A NOTE ON ORTHOGONAL SYSTEMS

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1. Let $\omega_n(x)$, $n=1, 2, \cdots$, $0 \leqq x \leqq 1$, be an orthonormal set of functions which are uniformly bounded,

$$(1) \qquad |\omega_n(x)| \leqq M \qquad (n=1, 2, \cdots, 0 \leqq x \leqq 1).$$

If $\sum_1^\infty |a_n| < \infty$, and if $\int_0^1 |g(x)| dx < \infty$ we may define

$$(2) \qquad f(x) = \sum_{n=1}^\infty a_n \omega_n(x), \qquad b_n = \int_0^1 g(x) \overline{\omega_n(x)} dx.$$

The following inequalities were established by R. E. A. C. Paley [1]:

$$(3) \qquad \begin{aligned} &\left[\int_0^1 |f(x)|^q dx\right]^{1/q} \leqq A_1'(q) \left[\sum_{n=1}^\infty |a_n|^q n^{q-2}\right]^{1/q} && (2 \leqq q < \infty); \\ &\left[\sum_{n=1}^\infty |b_n|^p n^{p-2}\right]^{1/p} \leqq A_2'(p) \left[\int_0^1 |g(x)|^p dx\right]^{1/p} && (1 < p \leqq 2); \\ &\left[\int_0^1 |f(x)|^p x^{p-2} dx\right]^{1/p} \leqq A_3'(p) \left[\sum_{n=1}^\infty |a_n|^p\right]^{1/p} && (1 < p \leqq 2); \\ &\left[\sum_{n=1}^\infty |b_n|^q\right]^{1/q} \leqq A_4'(q) \left[\int_0^1 |g(x)|^q x^{q-2} dx\right]^{1/q} && (2 \leqq q < \infty). \end{aligned}$$

In the present paper we shall establish some related results which are however a great deal simpler. We shall prove that

$$(4') \qquad \left[\int_0^1 |f(x)|^2 x^{-2\alpha} dx\right]^{1/2} \leqq A_1(\alpha) \left[\sum_{n=1}^\infty |a_n|^2 n^{2\alpha}\right]^{1/2} \qquad (0 \leqq \alpha < 1/2);$$

$$(4'') \qquad \left[\sum_{n=1}^\infty |b_n|^2 n^{-2\alpha}\right]^{1/2} \leqq A_2(\alpha) \left[\int_0^1 |g(x)|^2 x^{2\alpha} dx\right]^{1/2} \qquad (0 \leqq \alpha < 1/2).$$

As Paley pointed out, the inequalities (3) include the inequalities of F. Riesz which assert that

$$(5) \qquad \begin{aligned} &\left[\int_0^1 |f(x)|^q dx\right]^{1/q} \leqq B(p) \left[\sum_{n=1}^\infty |a_n|^p\right]^{1/p}, \\ &\left[\sum_{n=1}^\infty |b_n|^q\right]^{1/q} \leqq B(p) \left[\int_0^1 |g(x)|^p dx\right]^{1/p} \qquad (1 \leqq p \leqq 2,\ 1/p + 1/q = 1). \end{aligned}$$

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(The best values of the constants $B(p)$ cannot be obtained by this argument however). The inequalities (4′) and (4″) also include (5) (again not with the best values for $B(p)$). The demonstration of Riesz's theorem which one obtains in this way is unusually simple.

2. **We now proceed** to the demonstration of the inequalities 1(4). We assert that $A_1(\alpha)=A_2(\alpha)$ and thus that either of the inequalities implies the other. Suppose that the inequality 1(4′) holds. We define

$$F_N(x)=\sum_{n=1}^{N} b_n n^{-2\alpha}\omega_n(x)\,.$$

By assumption

$$\int_0^1 |F_N(x)|^2 x^{-2\alpha}\,dx \leqq A_1^2(\alpha)\sum_{n=1}^{N}|b_n n^{-2\alpha}|^2 n^{2\alpha} = A_1^2(\alpha)\sum_{n=1}^{N}|b_n|^2 n^{-2\alpha}\,.$$

