# ASYMPTOTIC LOWER BOUNDS FOR THE FREQUENCIES OF CERTAIN POLYGONAL MEMBRANES

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**1. Background.** Let the bounded, simply connected, open region $R$ of the $(x, y)$ plane have the boundary curve $C$. If a uniform elastic membrane of unit density is uniformly stretched upon $C$ with unit tension across each unit length, the square $\lambda = \lambda(R)$ of the fundamental frequency satisfies the conditions (subscripts denote differentiation)

$$(1a)\qquad \begin{cases} \Delta u \equiv u_{xx} + u_{yy} = -\lambda u \quad \text{in } R, \\ \lambda = \text{minimum}, \end{cases}$$

with the boundary condition

$$(1b)\qquad u(x, y) = 0 \quad \text{on } C.$$

The solution $u$ of problem (1) is unique up to a constant factor. It is known [**13**, p. 24] that $\lambda$ is the minimum over all piecewise smooth functions $u$ satisfying (1b) of the Rayleigh quotient

$$(2)\qquad \rho(u) = \iint_R |\nabla u|^2\, dxdy \Big/ \iint_R u^2\, dxdy,$$

where $|\nabla u|^2 = u_x^2 + u_y^2$. In many practical methods for approximating $\lambda$ one essentially determines $\rho(u)$ for functions $u$ satisfying (1b) which are close to a solution of the boundary value problem (1). See [**9**, p. 112; **6**, p. 276; **11**, and **12**]. By (2) these approximations are known to be *upper bounds* for $\lambda$; they can be made arbitrarily good with sufficient labor. It is obviously of equal importance to obtain close lower bounds for $\lambda$; cf. [**14**].

The lower bounds for $\lambda$ given by Pólya and Szegö [**13**] are ordinarily far

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from close. Those obtainable from $\rho(u)$, $\iint_R u^2 dxdy$, and $\iint_R |\Delta u|^2 dxdy$ by methods due to Temple [**15**], D. H. Weinstein [**17**], Wielandt [**18**], and Kato [**8**] (for expositions see [**3**] and [**16**]) are arbitrarily good, but presuppose knowledge of a lower bound for the second eigenvalue $\lambda_2$ of the problem (1). The same is true of Davis's proposals in [**4**]. It is possible, following Aronszajn and Zeichner [**1**], to get close lower bounds for $\lambda$ by minimizing $\rho(u)$ over a class of functions $u$ permitted some discontinuity in $R$ (method of A. Weinstein); the author has no knowledge of the practicability of the method.

A common method of approximating $\lambda$ is to replace the boundary value problem (1) by a similar problem in finite differences. Divide the plane into *squares* of side $h$ by the network of lines $x = \mu h$, $y = \nu h$ ($\mu$, $\nu = 0, \pm 1, \pm 2, \cdots$). The points $(\mu h, \nu h)$ are the *nodes* of the net. A *half-square* is an isosceles right triangle whose vertices are three nodes of one square of the net. Assume that

(3) $\quad R$ is the union of a finite number of squares and half-squares.

Then every interior node of $R$ has four neighboring nodes in $R \cup C$.

Define $\Delta_h$, a finite-difference approximation to $\Delta$, by the relation

$$h^2 \Delta_h v(x,y) = v(x+h,y) + v(x-h,y) + v(x,y+h) + v(x,y-h) - 4v(x,y).$$

Let $\lambda_h$ be the least number satisfying the following difference equation for a net function $v$ defined on the nodes $(x,y)$ of the net:

$$\Delta_h v = -\lambda_h v \quad \text{at the nodes in } R, \tag{4a}$$

with the boundary condition

$$v = 0 \quad \text{at the nodes on } C. \tag{4b}$$

One can interpret $\lambda_h$ as the square of the fundamental frequency of a network of massless strings with uniform tension $h$, fastened to $C$, and supporting a particle of mass $h^2$ at each node. That is, a certain lumping of the distributed masses and tensions of problem (1) yields problem (4).

