion principle, the convergence is uniform even in a domain slightly extended beyond a_0 and a_n. From this we conclude that all derivatives of u_m converge in the closure of G_n.

From the uniform convergence it follows that u takes on the value f on a_0. The condition $2°$ is guaranteed by (5). In order to study the condition $3°$ we observe that, for $p > 0$,

$$(8) \qquad D(u) = \lim_{n \to \infty} D_n(u) = \lim_{n \to \infty} \lim_{p \to \infty} D_n(u_{n+p}),$$

where D refers to G and D_n to G_n. We have

$$(9) \qquad D_n(u_{n+p}) \leq D_{n+p}(u_{n+p}) = \int_{-a_0+a_{n+p}} u_{n+p} \, dv_{n+p}.$$

In this expression, we have

$$(10) \qquad \int_{a_{n+p}} u_{n+p} \, dv_{n+p} = c_{n+p} \int_{a_{n+p}} dv_{n+p} = c_{n+p} \int_{a_0} dv_{n+p} = 0.$$

Since the integral on the right in (9) extended over a_0 converges because of the uniform convergence of the u_n and grad v_n, this integral is uniformly bounded. Hence, the condition $3°$ is fulfilled.

The condition $4°$ follows again by the uniform convergence of grad v_n on a_0.

This completes the proof of Lemma 1.

3. Functions with nonvanishing conjugate a_0-periods. Suppose now that the region G is not compact.

LEMMA 2. *There exists always a single-valued harmonic function u on G which coincides with f on a_0, and whose conjugate function v has the period*

$$\int_{a_0} dv = 1. \tag{11}$$

Proof. It suffices to consider the case $f \equiv 0$; in order to pass to the general case we have only to add to the constructed function a function furnished by Lemma 1.

Let now u_n be a harmonic function in G_n which vanishes on a_0 and assumes a constant value d_n on a_n, such that the period of the conjugate function v_n of u_n is

$$\int_{a_0} dv_n = 1. \tag{12}$$

This choice is always possible, since the value of the foregoing integral is proportional to d_n. Obviously u_n is a multiple of the harmonic measure of a_n.

By Green's formula

$$\int_{-a_0+a_n} (u_{n+p}\, dv_n - u_n\, dv_{n+p}) = 0,$$

we have

$$\int_{a_n} u_{n+p}\, dv_n = d_n. \tag{13}$$

On the other hand, for the functions u_{n+p}, positive in G_{n+1}, we can use Harnack's principle, which can be expressed, in the present case, as follows. For all the functions u_{n+p}, there is a constant $M < \infty$ such that, on a_n, interior to G_{n+1},

$$\max_{a_n} u_{n+p} < M \min_{a_n} u_{n+p}. \tag{14}$$

Hence, by (13) and

$$\int_{a_n} dv_n = 1,$$

we have

$$\max_{a_n} u_{n+p} < M \int_{a_n} u_{n+p} \, dv_n = M \, d_n. \tag{15}$$

Thus, by the maximum principle, the functions u_{n+p} are uniformly bounded in G_n and form a compact family.

4. Oscillation of functions. In order to prove the convergence of the alternating functions, we still need a lemma concerning oscillations of functions.

Let R be an arbitrary Riemann surface and R_0 a compact closed point-set on R. Consider all single-valued harmonic functions u on R.

LEMMA 3. *There exists a positive constant $q < 1$, independent of u, such that for every u the oscillations of u on R and R_0,*

$$\begin{aligned} S(u, R) &= \sup_R u - \inf_R u \\ S(u, R_0) &= \max_{R_0} u - \min_{R_0} u, \end{aligned} \tag{16}$$

satisfy the inequality

$$S(u, R_0) \leq q \, S(u, R). \tag{17}$$

Proof. For the two cases $S(u, R) = 0$ and $S(u, R) = \infty$, the proposition (17) is evident; thus, it suffices to consider bounded nonconstant functions u. We normalize these functions, without loss of generality, by adding a constant and multiplying by a constant such that

$$\sup_R u = 1, \quad \inf_R u = 0. \tag{18}$$

This being done, we have to prove the existence of a constant $q < 1$ such that $S(u, R_0) < q$. If such a constant did not exist, there would be a sequence of functions $u_1, u_2, u_3, \cdots$ such that

$$\lim_{n \to \infty} S(u_n, R_0) = 1, \tag{19}$$

and, consequently,

$$\max_{R_0} u_n \longrightarrow 1, \quad \min_{R_0} u_n \longrightarrow 0. \tag{20}$$