We have

$$\sum_{n=1}^{N}|b_n|^2 n^{-2\alpha}=\int_0^1 F_N(x)g(x)dx \leqq \left[\int_0^1 |F_N(x)|^2 x^{2\alpha}\,dx\right]^{1/2}\left[\int_0^1 |g(x)|^2 x^{-2\alpha}dx\right]^{1/2}$$

$$\leqq\left[A_1^2(\alpha)\sum_{n=1}^{N}|b_n|^2 n^{-2\alpha}\right]^{1/2}\left[\int_0^1 |g(x)|^2 x^{2\alpha}\,dx\right]^{1/2},$$

$$\left[\sum_{n=1}^{N}|b_n|^2 n^{-2\alpha}\right]^{1/2}\leqq A_1(\alpha)\left[\int_0^1 |g(x)|^2 x^{2\alpha}\,dx\right]^{1/2}.$$

Allowing N to increase without limit we see that $A_2(\alpha)\leqq A_1(\alpha)$. Suppose now that 1(4″) holds. Set

$$b_n=\int_0^1 f(x)\,x^{-2\alpha}\overline{\omega_n(x)}dx\,.$$

By assumption

$$\sum_{n=1}^{\infty}|b_n|^2 n^{-2\alpha}\leqq A_2^2(\alpha)\int_0^1 |f(x)x^{-2\alpha}|^2 x^{2\alpha}dx=A_2^2(\alpha)\int_0^1 |f(x)|^2 x^{-2\alpha}dx\,.$$

We have

$$\int_0^1 |f(x)|^2 x^{-2\alpha}\,dx=\int_0^1\left[\sum_{n=1}^{\infty}a_n\omega_n(x)\right]\overline{f(x)}x^{-2\alpha}dx=\sum_{n=1}^{\infty}a_n\int_0^1\overline{f(x)}x^{-2\alpha}\omega_n(x)dx$$

$$=\sum_{n=1}^{\infty}a_n\bar{b}_n\leqq\left[\sum_{n=1}^{N}|a_n|^2 n^{2\alpha}\right]^{1/2}\left[\sum_{n=1}^{\infty}|b_n|^2 n^{-2\alpha}\right]^{1/2}$$

$$\leqq\left[A_2^2(\alpha)\int_0^1 |f(x)|^2 x^{-2\alpha}\,dx\right]^{1/2}\left[\sum_{n=1}^{\infty}|a_n|^2 n^{2\alpha}\right]^{1/2},$$

$$\left[\int_0^1 |f(x)|^2 x^{-2\alpha}\,dx\right]^{1/2}\leqq A_2(\alpha)\left[\sum_{n=1}^{\infty}|a_n|^2 n^{2\alpha}\right]^{1/2},$$

and thus $A_1(\alpha)\leqq A_2(\alpha)$. Since $A_1(\alpha)=A_2(\alpha)$ we may write $A(\alpha)$ for $A_1(\alpha)$

and $A_\lambda(\alpha)$.

It is evidently sufficient to prove either 1(4′) or 1(4″). We shall prove 1(4′). There are four cases: (i) $\alpha=0$, (ii) $0<\alpha<1/4$, (iii) $\alpha=1/4$ and (iv) $1/4<\alpha<1/2$.

Case (i). The desired conclusion follows from Bessel's inequality.

To demonstrate the remaining cases we set

$$\Omega_\mu(x)=\sum_{2^{\mu-1}}^{2^\mu-1} a_n\omega_n(x), \qquad W_\mu=\sum_{2^{\mu-1}}^{2^\mu-1}|a_n|^2n^{2\alpha},$$

$$I_{\nu\mu}=\int_0^1 |\Omega_\mu(x)\Omega_\nu(x)|\, x^{-2\alpha}dx.$$

We further define

$$I_{\nu\mu}^{(1)}=\int_0^\varepsilon |\Omega_\mu(x)\Omega_\nu(x)|\, x^{-2\alpha}dx, \qquad I_{\nu\mu}^{(2)}=\int_\varepsilon^1 |\Omega_\mu(x)\Omega_\nu(x)|\, x^{-2\alpha}dx.$$