It is easily verified for a rectangular region of commensurable sides $\pi/p$, $\pi/q$, and for $h$ such that (3) holds, that one has $u = v = \sin px \sin qy$, and that

$$\frac{\lambda_h}{\lambda} = \frac{\sin^2(ph/2) + \sin^2(qh/2)}{(ph/2)^2 + (qh/2)^2} = 1 - \frac{p^4+q^4}{p^2+q^2}\,\frac{h^2}{12} + o(h^2) \quad (h \longrightarrow 0). \tag{5}$$

Hence $\lambda_h < \lambda$ for all $h$, and one can use $\lambda_h$ as a lower bound for $\lambda$. However, since $\lambda$ is known exactly for rectangular regions, relation (5) contributes nothing to its computation. For general regions $R$, it was stated [**3**, p. 405] in 1949 that nothing could be said about the relation of $\lambda_h$ to $\lambda$.

**2. A new result.** An asymptotic relation resembling (5) will now be established for any *convex* polygonal region $R$ satisfying (3). Such regions are polygons of at most eight sides, having interior vertex angles of 45°, 90°, or 135°. The following theorem[1] will be proved in § 3 by use of the lemmas of § 4:

THEOREM. *Let $R$ be a convex region which is a finite union of squares and half-squares for all $h$ under consideration. Let $u$ solve problem* (1) *for $R$, and let*

$$a = a(R) = \frac{\iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy}{\iint_R (u_x^2 + u_y^2)\,dxdy}.$$

*Then, as $h \longrightarrow 0$, one has*

$$\frac{\lambda_h}{\lambda} \leq 1 - \frac{a}{12} h^2 + o(h^2) \qquad (h \longrightarrow 0). \tag{6}$$

It is a consequence of the theorem that, for all sufficiently small $h$, say for $h \leq h_0$, $\lambda_h$ is a *lower bound* for $\lambda$. The ordinary finite-difference method thus complements any method based on Rayleigh quotients; and, since $\lambda_h \longrightarrow \lambda$ as $h \longrightarrow 0$, together two such methods can confine $\lambda$ to an arbitrarily short interval. In particular, Pólya [**11** and **12**] devises modified finite-difference approximations to problem (1) which furnish upper bounds to $\lambda$ for all $h$. Hence arbitrarily good two-sided bounds to $\lambda$ can be found by finite-difference methods alone.

The constant $a$ of the theorem is the best possible for a rectangle $R$ of sides $\pi/p$, $\pi/q$. For this region, we have $a = (p^4 + q^4) \cdot (p^2 + q^2)^{-1}$, and (6) is seen by (5) to be actually an equality up to terms $o(h^2)$.

Using heuristic reasoning, Milne [**9**, p. 238, (97.5)] finds an approximate formula which, specialized to the fundamental eigenvalue and set in our notation, says

$$\frac{\lambda_h}{\lambda} \doteq 1 - \frac{\lambda h^2}{24} + o(h^2) \qquad (h \longrightarrow 0). \tag{7}$$

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[1] The author gratefully acknowledges many helpful conversations with his colleague Dr. Wolfgang Wasow on the subject of this paper.

For a rectangle of sides $\pi/p$, $\pi/q$, the coefficient of $-h^2/12$ in (7) is $(p^2+q^2)/2$. Since

$$\frac{p^2+q^2}{2} + \frac{(p^2-q^2)^2}{p^2+q^2} = \frac{p^4+q^4}{p^2+q^2},$$

the coefficient of $h^2$ in (7) is low for all rectangles with $p \neq q$, and exact for squares. Hence (7) cannot ordinarily be expected to be exact in its $h^2$ term.

The use of the theorem to bound $\lambda$ is limited by our lack of knowledge of $h_0$. However, it is the author's conjecture that, for the regions $R$ of the theorem, $\lambda_h < \lambda$ for all $h$.

The convexity of $R$ is vital to the statement and proof of the theorem; in fact, by the remark after Lemma 4, $a = \infty$ for nonconvex polygons. A heuristic argument, supported by the numerical example of § 5, has in fact convinced the author that, for nonconvex polygons, $\lambda_h > \lambda$ for all sufficiently small $h$.

The restriction of $R$ and $h$ to satisfy (3) is less essential, but is used in two ways: (i) to be sure that no interior node has a neighboring node outside $R$; (ii) to prove that $\Gamma = 0$ in Lemma 7. With an appropriate alteration of $\Delta_h$ near $C$, and with a modification of Lemma 7, one can extend the present method to obtain formulas of type (6) without assuming (3)—and even for convex regions $R$ bounded by piecewise analytic curves $C$. See [5]. Analogous results can be expected in $n$ dimensions.

