ON LINEAR SYSTEMS WITH INTEGRAL VALUED SOLUTIONS

I. HELLER

1. Introduction. We consider a system of linear equations and inequalities in k variables

$$Ax=b\,, \quad x \geqq o\,, \tag{1.1}$$

where the matrix A has r rows, k columns, and rank less than k.

Assuming the system consistent, the solution set is a convex polyhedron P in k-space. A solution x^0 that satisfies k independent relations of (1.1) as equations, is a vertex of P, and conversely. Such solution is generally called basic or extremal, and is equivalently defined by the property, that the columns of A corresponding to nonzero coordinates of x^0 are independent. Basic solutions are of particular interest in problems where a linear functional is extremised over P, the extremum then being assumed at a vertex or at all points of a positive dimensional face F of P, that is, the convex hull of the vertices of F. In such problems the interest is often restricted to the integral valued basic solutions as the only ones that have meaning in the application. Now given P, any vertex of P can appear as solution of an extremum problem for some linear functional, and a question of interest is: when, that is for which systems (1.1), are all the vertices of P integral valued.

Directing the attention to the system

$$Ax=b\,, \tag{1.2}$$

we may, slightly generalizing, respectively specializing, carry over the definition and the question:

(1.3) DEFINITION. A solution x^0 of (1.2) is *basic*, when its nonzero coordinates correspond to linearly independent columns of A.

(1.4) QUESTION. Which systems (1.2) have all their basic solutions integral valued?

Obviously (1.4) is not equivalent to the same question for systems (1.1); the basic solutions of (1.2) contain those of (1.1); but they may also contain others, namely such with negative integral coordinates. Hence (1.4) asks more and will therefore yield a smaller family of

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systems as answer.

A further specialization in the same direction is obtained, when the attention is restricted to the matrix A above and the question varied as follows:

QUESTION. Which matrices A have the property that

(1.5) whenever b is such that (1.2) has an integral solution (that is whenever b belongs to the integral span of A), then all basic solutions of (1.2) are integral?

The subject of this note is precisely the question above, which will receive a partial answer.

We note first that (1.5) is equivalent to

(1.6) If a column of A is a linear combination of a set of independent columns of A, then the coefficients in the combination are integers.

The proof is nearly obvious: If d is a column of A, d is certainly in the integral span of A; hence, when A satisfies (1.5), the basic solutions of $Ax=d$ are integral, which is precisely (1.6). Conversely, if A satisfies (1.6), let x^0 be some (not necessarily basic) integral and y^0 an arbitrary basic solution of (1.2); let B and C be the set of columns of A corresponding to nonzero coordinates of x^0 and y^0 respectively, that is,

$$b=L(B)=M(C),$$

where L, M denote linear combinations. Extending C in A to a basis, say C^*, for the span of A, and substituting in $L(B)$ for each column of B its (certainly integral) representation in C^*, yields an integral representation of b in C^*, which representation, because of uniqueness, is identical with $M(C)$.

Next we observe that (1.6) is equivalent to

(1.7) THE DANTZING PROPERTY. If a column of A is a linear combination of a set of independent columns of A, then the coefficients in the combination are 1, -1, or 0.

To see that (1.6) implies (1.7): a representation of a column d where a column c enters with coefficient $\alpha \neq 0$, yields a representation of c where d enters with coefficient $1/\alpha$.

After these remarks the question can be rephrased as: which matrices A satisfy (1.7)?

Recent investigations on the subject comprise the following.

In the so-called Transportation Problem, there appears a matrix D, which G. Dantzig [1] showed to have the property (1.7). This fact was used by T. C. Koopmans and Dantzig to prove the existence of integral solutions to the mentioned problem, and by Dantzig [1] to establish a simplified computational procedure for solving the problem.