We begin by proving two inequalities we shall use repeatedly:

$$\begin{aligned}\operatorname*{l.u.b.}_{0\leqq x\leqq 1} |\Omega_\mu(x)| &\leqq M\sum_{2^{\mu-1}}^{2^\mu-1}|a_n| \\ &\leqq M\left[\sum_{2^{\mu-1}}^{2^\mu-1}|a_n|^2n^{2\alpha}\right]^{1/2}\left[\sum_{2^{\mu-1}}^{2^\mu-1}n^{-2\alpha}\right]^{1/2} \qquad (1)\\ &\leqq AW_\mu^{1/2}2^{-\alpha\mu+\mu/2};\end{aligned}$$

$$\begin{aligned}\left[\int_0^1|\Omega_\mu(x)|^2dx\right]^{1/2} &= \left[\sum_{2^{\mu-1}}^{2^\mu-1}|a_n|^2\right]^{1/2} \qquad (2)\\ &\leqq AW_\mu^{1/2}2^{-\alpha\mu}.\end{aligned}$$

Here and later A will be any constant depending only on M and α.

Case (ii). Suppose that $\nu\geqq\mu$. We have

$$\begin{aligned}I_{\nu\mu}^{(1)} &\leqq \left[\operatorname*{l.u.b.}_{0\leqq x\leqq\varepsilon}|\Omega_\mu(x)|\right]\left[\int_0^\varepsilon x^{-4\alpha}dx\right]^{1/2}\left[\int_0^\varepsilon|\Omega_\nu(x)|^2dx\right]^{1/2}\\ &\leqq A\varepsilon^{1/2-2\alpha}W_\mu^{1/2}W_\nu^{1/2}2^{-\alpha\mu-\alpha\nu+\mu/2},\end{aligned}$$

and

$$\begin{aligned}I_{\nu\mu}^{(2)} &\leqq \left[\operatorname*{l.u.b.}_{\varepsilon\leqq x\leqq 1}x^{-2\alpha}\right]\left[\int_\varepsilon^1|\Omega_\mu(x)|^2dx\right]^{1/2}\left[\int_\varepsilon^1|\Omega_\nu(x)|^2dx\right]^{1/2}\\ &\leqq A\varepsilon^{-2\alpha}W_\mu^{1/2}W_\nu^{1/2}2^{-\alpha\mu-\alpha\nu}.\end{aligned}$$

Setting $\varepsilon=2^{-\mu}$ we find that (for all μ and ν)

$$I_{\nu\mu} \leqq A W_{\mu}^{1/2} W_{\nu}^{1/2} 2^{-\alpha|\nu-\mu|}.$$

Since $f(x)=\sum_{\mu=0}^{\infty}\Omega_{\mu}(x)$ we have

$$\int_0^1 |f(x)|^2 x^{-2\alpha}\, dx \leqq \sum_{\mu,\,\nu=0}^{\infty} I_{\nu\mu} \leqq A \sum_{\mu,\,\nu=0}^{\infty} W_{\mu}^{1/2} W_{\nu}^{1/2} 2^{-\alpha|\nu-\mu|}$$

$$\leqq A\left[\sum_{\mu=1}^{\infty} W_{\mu} \sum_{\nu=0}^{\infty} 2^{-\alpha|\nu-\mu|}\right]^{1/2}\left[\sum_{\nu=1}^{\infty} W_{\nu} \sum_{\mu=0}^{\infty} 2^{-\alpha|\nu-\mu|}\right]^{1/2}.$$

We have

$$\sum_{\mu=1}^{\infty} 2^{-\alpha|\nu-\mu|} \leqq A$$

from which it follows that

$$\int_0^1 |f(x)|^2 x^{-2\alpha}\, dx \leqq A \sum_{n=1}^{\infty} |a_n|^2 n^{2\alpha}.$$

