ORTHONORMAL CYCLIC GROUPS

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In an earlier paper [1] a characterization was given of the Walsh functions in terms of their group structure and orthogonality. The object of the present note is to present a similar result concerning the complex exponentials.

THEOREM. *Let* $\{A_n(x)\}$ $(n = 0, \pm 1, \cdots; 0 \le x \le 1)$ *be a set of complex-valued measurable functions which is a multiplicative cyclic group. A necessary and sufficient condition that* $\{A_n(x)\}$ *be an orthonormal system over* $0 \le x \le 1$ *is that the generator of the group admit a representation* $\exp(2\pi i\, c(x))$ *almost everywhere, with* $c(x)$ *equimeasurable with* x.

As the sufficiency is immediate, we present only the proof of the necessity. Let the notation be chosen so that the generator of the group is $A_1(x)$, and

$$A_n(x) = (A_1(x))^n \qquad (n = 0, \pm 1, \cdots).$$

The normality implies $|A_1(x)| = 1$ almost everywhere. Hence there is a measurable $a(x)$, $0 \le a(x) < 1$, such that

$$A_1(x) = \exp(2\pi i\, a(x))$$

almost everywhere. Let $b(x)$ be a function [2, p. 207] monotonically increasing and equimeasurable with $a(x)$. Also let

$$c(x) = m\{u : 0 \le u \le 1,\ b(u) \le x\} \qquad (-\infty < x < \infty).$$

The orthonormal condition becomes

$$\delta_{0,n} = \int_0^1 \exp(2\pi ni\ b(x))dx = \int_{-\infty}^{\infty} \exp(2\pi niy)\, dc(y),$$

where the latter integral is a Lebesgue-Stieltjes integral. Thus for any $\epsilon > 0$,

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$$\delta_{0,n} = \int_{b(0)-\epsilon}^{b(1)} \exp(2\pi niy)\, dc(y)$$

$$= \int_{b(0)}^{b(1)} \exp(2\pi niy)\, dc(y) + \exp(2\pi ni\, b(0))\ \ m\{x : b(x) = b(0)\},$$

and the latter integral is interpretable as a Riemann-Stieltjes integral.

Integration by parts yields

$$(1) \qquad \delta_{0,n} = \exp(2\pi ni\, b(1)) - 2\pi ni \int_{b(0)}^{b(1)} c(y) \exp(2\pi niy)\, dy.$$

If $f(y) = y,\ 0 < y \leq 1$, and $f(y+1) = f(y)$, a direct calculation shows that

$$(2) \qquad \delta_{0,n} = \exp(2\pi ni\, b(1)) - 2\pi ni \int_0^1 f(y - b(1)) \exp(2\pi niy)\, dy.$$

Formulas (1) and (2), and the completeness of the complex exponentials, imply the existence of a constant k such that for almost all y, $0 < y \leq 1$,

$$f(y - b(1)) + k = \begin{cases} 0, & 0 < y \leq b(0) \\ c(y), & b(0) < y \leq b(1) \\ 0, & b(1) < y \leq 1. \end{cases}$$

Since the supremum of $c(y)$ is one, and $f(y)$ has no interval of constancy, one infers that $k = 0$, $b(0) = 0$, and $b(1) = 1$. Thus $c(y) = y$, $0 < y \leq 1$, which is equivalent to the proposition that was asserted.

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