The mentioned matrix D appears partitioned into an upper and a lower submatrix, and the columns of D consist of all possible vectors having a single 1 in each of the two submatrices and zeros everywhere else. If e_ν denotes the νth unit vector, then

$$D=\{e_i+e_j\} \qquad (i=1, 2, \cdots, m;\ j=m+1, \cdots, m+n=r) \tag{1.8}$$

Later C. Tompkins and the author [2] showed the property (1.7) to hold for a somewhat larger class of matrices:

If

$$u_1, u_2, \cdots, u_m, \quad v_1, v_2, \cdots, v_n$$

is a set of linearly independent vectors in r-dimensional vector space ($r \geqq m+n$), then the set

$$T=\{\pm u_i,\ \pm v_j,\ \pm(u_i+v_j),\ (u_i-u_{i^*}),\ (v_j-v_{j^*})\} \tag{1.9}$$

$$(i,\ i^*=1, 2, \cdots, m;\ j, j^*=1, 2, \cdots, n)$$

has property (1.7).

Finally A. J. Hoffman and J. Krushall [5] showed property (1.7) to hold for several classes of incidence matrices associated with partially ordered sets.

The property (1.7) will be referred to as *Dantzig property* throughout this note. The term *unimodular property* has also been proposed and used [5]. This term seems quite appropriate for the case of incidence matrices, as in [5], where nonsingular submatrices then represent unimodular transformations; in the general case it is the transition from one basis in the matrix to another that is a unimodular transformation.

2. Unification of prior results. This is achieved by a few trivial observations.

First, since the Dantzig property does not depend on the order in which the columns of A are arranged, it is convenient to interpret A simply as a set of vectors.

Second, the Dantzig Property is invariant under nonsingular linear transformations, hence if A has the property, so does the image of A under a nonsingular linear transformation.

Third, in (1.9) the partition of the set of vectors into two sets $\{u_i\}$ and $\{v_j\}$ is rather artificial. If, for instance, we substitute $-w_j$ for v_j, (1.9) becomes

$$T=\{\pm u_i,\ \pm w_j,\ \pm(u_i-w_j),\ (u_i-u_{i^*}),\ (w_{j^*}-w_j)\}$$

which shows that T can be simply described by

$$(2.1)\qquad T=\{\pm x_i,\ (x_i-x_j)\}\qquad (i\neq j;\ i,j=1,2,\cdots,r),$$

or

$$(2.2)\qquad T=\{x_i-x_j\}\qquad (i\neq j;\ i,\ j=0,1,\cdots,r),$$

where x_0 denotes the null vector, and $x_1, x_2, \cdots, x_r$ are linearly independent vectors.

In the last formulation T is the set of differences of the x_i. Since differences are invariant under translations, the x_i in (2.2) may also be specified as a set of $r+1$ vectors whose affine span (all linear combinations with coefficients sum equal 1) is of dimension r; in other words, the x_i are the vertices of an r-simplex. This reduces the result (1.9) of [2] to the simple statement:

(2.3) *The set of edges (that is, one-dimensional faces, taken in both orientations and interpreted as vectors) of a simplex has the Dantzig property.*

In this form the statement is nearly obvious. Clearly, a basis B among the edges:

(i) contains all the $r+1$ vertices (otherwise the vectors of B would be among the edges of a lower-dimensional simplex, and hence not a basis for the span of all edges),

(ii) is connected (otherwise the vectors of B would be among the edges of two simplices of s and $r+1-s$ vertices, so that $\dim B \leqq s-1+r-s=r-1$),

(iii) is free of cycles (the vectors of a cycle being linearly dependent).

Hence B is a tree containing all vertices and r oriented segments. Any edge not in B closes a chain in B, which proves the statement.

Using the statement (2.3) one can show the Dantzig property to hold for a series of incidence matrices (incidence matrices are defined here simply as having only 0's and ± 1's as entries), some of which can be identified with matrices exhibited in [5]. Let E be Euclidean n-space, S an n-simplex in E, T the set of edges of S and B a maximal independent subset of T, hence a basis in S. If B is taken as the basis for the coordinate system, the representation of T is the set of columns of an incidence matrix with Dantzig property.