Case (iii): $\alpha=1/4$. Suppose that $\nu \geqq \mu$. We have

$$I_{\nu\mu}^{(1)} \leqq \left[\operatorname*{l.u.b.}_{0\leqq x\leqq \varepsilon} |\Omega_{\mu}(x)|\right]\left[\operatorname*{l.u.b.}_{0\leqq x\leqq \varepsilon} |\Omega_{\nu}(x)|\right]\int_0^{\varepsilon} x^{-1/2} dx$$

$$\leqq A\varepsilon^{1/2} W_{\mu}^{1/2} W_{\nu}^{1/2} 2^{(\mu+\nu)/4};$$

$$I_{\nu\mu}^{(2)} \leqq \left[\int_{\varepsilon}^1 |\Omega_{\nu}(x)|^2\, dx\right]^{1/2}\left[\int_{\varepsilon}^1 x^{-1}\, |\Omega_{\mu}(x)|^2\, dx\right]^{1/2}$$

$$\leqq \left[\int_{\varepsilon}^1 |\Omega_{\nu}(x)|^2 dx\right]^{1/2}\left[\operatorname*{l.u.b.}_{\varepsilon\leqq x\leqq 1} |\Omega_{\mu}(x)|\right]^{1/2}\left[\int_{\varepsilon}^1 x^{-1}\, |\Omega_{\mu}(x)|\, dx\right]^{1/2}$$

$$\leqq \left[\int_{\varepsilon}^1 |\Omega_{\nu}(x)|^2 dx\right]^{1/2}\left[\operatorname*{l.u.b.}_{\varepsilon\leqq x\leqq 1} |\Omega_{\mu}(x)|\right]^{1/2}\left[\int_{\varepsilon}^1 |\Omega_{\mu}(x)|^2 dx\right]^{1/4}\left[\int_{\varepsilon}^1 x^{-2} dx\right]^{1/4}$$

$$\leqq A\varepsilon^{-1/4} W_{\mu}^{1/2} W_{\nu}^{1/2} 2^{-\nu/4}.$$

Choosing $\varepsilon=2^{-\mu/3-2\nu/3}$ we obtain (for all μ and ν)

$$I_{\nu\mu} \leqq A W_{\mu}^{1/2} W_{\nu}^{1/2} 2^{-|\nu-\mu|/12},$$

and the proof may be completed as before.

Case (iv): $1/4<\alpha<1/2$. We again suppose $\nu \geqq \mu$. We have

$$I_{\nu\mu}^{(1)} \leqq \left[\operatorname*{l.u.b.}_{0\leqq x\leqq \varepsilon} |\Omega_{\mu}(x)|\right]\left[\operatorname*{l.u.b.}_{0\leqq x\leqq \varepsilon} |\Omega_{\nu}(x)|\right]\int_0^{\varepsilon} x^{-2\alpha} dx$$

$$\leqq A\varepsilon^{1-2\alpha} W_{\mu}^{1/2} W_{\nu}^{1/2} 2^{-\alpha\mu-\alpha\nu+\mu/2+\nu/2};$$

$$I_{\nu\mu}^{(2)} \leqq \left[\operatorname*{l.u.b.}_{\varepsilon\leqq x\leqq 1} |\Omega_{\mu}(x)|\right]\left[\int_{\varepsilon}^1 x^{-4\alpha} dx\right]^{1/2}\left[\int_{\varepsilon}^1 |\Omega_{\nu}(x)|^2 dx\right]^{1/2}$$

$$\leqq A\varepsilon^{1/2-2\alpha}W_\mu^{1/2}W_\nu^{1/2}2^{-\alpha\mu-\alpha\nu+\mu/2}.$$

Choosing $\varepsilon=2^{-\nu}$ we find that (for all μ and ν)

$$I_{\mu\nu}\leqq AW_\mu^{1/2}W_\nu^{1/2}2^{(1/2-\alpha)|\nu-\mu|}.$$

The proof of the inequality 1(4) is now complete.

It is evident that 1(4′) remains valid if the condition $\sum_1^\infty |a_n|<\infty$ is abandoned, provided that $f(x)$ is interpreted as limit in the mean.

Let $a_1, a_2, \cdots$ be a sequence of complex constants which approach 0 as n approaches ∞. We denote by $a_1^*, a_2^*, \cdots$ the sequence $|a_1|$, $|a_2|, \cdots$ arranged in non-increasing order. Let $f(x)$ be a complex valued measurable function defined on [0,1]. We denote by $f^*(x)$ the function equimeasurable with $|f(x)|$ and non-increasing. A simple and well-known argument, see [2; pp. 207–211], enables us to restate our inequalities in the stronger form,

$$(3') \qquad \int_0^1 [f^*(x)]^2x^{-2\alpha}dx\leqq A(\alpha)\sum_{n=1}^\infty [a_n^*]^2n^{2\alpha} \qquad (0\leqq\alpha<1/2);$$

$$(3'') \qquad \sum_{n=1}^\infty [a_n^*]^2n^{-2\alpha}\leqq A(\alpha)\int_0^1 [f^*(x)]^2x^{-2\alpha}dx. \qquad (0\leqq\alpha<1/2).$$