Among the functions u_n, uniformly bounded on R, one can select a subsequence, say again $\{u_n\}$, which tends uniformly to a function u^*, harmonic and single-valued on R. The points P_n and Q_n where u_n assumes maximum and minimum values, respectively, on the closed set R_0, accumulate at some points P^* and Q^* of R_0,

$$(21) \qquad P_n \longrightarrow P^*, \qquad Q_n \longrightarrow Q^*.$$

It is easily seen that

$$(22) \qquad u^*(P^*) = 1 \text{ and } u^*(Q^*) = 0.$$

In fact, if $u^*(P^*)$ were < 1, let ϵ be a positive constant, $\epsilon < 1/2(1 - u^*(P^*))$. By the continuity of u^*, there would be a neighborhood K of P^* such that, at each point P of K,

$$u^*(P) < u^*(P^*) + \epsilon.$$

On the other hand, by the definition of P_n, for sufficiently large n,

$$u_n(P_n) > 1 - \epsilon,$$

and the points P_n lie on K. Thus, at these points P_n, one would have

$$u_n(P_n) - u^*(P_n) > 1 - u^*(P^*) - 2\epsilon = \text{const.} > 0,$$

in contradiction to the uniform convergence of the u_n to u^*. This proves the first equality (22). The second one is proved in the same manner.

Consequently, the function u^* would be harmonic, single-valued, and nonconstant on R, and would assume its maximum and minimum values at interior points of R. This violation of the maximum principle disproves our antithesis. The lemma follows.

5. The existence theorem. After these preparations we are able to establish existence of the harmonic functions in question on the whole surface. Let R_0 be a subdomain of R whose relative boundary, that is, the set of boundary points interior to R, consists of a finite set of closed analytic Jordan curves. The complement $G = R - R_0$ then consists of a finite number m of disjoint domains G_i $(i = 1, 2, \cdots, m)$, compact or not. Let now a_i be the common part of the boundaries of R_0 and G_i.

In each G_i, let u_i be a given function, vanishing on a_i, harmonic, single-valued and nonconstant in a neighborhood of a_i, having otherwise arbitrary singularities and, in case G_i is noncompact, an arbitrary behaviour at the common (ideal) part of the boundaries of R and G_i. Denote by ds an arc element of a_i, and by $\partial u_i/\partial n$ the normal derivative of u_i in the interior direction of G_i.

THEOREM. *If R_0 is compact, then the condition*

$$\sum_{i=1} \int_{a_i} \frac{\partial u_i}{\partial n} ds = 0 \tag{23}$$

guarantees the existence of a function f on the whole surface R, satisfying the following conditions:

1°. *The function is harmonic, single-valued and nonconstant outside the possible singularities of the u_i.*

2°. *The difference $f - u_i$ is harmonic, single-valued, and bounded in the whole region G_i, and has a finite Dirichlet integral over G_i.*

In case R_0 is noncompact, the existence of f satisfying 1° and 2° is always assured, independently of the condition (23). If this is satisfied, f is bounded in R_0 and has there a finite Dirichlet integral.

Proof. Consider first the case where R_0 is compact. Let R' be another compact region ($\subset R$), containing the closure of R_0 in its interior, and bounded by a finite number of closed analytic Jordan curves. The intersection $H_i = R' \cap G_i$ is supposed to consist of one single region, bounded by a_i and the intersection b_i of G_i and the boundary of R'. Denote, for the time being, u_i by u_{i0}. In R', let f_0 be the harmonic function coinciding with u_{i0} on b_i. In G_i, form, by the procedure of Lemma 1, a function h_{i1}, harmonic and single-valued in G_i, coinciding with f_0 on a_i, bounded by the inequalities