It is worthwhile to follow this somewhat closer. Since choosing a basis among the vectors of T amounts to choosing, in the net of vertices and edges of S, a tree containing all $n+1$ vertices and n oriented segments, the construction leads to as many essentially different incidence matrices as there are graphically different trees of $n+1$ vertices (note that permutation of columns or rows in a matrix preserves the Dantzig property, so that matrices obtained from each other by such permutations may be considered as equivalent; by *essentially different* we then mean *not equivalent*).

We point out two particular choices.

(i) The star consisting of all edges radiating from a given vertex and oriented from this vertex to the remaining vertices. This yields the set T of (2.1) with the x_i as unit vectors.

(ii) The oriented chain obtained by numbering the vertices from 0 to n and taking the set of oriented edges

$$x_1-x_0,\ x_2-x_1,\ \cdots,\ x_n-x_{n-1}\,.$$

If these vectors are taken as basis in the listed order, then the representation of all edges in this basis is the set of all columns that have a consecutive string of 1's or (-1)'s, and 0's everywhere else. This is a result of [5].

Obviously the transition from one basis to another is a unimodular transformation.

3. Maximal Dantzig sets.

Since with a set D each subset of D has the Dantzig property, or briefly is a *Dantzig set*, the interest lies in determining maximal Dantzig sets.

Obviously a maximal Dantzig set contains with each vector x also $-x$. Further, it should contain, but we agree to exclude, the null vector.

(3.1) *A set T consisting of the edges of a simplex is a Dantzig set which is maximal for its dimension (in the sense that there is no Dantzig set of the same dimension properly containing T).*

Proof. We have to show that when a vector x not belonging to T is adjoined to T, the new set does not have the Dantzig property. In the representation (2.1) with the x_i as basis vectors, x will have at least two coordinates of the same sign (both $=1$ or both $=-1$), since all other possibilities are already in T. Say

$$x=x_1+x_2+L(x_3,\ \cdots,\ x_n)\,,$$

where L denotes linear combination. But then

$$x=(x_1-x_2)+2x_2+L(x_3,\ \cdots,\ x_n)\,,$$

that is, the representation of x in the basis

$$x_1-x_2,\ x_2,\ x_3,\ \cdots,\ x_n$$

does not satisfy the Dantzig property, since the coefficient of x_2 equals $2\neq 0,\ \pm 1$.

The question whether every maximal Dantzig set is the set of edges of a simplex will obtain a negative answer by an example. We first note that in order to test whether a Dantzig set D can be extended to contain an additional vector b without losing the Dantzig property, it is sufficient to test the representation of b in every basis of D. That is:

(3.2) *Let D be a Dantzig set, b a vector not in D, and C the union of D and $\{b\}$. Then C has the Dantzig property if and only if the coordinates of b with respect to every basis in D consist of* 0's *and* ± 1's.

To see (indirectly) that the condition is sufficient, let d be a vector of C, B a basis in C, and let the representation of d in B have a coefficient $\neq 0, \pm 1$. Then obviously $d \neq b$, b is in B, and the coefficient of b is not 0:

$$d = \lambda_1 b + \lambda_2 b_2 + \cdots + \lambda_n b_n \quad (\lambda_1 \neq 0;\ \text{some } \lambda_i \neq 0,\ \pm 1)$$

But then the representation of b in the basis $\{d, b_2, \cdots, b_n\}$ contradicts the condition. This proves (3.2), since the necessity of the condition is obvious.

Further we formulate a necessary consistency condition for the Dantzig property which will be helpful in the sequel. Let b and d be two vectors in a Dantzig set D, and C a basis in D. Comparing the representations of b and d in C, we consider those vectors of C (if any) that enter with nonzero coefficients in both representations; say these are $c_1, c_2, \cdots, c_s$, so that

$$b = \beta_1 c_1 + \beta_2 c_2 + \cdots + \beta_s c_s + \cdots; \quad d = \gamma_1 c_1 + \gamma_2 c_2 + \cdots + \gamma_s c_s + \cdots$$

$$(\beta_i \neq 0 \neq \gamma_i:\ i = 1, 2, \cdots, s)$$

Obviously $\beta_i = {}_i\gamma \epsilon_i$; $\epsilon_i = \pm 1$. However, we confirm that ϵ_i remains constant, that is

$$\beta_i = \epsilon \gamma_i \qquad (i = 1, 2, \cdots, s) \tag{3.3}$$

where $\epsilon = \text{constant} = \pm 1$.