**3. Proof of the theorem.** Let $K$ be the class of functions $u$ which vanish on $C$, such that $(uu_x)_x$ and $(uu_y)_y$ are continuous in $R \cup C$. Applying Gauss's divergence formula (27) with $p = uu_x$, $q = uu_y$, one finds that, for all $u$ in $K$, Green's formula is valid in the form

$$\iint_R |\nabla u|^2 dxdy = -\iint_R u\Delta u dxdy.$$

Hence, for all $u \in K$, $\rho(u)$ in (2) can be rewritten with $-\iint_R u\Delta u\, dxdy$ in the numerator.

Since, by Lemma 1, the function $u$ which minimizes (2) and solves (1) belongs to $K$, and since any function in $K$ is piecewise smooth, one may alternatively define $\lambda$ as the minimum, over all functions in $K$, of the quotient

$$\rho(u) = -\iint_R u\Delta u dxdy \Big/ \iint_R u^2 dxdy.$$

Analogously, without having to worry about function classes, one can show that $\lambda_h$ is the minimum, over all net functions $v$ satisfying (4b), of the quotient

$$(8) \qquad \rho_h(v) = -h^2 \sum\sum_{N_h} v\Delta_h v \Big/ h^2 \sum\sum_{N_h} v^2,$$

where the sums are extended over all nodes $N_h$ of the net inside $R$.

The key to proving the theorem is to set the solution $u$ of problem (1) into the Rayleigh quotient (8) of problem (4). It will be shown that

$$(9) \qquad \frac{\rho_h(u)}{\lambda} = 1 - \frac{1}{12} ah^2 + o(h^2) \qquad (h \longrightarrow 0).$$

Since $\lambda_h \leq \rho_h(u)$, the theorem follows from (9). Henceforth $u$ will always denote a solution of problem (1).

The denominator of $\rho_h(u)$ is a Riemann sum for $\iint_R u^2\,dxdy$. Since $u^2$ is continuous and hence Riemann integrable over $R$,

$$(10) \qquad h^2 \sum\sum_{N_h} u^2 = \iint_R u^2\,dxdy + o(1) \qquad (h \longrightarrow 0).$$

(It can be shown that one can replace $o(1)$ by $o(h^2)$ in (10), but we shall not need to do this.)

The nodes $N_h$ inside $R$ are divided into two classes:

$N_h'$ : those at a distance $h$ from some 135° vertex of $C$;

$N_h''$: the other nodes of $N_h$.

Split the numerator of $\rho_h(u)$ accordingly:

$$(11) \qquad -h^2 \sum\sum_{N_h} u\Delta_h u = -h^2 \sum\sum_{N_h'} u\Delta_h u - h^2 \sum\sum_{N_h''} u\Delta_h u = S_h'(u) + S_h''(u).$$

To estimate $S_h'(u)$ note that, since there are at most eight 135° vertices, the number of nodes in $N_h'$ is at most 8, for any $h$. At any node in $N_h'$,

$$h^2 |u\Delta_h u| \leq h^2 \left(\frac{u-0}{h}\right) \sum_{i=1}^{4} \left|\frac{u-u_i}{h}\right| \leq 4h^2 \max |\nabla u|^2,$$

where the maximum of $|\nabla u|^2$ is taken for all points $(x, y)$ within a distance $2h$ of some $135^\circ$ vertex. Hence, by Lemma 2, as $h \longrightarrow 0$ through values such that (3) holds,

$$(12) \qquad |S_h'(u)| \leq 32h^2 \max |\nabla u|^2 = o(h^2) \qquad (h \longrightarrow 0).$$

Now, using the notation and assertion of Lemma 5, one obtains

$$(13) \qquad S_h''(u) = -h^2 \sum_{N_h''}\sum u\Delta u - \frac{h^4}{12} \sum_{N_h''}\sum u(u_{xxxx}' + u_{yyyy}'').$$

Since $u$ satisfies (la),

$$(14) \qquad -h^2 \sum_{N_h''}\sum u\Delta u = \lambda h^2 \sum_{N_h''}\sum u^2 = \lambda h^2 \sum_{N_h}\sum u^2 + o(h^2) \quad (h \longrightarrow 0);$$

the last step is correct because $u(x, y) \longrightarrow 0$ as $(x, y) \longrightarrow C$.

