COMMUTING SPECTRAL MEASURES ON HILBERT SPACE

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1. Introduction. By a "spectral measure" on Hilbert space H we mean a family of bounded operators $E(\sigma)$ on H defined for all Borel sets σ in the plane. We suppose:

(i) If σ_0 denotes the empty set and σ_1 the whole plane, then

$$E(\sigma_0) = 0, \quad E(\sigma_1) = I,$$

where I is the identity.

(ii) For all σ_1, σ_2,

$$E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2);$$

and for disjoint σ_1, σ_2,

$$E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2).$$

(iii) There exists a constant M with $\|E(\sigma)\| \leq M$, all σ. It follows that $E(\sigma)^2 = E(\sigma)$ for each σ, and $E(\sigma_1)E(\sigma_2) = 0$ if σ_1, σ_2 are disjoint.

Mackey has shown in [3], as part of the proof of Theorem 55 of [3], that if $E(\sigma)$ is a spectral measure with the properties just stated, then there exists a bicontinuous operator A such that $A^{-1}E(\sigma)A$ is self-adjoint for every σ. In a special case this result was proved by Lorch in [2]. We shall prove:

THEOREM 1. *Let $E(\sigma)$ and $F(\eta)$ be two commuting spectral measures on H; that is,*

$$E(\sigma)F(\eta) = F(\eta)E(\sigma)$$

for every σ, η. Then there exists a bicontinuous operator A such that $A^{-1}E(\sigma)A$ and $A^{-1}F(\eta)A$ are self-adjoint for every σ, η.

As a corollary of Theorem 1, we shall obtain:

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THEOREM 2. *If T_1, T_2 are spectral operators on H, in the sense of Dunford* [1], *and $T_1 T_2 = T_2 T_1$, then $T_1 + T_2$ and $T_1 T_2$ are again spectral operators.*

2. Lemmas. We shall use two lemmas in proving Theorem 1.

LEMMA 1. *Let $P_1, P_2, \cdots, P_n$ be operators on Hilbert space with*

$$P_i P_j = 0 \qquad (i \neq j), \qquad P_i^2 = P_i, \qquad \sum_{i=1}^{n} P_i = I.$$

Suppose that, for every set $\delta_1, \delta_2, \cdots, \delta_n$ of zeros and ones,

$$\left\| \sum_{i=1}^{n} \delta_i P_i \right\| \leq M.$$

Then for every x we have

$$\frac{1}{4M^2} \|x\|^2 \leq \sum_{i=1}^{n} \|P_i x\|^2 \leq 4M^2 \|x\|^2$$

This Lemma is proved in [3, p. 147]; we include the proof for completeness.

Proof. We note that

$$\sum_{i=1}^{n} \|P_i x\|^2 = \frac{1}{2^n} \sum \|\epsilon_1 P_1 x + \cdots + \epsilon_n P_n x\|^2,$$

where the sum is taken over all possible sets $(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$, where $\epsilon_i = \pm 1$. Hence

$$a_x = \|\epsilon_1' P_1 x + \cdots + \epsilon_n' P_n x\|^2 \leq \sum_{i=1}^{n} \|P_i x\|^2$$

$$\leq \|\epsilon_1 P_1 x + \cdots + \epsilon_n P_n x\|^2 = b_x$$

for some choice of the ϵ_i' and ϵ_i. Now

$$b_x = \left\| \sum_{i=1}^{n} \delta_i^+ P_i x - \sum_{i=1}^{n} \delta_i^- P_i x \right\|^2,$$

where the δ_i^+ and the δ_i^- are 1 or 0.

Hence

$$\sum_{i=1}^{n} ||P_i x||^2 \leq 4M^2 \cdot ||x||^2.$$

Let now $P^+ = \sum P_i$, summed over those i with $\epsilon_i' = 1$; and let $P^- = \sum P_i$, summed over those i with $\epsilon_i' = -1$. Then

$$(P^+ - P^-)^2 = P^+ + P^- = I \text{ and } ||P^+x - P^-x||^2 = a_x.$$

hence

$$||x||^2 = ||(P^+ - P^-)^2 x||^2 \leq ||P^+ - P^-||^2 \cdot ||P^+x - P^-x||^2.$$