Proof (indirect). Assume

$$b = c_1 + c_2 + \cdots, \quad d = c_1 - c_2 + \cdots .$$

Replacing c_1 by d yields a new basis in which b is represented by $b = d + 2c_2 + \cdots$, contradicting the Dantzig Property. This proves (3.3), which excludes "mixed incidences" (and permits to assign an "incidence number" 0, 1, -1 to every pair of vectors, with respect to a given basis in D).

Finally we give the example:

(3.4) *Let e_1, e_2, e_3, e_4 be independent vectors and*

$$A=\{\pm e_1,\ \pm e_2,\ \pm e_3,\ \pm e_4,\ \pm(e_1+e_2+e_3+e_4)\ ,$$
$$\pm(e_1+e_2),\ \pm(e_2+e_3),\ \pm(e_3+e_4),\ \pm(e_4+e_1)\}\ .$$

Then A is a maximal Dantzig set which is not the set of edges of a simplex.

To see that A has the Dantzig property, we note that the subset A^*, obtained from A by deleting $\pm(e_1+e_2+e_3+e_4)$, consists of (not all) edges of the simplex S given by the vertices

$$o,\ e_1,\ -e_2,\ e_3,\ -e_4\ .$$

Hence A^* is a Dantzig set. The deleted vector is represented with coefficients 0, ± 1 in every basis of A^*, as seen by direct verification. By (3.2) this implies that A has the Dantzig property.

To see that A is maximal, assume a vector h can be adjoined to A without disturbing the Dantzig property. If h is expressed in the basis e_1, e_2, e_3, e_4, then the nonzero coefficients are all equal, otherwise h would have "mixed incidence" with $d=e_1,+e_2+e_3+e_4$ in that basis and contradict (3.3). This leaves for h the following possibilities:

$$\pm h=e_1+e_3$$

$$\pm h=e_2+e_4$$

$\pm h=e_1+e_2+e_3=d-e_4$ and the equivalents.

However, each of these possibilities contradicts the Dantzig property, since, after adequate choice of bases, we obtain:

$$e_1+e_3=(e_1+e_2)+(e_2+e_3)-2e_2$$

$$e_2+e_4=(e_1+e_2)+(e_1+e_4)-2e_1$$

$$e_1+e_2+e_3=(e_1+e_4)+e_2+(e_3+e_4)-2e_4\ .$$

Finally A has 18 elements and therefore is not the set of 20 edges of a simplex (of dimension 4).

4. The two theorems in this section are prepared by the following lemma:

(4.1) *The image D' of a Dantzig set D under a projection, along a subspace N spanned by vectors of D, is a Dantzig set.*

Proof. Let D be in a vectorspace V, both of dimension n,

N the span of $\{d_1, d_2, \cdots, d_k\} \subset D$, ($k < n$; for $k=n$ the lemma is trivial),
M the range of the projection (some complement of N in V),
$\{b'_1, b'_2, \cdots, b'_s\}$ a basis (for M) in D' (hence $k+s=n$),
$\{b_1, b_2, \cdots, b_s\}$ some set of originals in D (that is b'_i is image of b_i),
b' an arbitrary vector in D',
b an original of b' in D, and
$b'=\beta_1 b'_1+\beta_2 b'_2+\cdots+\beta_s b'_s$.

Clearly the set $B=\{d_1, d_2, \cdots, d_k, b_1, b_2, \cdots, b_s\}$ is a basis (for V) in D (a nontrivial representation of o could not have all its nonzero coefficients attached to the d_i alone, since these are independent; on the other hand, nonzero coefficients of the b_i would imply dependence for the b'_i). Therefore b is representable in B:

$$b=\gamma_1 d_1+\cdots+\gamma_k d_k+\beta_1 b_1+\cdots+\beta_s b_s\ ,$$

where all coefficients, and hence in particular the β_i, are 0, ± 1, which proves the lemma.