Combining (13) and (14), one finds that, as $h \longrightarrow 0$,

$$(15) \qquad \begin{aligned} S_h''(u) &= \lambda h^2 \sum_{N_h}\sum u^2 - \frac{h^4}{12} \sum_{N_h''}\sum u(u_{xxxx}' + u_{yyyy}'') + o(h^2) \\ &= \lambda h^2 \sum_{N_h}\sum u^2 - \frac{h^2}{12} \iint_R u(u_{xxxx} + u_{yyyy})\,dxdy + o(h^2), \end{aligned}$$

by Lemma 6. The integrals used in this proof exist, by Lemma 3. Using (11), (12), (15), and Lemma 7, one finds that

$$(16) \qquad \begin{aligned} &-h^2 \sum_{N_h}\sum u\Delta_h u \\ &= \lambda h^2 \sum_{N_h}\sum u^2 - \frac{h^2}{12} \iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy + o(h^2) \qquad (h \longrightarrow 0). \end{aligned}$$

Dividing (16) by the denominator of $\rho_h(u)$, one gets

$$\rho_h(u) = \lambda - \frac{h^2}{12} \frac{\iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy}{h^2 \sum\sum_{N_h} u^2} + o(h^2).$$

Hence, by (10),

$$(17) \qquad \rho_h(u) = \lambda - \frac{h^2}{12} \frac{\iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy}{\iint_R u^2\,dxdy} + o(h^2) \qquad (h \longrightarrow 0).$$

If one divides (17) by $\lambda$, and notes from (2) that $\lambda \iint_R u^2 dxdy = \iint_R |\nabla u|^2 dxdy$, it is seen that

$$\frac{\rho_h(u)}{\lambda} = 1 - \frac{h^2}{12} \frac{\iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy}{\iint_R |\nabla u|^2\,dxdy} + o(h^2) \qquad (h \longrightarrow 0).$$

By the definition of $a$ we have proved (9) and hence the theorem.

**4. Some lemmas.** Lemma 1, suggested to the author by Professor Max Shiffman, is used to establish Lemmas 2 to 7, which were applied to prove the theorem. In all the lemmas $R$ is the convex union of squares and half-squares of the network, while $u = u(x, y)$ is a function solving problem (1) in $R$.

LEMMA 1. *The function $u$ is an analytic function of $x$ and $y$ in $R \cup C$, except at the $135°$ vertices of $C$. Let $r$, $\theta$ be local polar coordinates centered at a $135°$ vertex $P_k$, with $0 < \theta < 3\pi/4$ in $R$. Then*

$$(18) \qquad u = \gamma_k r^{4/3} \sin(4\theta/3) + r^{7/3} E_k(r, \theta),$$

*where $\gamma_k$ is a constant, and where $E_k(r, \theta)$, together with all its derivatives, is bounded in a neighborhood of $P_k$.*

*Proof.* By reflection one can continue $u$ antisymmetrically across each straight segment of $C$, and (1a) is satisfied by the extended $u$ at all points of $R \cup C$ except the $135°$ vertices. The first sentence of the lemma then follows from [2, p. 179].

For $(\xi, \eta) \in R$, write $t = \xi + i\eta$. For each $t$, let $w = f(z, t)$ be an analytic function of the complex variable $z = x + iy$ which maps $R$ into the unit circle $|w| < 1$, with $f(t, t) = 0$. To study $f$ near a vertex $z_k$ of $C$, one may assume that $f(z_k, t) = 1$. Let the interior vertex angle of $C$ at $z_k$ be $\pi/\alpha_k$ ($\alpha_k = 4, 2$, or $4/3$). It is a property of the Schwarz-Christoffel transformation [10, p. 189] that

$$f(z,t) = 1 + (z - z_k)^{\alpha_k} g_k(z,t), \tag{19}$$

where $g_k$ is an analytic function of $z$ regular at $z_k$.

Let $G(z,t) = G(x,y;\xi,\eta)$ be Green's function for $\Delta u$ in $R$. Now $G(z,t) = -(2\pi)^{-1} \log |f(z,t)|$; see [10, p. 181]. It then follows from (19) that, in the notation of the lemma, when $\alpha_k = 4/3$,

$$G(z,t) = \gamma_k(t) r^{4/3} \sin(4\theta/3) + r^{7/3} E_k(r,\theta,t). \tag{20}$$

Moreover, $\gamma_k(t)$ and $E_k(r,\theta,t)$ are integrable over $R$, since the only discontinuity of $G(z,t)$ is a logarithmic one at $t = z$.