$$\min_{a_i} f_0 \le h_{i1} \le \max_{a_i} f_0,$$

possessing a finite Dirichlet integral over G_i,

$$D_i(h_{i1}) < \infty,$$

and satisfying the condition

$$\int_{a_i} dk_{i1} = 0,$$

where k_{i1} is the harmonic conjugate of h_{i1}. Write, in G_i,

$$u_{i1} = u_{i0} + h_{i1}. \tag{24}$$

Let f_1 be the harmonic function in R' coinciding with u_{i1} on b_i. We then form again by the procedure of Lemma 1 a harmonic function h_{i2} in G_i which assumes the values f_1 on a_i and has the corresponding boundedness properties. We thus obtain successively a sequence of functions h_{in} and u_{in} in G_i, and f_n in R', determined by the conditions

$$(25) \qquad \begin{cases} f_n = u_{in} \text{ on } b_i, \\ h_{i(n+1)} = f_n \text{ on } a_i \\ u_{i(n+1)} = u_{i0} + h_{i(n+1)} \text{ in } G_i, \end{cases} \qquad (n = 0, 1, 2, \cdots),$$

and having the properties

$$(26) \qquad \min_{a_i} f_n \leq h_{i(n+1)} \leq \max_{a_i} f_n,$$

$$(27) \qquad D_i(h_{in}) < \infty,$$

$$(28) \qquad \int_{a_i} dk_{in} = 0,$$

where k_{in} is the conjugate function of h_{in}.

One has to prove the convergence of the functions f_n and u_{in} toward a desired common function f. We shall show first the convergence of the functions f_n on the closure $\overline{R}'$ of R'.

By Cauchy's criterion, this convergence is assured as soon as the difference $f_{n+p} - f_n$ tends, for $n, p \longrightarrow \infty$, toward zero on the boundary b of R'. In order to use Lemma 3, we shall reduce estimation of this difference to that of its oscillation on b,

$$(29) \qquad |f_{n+p} - f_n| \leq S(f_{n+p} - f_n;\ b),$$

this inequality being valid as soon as

$$(30) \qquad \min_{R'} |f_{n+p} - f_n| = 0.$$

We shall now prove the latter relation.

Let x_i be, in $H_i = R' \cap G_i$, the harmonic function vanishing on a_i and assuming the constant value 1 on b_i. Let y_i be the conjugate function of x_i. The condition (30) is satisfied if

$$(31) \qquad \sum_i \int_{b_i} (f_{n+p} - f_n)\, dy_i = 0.$$

In order to establish this equation, we make use of Green's formula

$$\int_{-a_i + b_i} (f_n \, dy_i - x_i \, dg_n) = 0,$$

where g_n is the harmonic conjugate of f_n. It follows, by

$$\sum \int_{b_i} dg_n = 0,$$

that

$$(32) \qquad \sum \int_{a_i} f_n \, dy_i = \sum \int_{b_i} f_n \, dy_i.$$

On the other hand, the formula

$$\int_{-a_i + b_i} (u_{i(n+1)} \, dy_i - x_i \, dv_{i(n+1)}) = 0$$

gives, in view of (23), (28), and, accordingly, of

$$\sum \int_{b_i} dv_{i(n+1)} = \sum \int_{b_i} dv_{i0} = 0,$$

the relation

$$(33) \qquad \sum \int_{a_i} u_{i(n+1)} \, dy_i = \sum \int_{b_i} u_{i(n+1)} \, dy_i.$$

By (25), (32), and (33), we have

$$(34) \qquad \sum \int_{b_i} f_n \, dy_i = \sum \int_{b_i} f_{n+1} \, dy_i.$$

This yields the desired equality (31).

The problem of convergence of f_n has herewith been reduced to the estimation of the oscillation $S(f_{n+p} - f_n;\, b)$. We have first

$$(35) \qquad S(f_{n+p} - f_n;\ b) \le \sum_{m=1}^{p} S(f_{n+m} - f_{n+m-1};\ b).$$

To estimate $S(f_{n+1} - f_n;\ b)$, note first that

$$(36) \qquad f_{n+1} - f_n = h_{i(n+1)} - h_{in} \text{ on } b_i,$$

$$(37) \qquad f_n - f_{n-1} = h_{i(n+1)} - h_{in} \text{ on } a_i.$$

Since the functions $h_{i(n+1)}$ and h_{in} were constructed by the procedure of Lemma 1 as limits of certain harmonic functions coinciding with f_n and f_{n-1}, respectively, on a_i, and satisfying the condition (26), the difference $h_{i(n+1)} - h_{in}$ can be considered as defined by the procedure of Lemma 1, with the boundary values $f_n - f_{n-1}$ on a_i. Thus, this difference satisfies the corresponding condition in the whole G_i:

$$(38) \qquad \min_{a_i} (f_n - f_{n-1}) \le h_{i(n+1)} - h_{in} \le \max_{a_i} (f_n - f_{n-1}).$$