(4.2) THEOREM. *A Dantzig set of dimension n contains at most $n(n+1)$ elements* (not counting the nullvector); *that is, if it contains $n(n+1)$ elements, then it is maximal.*

The proof is by induction on the dimension n. For $n=1$ the theorem is obvious. Assuming it holds for dimensions $<n$, we prove it to hold for $n (n \geqq 2)$.

Let D be a Dantzig set of dimension $n \geqq 2$ containing at least $n(n+1)$ elements. We may assume that D contains with each vector also its negative (otherwise we extend D to that effect, since adjoining the negatives does not remove the Dantzig property).

After choosing a basis $B=\{b_1, b_2, \cdots, b_n\}$ in D, D is projected along b_1 on the span of $\{b_2, b_3, \cdots, b_n\}$. Then the image D' of D is of dimension $\leqq n-1$, has the Dantzig property (by Lemma 4.1), and, excluding the nullvector, has at most $n(n-1)$ elements (by the induction's assumption that the theorem holds for dimensions $<n$).

We prove that D has at most, and hence exactly, $n(n+1)$ elements, in showing that the number of nonzero elements cannot be reduced, by the projection, by more than $2n=(n+1)n-n(n-1)$; this will be shown in two steps, namely:

(i) that a vector in D' is image of at most two originals in D, and
(ii) that the set of nonzero vectors with double originals consists of a linearly independent set and its negatives.

If distinct vectors x and y of D have the same nonzero image, then, with respect to the basis B, they coincide in all but their first coordinates. Further they cannot both have nonzero values for the first coordinate, since these would then have to be 1 and -1 and contradict the consistency condition (3.3). Therefore the first coordinate of the two vectors is 0 and ± 1 respectively. This implies that no three vectors can have the same nonzero image. If the image is 0, the only two originals are $\pm b_1$. Hence

(4.3) a vector x' in D' is the image of at most two vectors x and y in D; if $x \neq y$ and $x'=y' \neq 0$, then $x=x'$ and $y=x'\pm b_1$ (if $x'=0$, then $x=\pm b_1$, $y=\mp b_1$)

Denoting by D^* the set obtained from D' after removal of the null vector, let E^* be the set of vectors in D^* that have double originals in D. Since D contains with each vector also its negative, so does E^*. Furthermore E^* is also in D. If x' is in E^*, then its originals are

$$x' \text{ and } y=x'+\epsilon b_1 \qquad (\epsilon=\pm 1)$$

while the originals of $-x'$ are $-x'$ and $-y=-x'-\epsilon b_1$.
From the pair $-x'$, x' we choose one vector, call it d' so, that its originals are d' and $d=d'+b_1$. Making this choice from each such pair in E^*, we obtain the set $F^*=\{d_1', d_2', \cdots, d_s'\}$, where certainly $d_i' \neq \pm d_j'$ for $i \neq j$, and d_i' and $d_i=d_i'+b_1$ are the originals of d_i' in D.

An indirect proof will establish that the vectors of F^* are linearly independent. Obviously a linear relation between them must involve at least 3 vectors, say the first 3, with nonzero coefficients (which implies in particular that the assertion is true when F^* contains less than 3 vectors). We consider separately each of the two following possibilities

$$\text{(i)} \qquad d_1'=\pm(d_2'+d_3')+L(d_4', \cdots, d_t')$$

$$\text{(ii)} \qquad d_1'=d_2'-d_3'+L(d_4', \cdots, d_t'),$$

where L denotes a linear combination with nonzero coefficients throughout. We assume to have chosen, among all existing linear relations, the one that involves the smallest number of vectors. Then the vectors appearing on the right hand side, that is $d_2', d_3', d_4', \cdots, d_t'$, are linearly independent. Therefore each of the following two sets in D is also linearly independent:

$$\text{(a)} \qquad b_1, d_2, d_3, d_4', d_5', \cdots, d_t'$$

$$\text{(b)} \qquad b_1, d_2', d_3, d_4', d_5', \cdots, d_t'.$$

We now obtain,

in case (i): $d_1' = \pm(-2b_1 + d_2 + d_3) + L(d_4', \cdots, d_t')$

in case (ii): $d_1 = 2b_1 + d_2' - d_3 + L(d_4', \cdots, d_t')$,

hence in either case a contradiction to the Dantzig property (note that all vectors are in D).