The function $u$ is representable by the integral [2, pp. 182-3]

$$u(x,y) = \lambda \iint_R G(x,y;\xi,\eta) u(\xi,\eta)\, d\xi d\eta. \tag{21}$$

Substituting (20) into (21) proves (18) and the lemma.

LEMMA 2. *$|\nabla u(x,y)| \longrightarrow 0$ as $(x,y) \longrightarrow$ any $135°$ vertex of $C$.*

*Proof.* By (18), $|\nabla u| = O(r^{1/3})$, as $(x,y) \longrightarrow$ any $135°$ vertex of $C$.

LEMMA 3. *The functions $u_{xx}^2$, $u_x u_{xxx}$, $u u_{xxxx}$, $u_{yy}^2$, $u_y u_{yyy}$, and $u u_{yyyy}$ are Lebesgue-integrable in $R$.*

*Proof.* By Lemma 1 these functions are continuous in $R \cup C$, except at the $135°$ vertices $P_k$. At these vertices (18) implies that they are $O(r^{-4/3})$ and are hence integrable.

LEMMA 4. *The Lebesgue integrals $\int_C u_y u_{yy} dx$ and $\int_C u_x u_{xx} dy$ exist.*

*Proof.* Analogous to that of Lemma 3.

REMARK. Lemmas 2, 3, and 4 are false for polygonal regions $R$ which are not convex, since in general the exponent in (18) is $\alpha_k$, where $\pi/\alpha_k$ is the interior angle at the vertex $P_k$.

LEMMA 5. *At each node $(x,y)$ in $R$ of the network of section 1, one has*

$$(22) \qquad \Delta_h u = \Delta u + \frac{1}{12} h^2 (u'_{xxxx} + u''_{yyyy}),$$

where

$$(23) \qquad \begin{cases} u'_{xxxx} = u_{xxxx}(x + \theta' h, y), \ -1 < \theta' < 1; \\ u''_{yyyy} = u_{yyyy}(x, y + \theta'' h), \ -1 < \theta'' < 1. \end{cases}$$

*Proof.* By Lemma 1, $u_{xxxx}$ is continuous in the open line segment from $(x - h, y)$ to $(x + h, y)$ (though infinite at any 135° vertex). Since $u$ is continuous in $R \cup C$, it follows from Taylor's formula [7, p. 357] that, if we fix $y$ and set $\phi(x) = u(x, y)$,

$$\phi(x + h) + \phi(x - h) - 2\phi(x)$$
$$= h^2 \phi''(x) + \frac{1}{24} h^4 [\phi''''(x + \theta_1 h) + \theta''''(x - \theta_2 h)],$$

where $0 < \theta_i < 1$ $(i = 1, 2)$. By the continuity of $\phi''''$, the last bracket equals $2\phi''''(x + \theta' h)$, where $-1 < \theta' < 1$.

A similar formula for $\psi(y) = u(x, y)$, when added to the above and divided by $h^2$, yields (22) and (23).

LEMMA 6. *Define $N''_h$ as in § 3. For each node $(x, y)$ in $N''_h$, use the notation of (23). Then, as $h \longrightarrow 0$ over values such that (3) holds, one has*

$$(24) \quad h^2 \sum\sum_{N''_h} u(u'_{xxxx} + u''_{yyyy}) = \iint_R u(u_{xxxx} + u_{yyyy}) dx dy + o(1) \quad (h \longrightarrow 0).$$

*Proof.* For all $(x, y)$ in the entire plane $E_2$ define

$$f(x, y) = \begin{cases} u(u_{xxxx} + u_{yyyy}), & \text{if } (x, y) \in R; \\ 0, & \text{elsewhere}. \end{cases}$$