3. **We now deduce** the first of Riesz's inequalities. Let $b_1, b_2 \cdots$ be given such that $B=(\sum_1^\infty |b_n|^p)^{1/p}$ is finite where $1<p\leqq 2$. We may write $B^p=\sum_1^\infty[b_n^*]^p$.

Since b_n^* is non-increasing $n[b_n^*]^p\leqq B^p$ or $b_n^*\leqq n^{-1/p}B$. It follows that

$$\sum_{n=1}^\infty [b_n^*]^2n^{(2-p)/p}\leqq B^{2-p}\sum_{n=1}^\infty [b_n^*]^p=B^2.$$

By 2(3′) we have, if $f(x)=\sum_{n=1}^\infty b_n\omega_n(x)$,

$$\int_0^1 [(f^*(x)]^2x^{-(2-p)/p}dx\leqq\left[A\left(\frac{2-p}{2p}\right)\right]^2B^2.$$

Let $F=\left[\int_0^1 |f(x)|^q dx\right]^{1/q}$ where $p^{-1}+q^{-1}=1$. We have $F^q=\int_0^1[f^*(x)]^q dx$.

Since $f^*(x)$ is non-increasing $x[f^*(x)]^q\leqq F^q$ or $f^*(x)\leqq x^{-1/q}F$. It follows that

$$F^2=F^{2-q}\int_0^1 [f^*(x)]^q dx\leqq\int_0^1 [f^*(x)]^2x^{-(q-2)/q}dx.$$

Since $(q-2)/q=(2-p)/p$ we obtain

$$\left[\int_0^1 |f(x)|^q dx\right]^{1/q} \leq B(p)\left[\sum_1^\infty |b_n|^p\right]^{1/p},$$

where $B(p)$ may be taken as $\left[A\left(\frac{2-p}{2p}\right)\right]$. A similar argument serves to establish the other Riesz inequality.

4. It is natural to conjecture the existence of a general inequality which includes Paley's inequalities 1(3) and the inequalities 1(4). We shall prove that

$$(1)\quad \int_0^1 |f(x)|^r x^{-r\gamma}\, dx \leq A_1''(r, \gamma) \sum_{n=1}^\infty |a_n|^r n^{r-2+r\gamma} \qquad (2 \leq r < \infty,\ 0 \leq \gamma < 1/r),$$

$$(2)\quad \sum_{n=1}^\infty |a_n|^r n^{-r\gamma} \leq A_2''(r, \gamma) \int_0^1 |f(x)|^r x^{r-2+r\gamma} dx \qquad (2 \leq r < \infty,\ 0 \leq \gamma < 1/r),$$

$$(3)\quad \int_0^1 |f(x)|^r x^{r-2-r\gamma}\, dx \leq A_3''(r, \gamma) \sum_{n=1}^\infty |a_n|^r n^{r\gamma} \qquad (1 < r \leq 2,\ 0 \leq \gamma < 1-1/r),$$

$$(4)\quad \sum_{n=1}^\infty |a_n|^r n^{r-2-r\gamma} \leq A_4''(r, \gamma) \int_0^1 |f(x)|^r x^{r\gamma} dx \qquad (1 < r \leq 2,\ 0 \leq \gamma < 1-1/r).$$

Let us prove (1). Choose q, $2 < r < q$. We have

$$\left[\int_0^1 |f(x)|^2 x^{-2\alpha}\, dx\right]^{1/2} \leq A(\alpha)\left[\sum_{n=1}^\infty |a_n|^2 n^{2\alpha}\right]^{1/2},$$

$$\left[\int_0^1 |f(x)|^q\, dx\right]^{1/2} \leq A_1'(q)\left[\sum_{n=1}^\infty |a_n|^q n^{q-2}\right]^{1/q}.$$