This completes the proof that the vectors of F^* are linearly independent, which implies, because of $\dim D^* \leqq n-1$, that F^* contains at most $n-1$ vectors. Hence E^* contains at most $2(n-1)$ vectors.

Now, since E^* consists of all nonnull vectors with double originals and the null vector has two originals (namely $\pm b_1$), it follows that the number of vectors in D exceeds the number of vectors in D^* by at most $2n$. Since D^*, as a Dantzig set of dimension $\leqq n-1$, contains at most $n(n-1)$ vectors, it follows that D contains at most $n(n-1)+2n = n(n+1)$ vectors.

This completes the proof of Theorem (4.2), and, in addition yields the following conclusions, which will be used in the proof of next theorem.

From the assumption that D contains at least $n(n+1)$ vectors it now follows that

(4.4) D contains exactly $n(n+1)$ vectors
D^* contains exactly $n(n-1)$ vectors
F^* contains exactly $n-1$ vectors,

and hence

(4.5) $$F^* = \{d_1', d_2', \cdots, d_{n-1}'\} \text{ is a basis in } D^*.$$

(4.6) THEOREM. *If a Dantzig set D of dimension n contains $n(n+1)$ vectors (not counting the null vector), then D is the set of edges of an n-simplex.*

Proof. We will construct a basic $H = \{h_1, h_2, \cdots, h_n\}$ in D, such that every element of D which is not in H, is a difference of two elements of H. The mechanism that governs the construction is based on the obvious geometrical picture (assuming the theorem true).

We take over the projection, notation and facts from the proof of theorem (4.2); the assumptions made in that proof contain the assumptions of the present theorem as special case (note, that the induction's hypothesis made there, is now a true statement).

For ease of writing we renumber the vectors of F^* in (4.5) to

(4.7) $$F^* = \{d_2', d_3', \cdots, d_n'\} ,$$

and first show that

(4.8) *the representation of an element of* D^* *in the basis* F^* *has at most two nonzero coefficients.*

Proof (indirect). Let x' be in D^*, and

$$x' = \epsilon_2 d_2' + \epsilon_3 d_3' + \epsilon_4 d_4' + L(d_5', \cdots, d_n'); \qquad \epsilon_i = \pm 1.$$

We distinguish whether x' is, or is not, in D.

(i) x' is in D: We use the fact, that two of the ϵ_i are equal, say $\epsilon_2 = \epsilon_3 = 1$ (if $= -1$, we take $-x'$), and consider the basis in D (see page 1356):

$$b_1,\ d_2,\ d_3,\ d_4',\ d_5',\ \cdots,\ d_n' .$$

Then

$$x' = -2b_1 + d_2 + d_3 + \epsilon_4 d_4' + L ,$$

which contradicts the Dantzig property.

(ii) x' is not in D: Then its original $x = x' + \epsilon b_1$ ($\epsilon = \pm 1$) is in D, and we distinguish whether all three ϵ_i are equal or not. In the first case we may assume all $\epsilon_i = 1$ (otherwise we take $-x$), and obtain, after adequate choice of basis

$$x = (\epsilon - 3) b_1 + d_2 + d_3 + d_4 + L ,$$

where $\epsilon - 3 = -2$ or -4 contradicts the Dantzig property. In the second case, ϵ and one of the ϵ_i, say ϵ_2, have opposite sign. Then a contradiction is obtained by the coefficient of b_1 in the representation

$$x = (\epsilon - \epsilon_2) b_1 + \epsilon_2 d_2 + \epsilon_3 d_3' + \epsilon_4 d_4' + L .$$