By the proof of Lemma 3 one sees that $f(x, y)$ is $O(r^{-4/3})$ in the neighborhood of each 135° vertex $P_k$ of $C$, and continuous elsewhere. Divide the nodes $(x, y) = (\mu h, \nu h)$ of $N''_h \subset R$ into four classes $K^{(i)}$ $(i = 1, 2, 3, 4)$ according to the parity of $(\mu, \nu)$. Fix any class $K^{(i)}$. For each vertex $(x, y)$ in $K^{(i)}$ let $S(x, y)$ be the union of the four closed network squares of $E_2$ which contain $(x, y)$. The area of each $S(x,y)$ is $4h^2$; ordinarily certain of the $S(x,y)$ contain points not in $R$. Define

$$f_h^{(i)}(\xi,\eta) = \begin{cases} u(x,y)(u'_{xxxx} + u''_{yyyy}), & \text{for } (\xi,\eta) \in S(x,y); \\ 0, & \text{for } (\xi,\eta) \notin \bigcup S(x,y). \end{cases}$$

Then $f_h^{(i)}(\xi,\eta) \longrightarrow f(\xi,\eta)$, as $h \longrightarrow 0$, for almost all $(\xi,\eta)$ in the plane. Using the fact that no node of $N_h''$ is adjacent to a $135°$ vertex of $C$, one can show that for all $i$, uniformly in $h$, $|f_h^{(i)}(\xi,\eta)| \leq F(\xi,\eta)$, where $F$ is an integrable function in $E_2$.

Each term of the sum (24) for which $(x,y) \in K^{(i)}$ is equal to

$$\frac{1}{4}\iint_{S(x,y)} f_h^{(i)}(\xi,\eta)\,d\xi\,d\eta.$$

Hence, applying Lebesgue's convergence theorem, one sees that, as $h \longrightarrow 0$, for each $i$,

$$\begin{aligned} \sum_{N_h'' \cap K^{(i)}}\sum u(u'_{xxxx} + u''_{yyyy}) &= \frac{1}{4}\iint_{E_2} f_h^{(i)}(\xi,\eta)\,d\xi\,d\eta \\ &\longrightarrow \frac{1}{4}\iint_{E_2} f(\xi,\eta)\,d\xi\,d\eta \qquad (h \longrightarrow 0). \end{aligned} \tag{25}$$

Summing (25) over $i = 1, 2, 3, 4$ proves (24) and the lemma.

LEMMA 7. *One has*

$$\iint_R u(u_{xxxx} + u_{yyyy})\,dxdy = \iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy. \tag{26}$$

*Proof.* The following applications of Gauss's divergence theorem in the form

$$\iint_R (p_x + q_y)\,dxdy = \int_C (p\,dy - q\,dx) \tag{27}$$

can be justified by integrating over the region $R^*$ interior to a smooth convex curve $C^*$ inside $R$, and then letting $C^* \longrightarrow C$ appropriately. The continuity of the integrals in the limit follows from Lemmas 1, 3, and 4.

In the divergence theorem for $p = uu_{xxx}$, $q = uu_{yyy}$, the line integral vanishes, and one finds

$$(28) \qquad \iint_R u\,(u_{xxxx} + u_{yyyy})\,dxdy = -\iint_R (u_x u_{xxx} + u_y u_{yyy})\,dxdy.$$

A second application of the divergence theorem with $p = u_x u_{xx}$, $q = u_y u_{yy}$, combined with (28), shows that

$$(29) \qquad \iint_R u\,(u_{xxxx} + u_{yyyy})\,dxdy = \iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy + \Gamma,$$

where $\Gamma = \int_C (u_y u_{yy} dx - u_x u_{xx} dy)$.

By (1a), $u_{xx} = -u_{yy}$ on $C$, whence $\Gamma = \int_C u_{yy}(u_y dx + u_x dy)$. On the segments of $C$ parallel to the axes, $u_{xx} = u_{yy} = 0$, so that there the contribution to $\Gamma$ is zero.

Now the vector $\nabla u = (u_x, u_y)$ is perpendicular to $C$. On the segments of $C$ making a 45° or 135° angle with the $x$-axis, $(u_y, u_x)$ is parallel to $(u_x, u_y)$, whence $(u_y, u_x)$ is perpendicular to $C$. Thus $u_y dx + u_x dy \equiv 0$ when $(dx, dy)$ is tangent to $C$, so that the contribution to $\Gamma$ from these 45° and 135° segments of $C$ is also zero.

Hence $\Gamma = 0$, and the lemma follows from (29).