We write (formally)

$$|g(x)|^2 M(x) = |f(x)|^2 x^{-2\alpha}, \quad |g(x)|^q M(x) = |f(x)|^q.$$

These relations suggest that we define $g(x)$ and $M(x)$ by

$$g(x) = f(x) x^{2\alpha/(q-2)}, \quad M(x) = x^{-2\alpha q/(q-2)}.$$

Similarly from the (formal) relations

$$|b_n|^2 m(n) = |a_n|^2 n^{2\alpha}, \quad |b_n|^q m(n) = |a_n|^q n^{q-2},$$

we are lead to the definitions

$$b_n = a_n n^{1-2\alpha/(q-2)}, \quad m(n) = n^{2(q\alpha-q+2)/(q-2)}.$$

The mapping $T\{b_n\}_1^\infty = g(x)$ is a linear transformation, and we have

$$\int_0^1 |g(x)|^2 M(x)\, dx \leq A(\alpha) \sum_{n=1}^\infty |b_n|^2 m(n),$$

$$\int_0^1 |g(x)|^q M(x) dx \leq A'(q) \sum_{n=1}^\infty |b_n|^q m(n).$$

By the Riesz interpolation theorem,

$$(5)\qquad \left[\int_0^1 |g(x)|^r M(x)\,dx\right]^{1/r} \leqq C(\alpha, r, q)\left[\sum_{n=1}^{\infty} |b_n|^r m(n)\right]^{1/r}.$$

Now

$$\int_0^1 |g(x)|^r M(x)\,dx = \int_0^1 |f(x)|^r x^{2\alpha r/(q-2)} x^{-2\alpha q/(q-2)} dx\,,$$

$$\sum_{n=1}^{\infty} |b_n|^r m(n) = \sum_{n=1}^{\infty} |a_n|^r n^r n^{-2\alpha r/(q-2)} n^{2(q\alpha-q+2)/(q-2)}\,.$$

If γ is defined by the equation

$$(6)\qquad r\gamma = -\frac{2\alpha(r-q)}{q-2}$$

then (5) can be rewritten as

$$\int_0^1 |f(x)|^r x^{-r\gamma}\,dx \leqq C(\alpha, r, q)\sum_{n=1}^{\infty} |a_n|^r n^{r-2+r\gamma}\,.$$

It is evident from (6) that, by properly choosing α and q, γ can assume any value in the range $0 \leqq \gamma < 1/r$. Thus we have established (1). The relations (2), (3) and (4) can be dealt with similarly. For the special case of Fourier series these inequalities have been established by H. R. Pitt [3].

The "$*$" forms of these inequalities are also true.

5. In the present section we shall prove a result which is a slight variant of the Riesz-Thorin convexity theorem. While this is probably known I have not been able to find a reference for it.

Let $[T_{ij}]$ $(i=1, \cdots, m; j=1, \cdots, n)$ be a complex matrix, and let

$$a_i = \sum_{j=1}^{n} T_{ij} b_j \qquad (i=1, \cdots, m)\,.$$

Let μ_i, σ_i be positive for $i=1, \cdots, m$ and ν_j, τ_j be positive for $j=1, \cdots, n$. For $1 \leqq p, q \leqq \infty$ let

$$(1)\qquad A(\alpha, \beta) = \text{l.u.b.}\left[\sum_{i=1}^{m} |a_i|^p \mu_i^{p\alpha} \sigma_i\right]^{1/p},$$

where the least upper bound is extended over all sets $(b_1, \cdots, b_n)$ such that

$$(2)\qquad \left[\sum_{j=1}^{n} |b_j|^q \nu_j^{q\beta} \tau_j\right]^{1/q} = 1\,.$$

We assert that $\log A(\alpha, \beta)$ is convex for $-\infty < \alpha, \beta < \infty$; that is, if

$$\alpha=(1-\theta)\alpha_1+\theta\alpha_2, \quad \beta=(1-\theta)\beta_1+\theta\beta_2 \qquad (0<\theta<1), \tag{3}$$

then

$$A(\alpha, \beta)\leq A(\alpha_1, \beta_1)^{1-\theta}A(\alpha_2, \beta_2)^{\theta}. \tag{4}$$