Now $||P^+|| \leq M$ and $||P^-|| \leq M$ and so

$$||x||^2 \leq (2M)^2 a_x \leq (2M)^2 \sum_{i=1}^{n} ||P_i x||^2.$$

LEMMA 2. *Let $E(\sigma)$ and $F(\eta)$ be commuting spectral measures on Hilbert space. Then there is a fixed K such that for any set $\sigma_1, \sigma_2, \cdots, \sigma_n$ of disjoint Borel sets, and set $\eta_1, \eta_2, \cdots, \eta_n$ of arbitrary Borel sets,*

$$\left\| \sum_{i=1}^{n} E(\sigma_i) F(\eta_i) \right\| \leq K.$$

Proof. Fix x. By (iii) there exist constants L and M, with $||E(\sigma)|| \leq M$, $||F(\eta)|| \leq L$ for any σ, η. Let σ_{n+1} be the complement of

$$\bigcup_{i=1}^{n} \sigma_i .$$

Then

$$\left\| \sum_{i=1}^{n} E(\sigma_i) F(\eta_i) x \right\|^2 \leq 4M^2 \sum_{\nu=1}^{n+1} \left\| E(\sigma_\nu) \left(\sum_{i=1}^{n} E(\sigma_i) F(\eta_i) x \right) \right\|^2 = C$$

by Lemma 1;

$$C = 4M^2 \sum_{\nu=1}^{n} ||E(\sigma_\nu)F(\eta_\nu)x||^2,$$

since $E(\sigma_\nu)E(\sigma_i) = E(\sigma_\nu \cap \sigma_i)$;

$$C = 4M^2 \sum_{\nu=1}^{n} ||F(\eta_\nu)E(\sigma_\nu)x||^2,$$

by commutativity of the $E(\sigma)$ and $F(\eta)$;

$$C \leq 4M^2 \cdot L^2 \sum_{\nu=1}^{n} ||E(\sigma_\nu)x||^2,$$

since $||F(\eta_\nu)|| \leq L$;

$$C \leq (4M^2)^2 \cdot L^2 ||x||^2,$$

by Lemma 1. Hence

$$\left\| \sum_{i=1}^{n} E(\sigma_i)F(\eta_i) \right\| \leq 4M^2 L.$$

In the proof of Theorem 1 we shall use the method of Mackey in [3], together with Lemmas 1 and 2.

3. Proof of Theorem 1. By a "partition" π of the plane we mean a finite family of Borel sets $\sigma_1, \sigma_2, \cdots, \sigma_n$, mutually disjoint and with union equal to the whole plane. If (x, y) denotes the given scalar product in H, and

$$\pi_1 = (\sigma_i)_{i=1}^{n} \quad \pi_2 = (\eta_j)_{j=1}^{m}$$

are two partitions, set

$$(x, y)_{\pi_1, \pi_2} = \sum_{i=1}^{n} \sum_{j=1}^{m} (E(\sigma_i)F(\eta_j)x, E(\sigma_i)F(\eta_j)y).$$

It is easily verified that the quantity $(x, y)_{\pi_1, \pi_2}$ is a scalar product in H. Further, it follows by Lemma 2 that the operators

$$P_{ij} = E(\sigma_i)F(\eta_j) \qquad (i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m,)$$

satisfy the hypotheses of Lemma 1.

Hence Lemma 1 yields

$$\frac{1}{4K^2} \|x\|^2 \le \sum_{i=1}^{n} \sum_{j=1}^{m} \|E(\sigma_i)F(\eta_j)x\|^2 \le 4K^2\|x\|^2,$$

where K depends only on $\sup_\sigma \|E(\sigma)\|$ and $\sup_\eta \|F(\eta)\|$. But

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \|E(\sigma_i)F(\eta_j)x\|^2 = (x, x)_{\pi_1, \pi_2} = \|x\|^2_{\pi_1, \pi_2}.$$

Finally, each $E(\sigma_i)$ and $F(\eta_j)$ $(i = 1, 2, \cdots n;\ j = 1, 2, \cdots, m)$ is self-adjoint in the scalar product $(x, y)_{\pi_1, \pi_2}$, as is readily verified.

For each pair of vectors $x, y \in H$, now, let S_{xy} be the disk in the complex plane consisting of all z with

$$|z| \le 4K^2\ \|x\| \cdot \|y\|.$$

If S denotes the cartesian product of the disks S_{xy} over all pairs x, y, then S is a compact topological space, by Tychonoff's theorem. Further, as we saw above,

$$\|x\|^2_{\pi_1, \pi_2} \le 4K^2\|x\|^2.$$

Hence by Schwarz's inequality, applied to the scalar product $(x, y)_{\pi_1, \pi_2}$, we see that the number $(x, y)_{\pi_1, \pi_2}$ lies in the disk S_{xy} for every pair x, y. Hence there is a point p_{π_1, π_2} in S whose x, y-coordinate is $(x, y)_{\pi_1, \pi_2}$.