The relations (36) - (38) yield

$$(39) \qquad S(f_{n+1} - f_n;\ b) \le S(f_n - f_{n-1};\ a),$$

where a is the boundary of R_0.

On the other hand, by Lemma 3, applied to the difference $f_n - f_{n-1}$, the domain R', and the boundary of R_0, we have

$$(40) \qquad S(f_n - f_{n-1};\ a) \le q \cdot S(f_n - f_{n-1};\ b),$$

q being a positive constant < 1. Thus,

$$(41) \qquad S(f_{n+1} - f_n;\ b) \le q\, S(f_n - f_{n-1};\ b).$$

By repetition of the same reasoning starting from $f_n - f_{n-1}$, and so on, we obtain the desired estimate

$$(42) \qquad S(f_{n+1} - f_n;\ b) \le q^n S_0,$$

where S_0 signifies the constant $S(f_1 - f_0;\ b)$.

Applied to (35), this yields

$$(43) \qquad S(f_{n+p} - f_n;\ b) < q^n \frac{S_0}{1-q}.$$

The right side tends to zero, independently of p. By (29), Cauchy's criterion is satisfied and the uniform convergence of the functions f_n to a single-valued harmonic function f in R' has been proved.

The convergence of the functions h_{in} follows immediately. In fact, the relation (38), applied to the difference $h_{i(n+p)} - h_{in}$, gives

$$\max_{G_i} |h_{i(n+p)} - h_{in}| \leq \max_{a} |f_{n+p-1} - f_{n-1}|. \tag{44}$$

This implies, by the convergence of f_n, that of h_{in}. The limit function h_i is harmonic and single-valued in G_i. The corresponding limit of the functions u_{in} is $u_i + h_i$, where we use again u_i instead of u_{i0}.

The functions f and $u_i + h_i$ are identical in $H_i = R' \cap G_i$. In fact, the difference $f_n - u_{in}$ vanishes on b_i and coincides with $f_n - f_{n-1}$ on a_i, thus tending to zero on $a_i + b_i$ and hence on H_i.

Denote in the sequel by f the function thus obtained on the whole surface R. It remains to show that it satisfies the conditions $1° - 2°$ of the theorem.

Since the difference $f - u_i = h_i$ is harmonic and single-valued in G_i, the same is true of the function f except at the singularities of u_i. We shall show that f does not reduce to a constant.

Let h_i' be the harmonic function in G_i, constructed by the procedure of Lemma 1 to coincide with $h_i = f - u_i$ on a_i. Then $h_i' - h_{in}$ is the function in G_i corresponding, by this procedure, to the values $h_i' - h_{in}$ on a_i. By the relations

$$\min_{a_i} (h_i - h_{in}) \leq h_i' - h_{in} \leq \max_{a_i} (h_i - h_{in}),$$

valid in G_i, and by the convergence $h_{in} \longrightarrow h_i$ on a_i, the functions h_{in} converge uniformly to h_i' in G_i; that is, $h_i' \equiv h_i$. By Lemma 1, this implies that

$$\min_{a_i} f \leq h_i \leq \max_{a_i} f. \tag{45}$$

If now f were constant, the same would be the case with h_i, hence also with $u_i = f - h_i$, contrary to our assumptions. This proves the property 1°.

Property 2° follows from the fact just mentioned that $h_i = f - u_i$ is a harmonic function in G_i constructed by the procedure of Lemma 1. This completes the proof of the theorem for the case where R_0 is compact.