To prove this let $(b_1, \cdots, b_n)$ be fixed, such that (2) is satisfied and let $(c_1, \cdots, c_m)$ be such that

$$\left[\sum_{i=1}^{m}|c_i|^{p'}\mu_i^{-p'\alpha}\sigma_i\right]^{1/p'}=1, \qquad 1/p+1/p'=1. \tag{5}$$

Consider

$$f(w)=\sum_{i=1}^{m}\left[\sum_{j=1}^{n}T_{ij}b_j\nu_j^{\beta-\beta_1+w(\beta_1-\beta_2)}\right]c_i\mu_i^{-\alpha+\alpha_1+w(-\alpha_1+\alpha_2)}\sigma_i .$$

The function $f(w)$ is entire and is bounded in every vertical strip. Let us set

$$b_j(w)=b_j\nu_j^{\beta-\beta_1+w(\beta_1-\beta_2)}, \quad c_i(w)=c_i\mu_i^{-\alpha+\alpha_1+w(-\alpha_1+\alpha_2)},$$

$$a_i(w)=\sum_{j=1}^{n}T_{ij}b_j(w),$$

so that

$$f(w)=\sum_{i=1}^{m}a_i(w)c_i(w)\sigma_i .$$

We have

$$\left[\sum_{j=1}^{n}|b_j(iv)|^q\nu_j^{q\beta_1}\tau_j\right]^{1/q}=1,$$

and thus

$$\left[\sum_{i=1}^{m}|a_i(iv)|^p\mu_i^{p\alpha_1}\sigma_i\right]^{1/p}\leq A(\alpha_1, \beta_1);$$

further

$$\left[\sum_{i=1}^{m}|c_i(iv)|^{p'}\mu_i^{-p'\alpha_1}\sigma_i\right]^{1/p'}=1.$$

Applying Hölder's inequality we obtain

$$|f(iv)|\leq A(\alpha_1, \beta_1).$$

We may similarly show that

$$|f(1+iv)|\leq A(\alpha_2, \beta_2).$$

By the three lines theorem

$$|f(\theta)| \leqq A(\alpha_1, \beta_1)^{1-\theta} A(\alpha_2, \beta_2)^\theta$$

which is equivalent to the inequality

$$\left|\sum_{i=1}^{m} a_i c_i \sigma_i\right| \leqq A(\alpha_1, \beta_1)^{1-\theta} A(\alpha_2, \beta_2)^\theta .$$

Since this holds for all $(c_1, c_2, \cdots, c_m)$ satisfying (5) this implies that

$$\left[\sum_{i=1}^{m} |a_i|^p \mu_i^{p\alpha} \sigma_i\right]^{1/p} \leqq A(\alpha_1, \beta_1)^{1-\theta} A(\alpha_2, \beta_2)^\theta ,$$

thus verifying our assertion. We have tacitly assumed above that p, $q < \infty$. The case where p or q or both are ∞ cae be dealt with by passing to the limit.

We shall now apply this to show that if $A(\alpha)$ is defined as in § 2 then $\log A(\alpha)$ is convex. Let

$$T_{ij} = \int_{(j-1)/n}^{j/n} \omega_i(x) dx \qquad (i=1, \cdots, m; j=1, \cdots, n)$$

and let $f(x)$ be a step function taking the value b_j for $(j-1)/n \leqq x < j/n$. If $a_i = \int_0^1 f(x) \omega_i(x) dx$ then

$$a_i = \sum_{j=1}^{m} T_{ij} b_j .$$

Let

$$A_{m,n}(\alpha) = \text{l.u.b.} \left[\sum_{i=1}^{m} |a_i|^2 i^{-2\alpha}\right]^{1/2}$$

the least upper bound being taken over all $b_1, \cdots, b_n$ such that

$$\left[\sum_{j=1}^{n} |b_j|^2 n^{-1} (j/n)^{-2\alpha}\right]^{1/2} = 1 .$$

For every m and n, $A_{m,n}$ is a logarithmically convex function of α. We have

$$\lim_{m, n \to \infty} A_{m,n}(\alpha) = A(\alpha)$$

and from this if follows that $A(\alpha)$ is logarithmically convex.

Because of this fact it is sufficient in § 2 to deal only with cases (i) and (iv), since (ii) and (iii) then follow by interpolation.

References

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