Let us now partially order the set of points p_{π_1, π_2} in S by saying that $p_{\pi_1', \pi_2'}$ is "greater than" p_{π_1, π_2} (in symbols $p_{\pi_1', \pi_2'} > p_{\pi_1, \pi_2}$) if π_1' is a refinement of the partion π_1, and π_2' is a refinement of the partition π_2. This ordering makes the set of points p_{π_1, π_2} in S into a directed system. Since S is a compact space, this directed system has a point of accumulation p. Let $(x, y)_p$ denote the (x, y) coordinate of p.

Then given a finite set of vector pairs (x_i, y_i), $i = 1, 2, \cdots, n$, and $\epsilon > 0$, and a pair π_1^0, π_2^0 of partitions, we have

$$|(x_i, y_i)_p - (x_i, y_i)_{\pi_1, \pi_2}| < \epsilon \qquad (i = 1, 2, \cdots, n)$$

for some

$$p_{\pi_1, \pi_2} > p_{\pi_1^0, \pi_2^0}.$$

Since $(x, y)_{\pi_1, \pi_2}$ is a scalar product for all π_1, π_2 it thus follows that $(x, y)_p$ is a scalar product, and since the norm $\|x\|_{\pi_1, \pi_2}$ is equivalent to the original norm with constants of equivalence independent of π_1, π_2, it follows that

$$\|x\|_p = \sqrt{(x, x)_p}$$

is also equivalent to the original norm.

Finally, fix a Borel set σ and vectors x, y. Let π_1^0 be the partition defined by σ and its complement, and π_2^0 be arbitrary. Then, if

$$p_{\pi_1, \pi_2} > p_{\pi_1^0, \pi_2^0},$$

we have

$$(E(\sigma)x, y)_{\pi_1, \pi_2} = (x, E(\sigma)y)_{\pi_1, \pi_2},$$

since π_1 is a refinement of π_1^0, and so σ is a finite union of sets involved in the partition π_1. Thus

$$(E(\sigma)x, y)_p = (x, E(\sigma)y)_p,$$

and so the $E(\sigma)$ are self-adjoint with respect to the scalar product $(x, y)_p$, and similarly the $F(\eta)$ are self-adjoint with respect to this scalar product.

Since $\|x\|_p$ is equivalent to the given norm, it now follows that there exists a bi-continuous operator A with $(x, y)_p = (Ax, Ay)$, and hence $AE(\sigma)A^{-1}$ and $AF(\eta)A^{-1}$ are all self-adjoint.

4. Proof of Theorem 2. By Theorem 8 of [**1**], an operator T is spectral if and only if there exist two commuting operators S and N such that N is quasi-nilpotent and S admits a representation:

$$S = \int \lambda E(d\lambda),$$

where $E(d\lambda)$ denotes integration with respect to a certain spectral measure.

Such an S is called in [1] a "scalar type operator."

Now, by hypothesis, T_1 and T_2 are commuting spectral operators. We write

$$T_1 = S_1 + N_1, \quad T_2 = S_2 + N_2,$$

in accordance with the preceding. Then by Theorem 5 of [1] the operators S_1, S_2, N_1, N_2 all commute with one another. We thus have

$$T_1 + T_2 = S_1 + S_2 + Q \quad \text{and} \quad T_1 T_2 = S_1 S_2 + Q',$$

where Q and Q' are quasi-nilpotent, Q commutes with $S_1 + S_2$, and Q' commutes with $S_1 S_2$. By Theorem 8, quoted above, it is thus sufficient to show that $S_1 + S_2$ and $S_1 S_2$ are spectral operators of type 0; that is, of scalar type.

Let $E^1(\sigma)$ and $E^2(\sigma)$ be the spectral measures for S_1 and S_2, respectively. By Theorem 5 of [1] it follows, from the fact that $S_1 S_2 = S_2 S_1$, that $E^1(\sigma)$ and $E^2(\sigma)$ commute with one another for all σ. By our Theorem 1, then, there exists an operator A such that the operators $AE^1(\sigma)A^{-1}$ and $AE^2(\sigma)A^{-1}$ are all self-adjoint. Hence

$$J_1 = AS_1A^{-1} \quad \text{and} \quad J_2 = AS_2A^{-1}$$

are normal operators. Also $J_1 J_2 = J_2 J_1$, since $S_1 S_2 = S_2 S_1$. It follows that $J_1 + J_2$ and $J_1 J_2$ are again normal operators, for they commute with their adjoints as we verify by direct computation, using the fact that J_1 and J_2^* commute and J_2 and J_1^* commute, since J_1 and J_2 commute.

Thus $A(S_1 + S_2)A^{-1}$ and $A(S_1 S_2)A^{-1}$ are normal operators and so of scalar type. But if J is a scalar type operator and A bi-continuous, then, as is easily seen, $A^{-1}JA$ is again a scalar type operator. Hence $S_1 + S_2$ and $S_1 S_2$ are scalar type operators, and all is proved.

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