The case where R_0 is noncompact reduces simply to the preceding case. We only have to isolate the common part of the boundaries of R_0 and R from $R - R_0$

by a finite set a_0 of simple analytic Jordan curves which divide R_0 into a compact domain R_0^* and a noncompact domain G_0. By Lemmas 1 and 2, there exists in G_0 a function u_0, harmonic and single-valued, vanishing on a_0 and having a prescribed period for the conjugate function v_0. We select this period in accordance with the condition

$$\int_{a_0} dv_0 = -\sum_{1}^{m} \int_{a_i} dv_i. \tag{46}$$

All the assumptions of the first part of our theorem are thus satisfied. Hence, there exists a function f on R which fulfills the conditions stated in the latter part of our theorem.

6. Applications. The theorem thus proved has applications to the classification of Riemann surfaces and to the theory of Abelian integrals as announced in [10; 11]. Here we confine our attention to some typical corollaries.

COROLLARY 1. *There are Green's functions on a Riemann surface R if and only if the boundary has a positive harmonic measure.*

Proof. Suppose there is a Green's function g on R. Let P: $z = 0$ be its logarithmic pole, in a parameter disc K: $|z| \leq 1$. In $G_1 = R - \overline{K}$, let u be the harmonic function constructed by the procedure of Lemma 1 for values $u = g$ on a_1: $|z| = 1$. Then $g - u$ is bounded in G_1,

$$|g - u| \leq M < \infty,$$

and has a nonvanishing conjugate period. This clearly implies the existence of a nonvanishing harmonic measure ω in G_1.

Conversely, suppose $\omega \not\equiv 0$ in G_1. Multiply ω by such a constant that the conjugate of the function u_1 thus obtained has the period 2π along a_1. Take as the domain R_0 of our theorem the annulus $1/2 < |z| < 1$. In G_2: $|z| < 1/2$ write $u_2 = \log 1/|z|$. By our theorem, there is a function g' on R with the pole $\log 1/|z|$ at $z = 0$ and such that $g' - u_1$ is bounded in G. The existence of a Green's function follows.

This result [10], proved later also by Virtanen [18] and Kuroda [2], shows that the classification of Riemann surfaces in those with "null-boundary" and "positive boundary" coincides with Riemann's classification on the basis of existence or nonexistence of Green's functions.

Another application of our theorem is a criterion for the existence of single-valued nonconstant harmonic functions which are bounded (HB) or have a finite Dirichlet integral (HD). It was stated by Nevanlinna [4] and Virtanen [17] that there are functions HB or HD on R if and only if R has a null-boundary. This assertion has been disproved by Ahlfors and Royden. A correct criterion follows:

COROLLARY 2. *There are functions* HB *or* HD *on a given Riemann surface* R *if and only if some function* u *of class* HB *or* HD *respectively in* G *satisfies the conditions* $u = 0$ *on* a, $\int_a dv = 0$.

The condition of a positive harmonic measure is equal to the first condition given above [8]. Thus, the inadequacy of Nevanlinna's statement is due to the lack of the second condition.

A further application deals with Abelian integrals. The following problem was stated by Myrberg in 1948 (October 13, at Helsinki University): Does there exist a nonconstant harmonic function with a finite Dirichlet integral on an arbitrary open Riemann surface R. The above theorem gives [11]:

COROLLARY 3. *On an arbitrary Riemann surface of positive genus there exist Abelian integrals of the first, second, and third kind which possess a finite Dirichlet integral outside a neighborhood of the singularities.*

The Abelian integrals, the existence of which was thus proved, have later been investigated by Virtanen and Nevanlinna. The existence proof can also be performed by adapting the classical reasoning of Weyl.

Another immediate consequence of the foregoing theorem is the following result, proved first by Nevanlinna [5] using integral equations. Let R be an open Riemann surface of parabolic type, and let A and B be two noncompact subdomains such that $A \cap B$ is a doubly connected region, bounded by two analytic Jordan curves. Let a and b be two single-valued harmonic functions in $A \cap B$.

COROLLARY 4. *If the difference of the conjugate functions of* a *and* b *is single-valued, then there exists a harmonic function* f *on* R *such that* $f - a$ *in* A, $f - b$ *in* B, *are harmonically continuable, single-valued, and bounded.*

To prove this, we have only to select as the domain R_0 of our theorem a region interior to $A \cap B$, separating the two boundary curves of the latter, and the existence of f is assured.

In several related problems, an extremal method [14] seems to be more powerful than the alternating methods. A comparative survey on these methods was given in [15].

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