**5. Numerical example.** Let $R_1$ be the six-sided, nonconvex, $L$-shaped region whose closure is the union of the three unit squares

$$\begin{cases} -1 \le x \le 0, & 0 \le y \le 1; \\ 0 \le x \le 1, & 0 \le y \le 1; \\ 0 \le x \le 1, & -1 \le y \le 0. \end{cases}$$

The fundamental frequencies $\lambda_h = \lambda_h(R_1)$ and corresponding net functions $v$ were computed by B. F. Handy on the SWAC (National Bureau of Standards Western Automatic Computer) for $1/h = 3, 4, \cdots, 8$. The computation used a *power method*; for some initial net function $v_0$, $(h^2\Delta_h + 5I)^m v_0$ was determined for large positive integers $m$, where $I$ is the identity operator. On the basis of Collatz's inclusion theorem [3, p. 289], the values in the accompanying table are believed to have errors less than $5 \times 10^{-6}$. Observe that $\lambda_h(R_1)$ is less for $h = 1/8$ than for $h = 1/7$.

TABLE

| $h$ | $\lambda_h(R_1)$ | $\lambda_h(R_2)$ |
|---|---|---|
| 1/2 | 9.07180 | 12.00000 |
| 1/3 | 9.52514 | 13.73700 |
| 1/4 | 9.64143 | 14.37340 |
| 1/5 | 9.67860 | 14.67081 |
| 1/6 | 9.69083 | 14.83259 |
| 1/7 | 9.69384 | 14.93003 |
| 1/8 | 9.69316 | 14.99315 |

Since $R_1$ is not convex, the theorem of § 2 does not apply, but a heuristic argument suggests that $\lambda_h(R_1) - \lambda(R_1) = O(h^{4/3})$. A least-squares fit to the values of $\lambda_h(R_1)$ for $1/8 \leq h \leq 1/4$ of a function of type

$$\lambda_h(R_1) \doteq \alpha_1 + \beta_1 h^{4/3} + \gamma_1 h^2 = \phi_1(h)$$

yielded the values

$$\alpha_1 = 9.63632, \qquad \beta_1 = 2.40286, \qquad \gamma_1 = -5.97212. \tag{30}$$

The maximum of $|\lambda_h(R_1) - \phi_1(h)|$ for the five values of $h$ is .00013. Hence $\alpha_1$ is a working estimate of $\lambda(R_1)$.

The fact that $\beta_1 > 0$ in (30) supports the author's conjecture that, for nonconvex polygonal domains satisfying (3), $\lambda_h > \lambda$ for all sufficiently small $h$.

The table also gives Handy's values for the second eigenvalues of $R_1$, which are the fundamental eigenvalues $\lambda_h(R_2)$ of the trapezoidal halfdomain $R_2$ of $R_1$ for which $x > y$. Since the theorem does apply to $R_2$, a least-squares fit to the values of $\lambda_h(R_2)$ for $1/8 \leq h \leq 1/4$ of a function of type

$$\lambda_h(R_2) \doteq \alpha_2 + \beta_2 h^2 = \phi_2(h)$$

seemed appropriate, and yielded the values

$$\alpha_2 = 15.19980, \qquad \beta_2 = -13.22219.$$

The maximum of $|\lambda_h(R_2) - \phi_2(h)|$ for the five values of $h$ was .00010. Hence $\alpha_2$ is a working estimate of $\lambda(R_2)$.

The value of $\beta_2$ is negative, in agreement with (6), but the quantity

$-12\beta_2/\alpha_2 = 10.4387$ is something like one-fifth larger than an estimate of the corresponding quantity $a(R_2)$ of the theorem. One therefore suspects that $a$ is not the best possible constant in (6) for the region $R_2$.

In the table, note the relative closeness of the values of $\lambda_h(R_2)$ to the working estimate, $\alpha_2$, of $\lambda(R_2)$, even for a coarse net. Thus the value 12 for $\lambda_{1/2}(R_2)$, which is obtained by pencil and paper from a simple quadratic equation, is comparable to the lower bounds 12.1 and $5\pi^2/4$ obtained respectively by comparison with $\lambda$ for the circular membrane of equal area [**13**, p. 8] and with $\lambda$ for the rectangular region $0 < x < 1$; $-1 < y < 1$. The value $\lambda_{1/3}(R_2) = 13.737$ requires getting the least eigenvalue of a 7th-order matrix, a relatively easy procedure with a desk machine.

The monotonicity of $\lambda_h(R_2)$ supports the author's conjecture[2] that, for the $R$ of the theorem, $\lambda_h < \lambda$ for all $h$.

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[2]See page 470.

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