This completes the proof of (4.8), and furthermore establishes the more specific assertions (i) and (ii) of the following statement:

(4.9) (i) *If* x' *of* $(D^* - E^*)$ *is in* D, *then* $x' = d_\mu' - d_\nu'$,
(ii) *If* y' *of* $(D^* - E^*)$ *is not in* D, *then* $\pm y' = d_\mu' + d_\nu'$,
(iii) *Conversely, for any two distinct* d_μ' *and* d_ν' *of* F^*, *either* $\pm x'$ *of* (i) *or* $\pm y'$ *of* (ii), *but not both, are in* $(D^* - E^*)$.

Part (iii) follows from the fact that D^* has $n(n-1)$ elements and the observation that the sum and the difference of d_μ' and d_ν' cannot both belong to the Dantzig set D^* because of the consistency condition (3.3).

By means of (4.9) F^* can be divided in (at most two) classes, by putting two distinct vectors of F^* into the same class when their difference is in D'.

We first prove that this is an equivalence relation. Reflexivity and symmetry are obvious. Transitivity is shown indirectly. Let only the first two of the following three differences be in D'

$$d'_i - d'_j,\ d'_j - d'_k,\ d'_k - d'_i\ .$$

Then in particular $d'_k - d'_i \neq o$, and hence by (4.9 iii), $d'_k + d'_i = d'$ is in D^*. But then

$$d' = (d'_i - d'_j) - (d'_j - d'_k) + 2d_j$$

violates the Dantzig property of D^*.

To see that there are at most two classes, we assume that d'_i, d'_j, d'_k belong to three distinct classes, which by (4.9 iii) implies that the sum of any two of the three vectors is in D^*. Then the representation

$$(d'_i + d'_k) = (d'_i + d'_j) + (d'_j + d'_k) - 2d'_j$$

violates the Dantzig property of D^*. This establishes that F^* decomposes in two classes

$$\mathrm{I} = \{d'_2,\ d'_3,\ \cdots,\ d'_k\}$$

$$\mathrm{II} = \{d'_{k+1},\ d'_{k+2},\ \cdots,\ d'_n\}$$

(where II may be empty), such that

(4.10) (i) the difference of two distinct vectors of the same class is in D^*

(ii) the (positive and negative) sum of two vectors of distinct classes is in D^*

(iii) the representations (i) and (ii) comprise all vectors of D^* which are not in E^*

We are now ready to construct the basis $H = \{h_1, h_2, \cdots, h_n\}$ of D, setting

(4.11) $\quad h_1 = b_1;\quad h_i = d'_i + b_1\ (2 \leqq i \leqq k);\quad h_j = -d'_j\ (k < j \leqq n)$.

That $h_i = d'_i + b_1 = d_i$ is in D, follows from the construction of the d'_i on page 1359.

To verify that every x of D is represented by either $x = \pm h_\nu$ or $x = h_\mu - h_\nu$ we consider the projection x' of x, so that $x = x' + \alpha b_1$ where α may be one of the values 0, 1, -1. We may disregard $\alpha = -1$ (which amounts to consider only one vector of each pair x, $-x$), and distinguish the following cases:

(a) $x' = o$

(b) $x' \neq o$ and $\alpha = 0$

(c) $x' \neq o$ and $\alpha = 1$.

(a) implies $x = b_1 = h_1$.

(b) implies $x = x'$, that is, x is in D^*; we distinguish (b1) x is in E^*, (b2) x is in $D^* - E^*$.

(b1) implies $\pm x = d'_\nu$; hence, according to whether d'_ν belongs to class I or II, we have either $\pm x = h_\nu - b_1 = h_\nu - h_1$ or $\pm x = -h_\nu$.

(b2) and (4.9 i) imply $x = d'_\mu - d'_\nu$, where the last two vectors are in the same class because of (4.10); hence either $x = h_\mu - h_\nu$ or $x = -(h_\mu - h_\nu)$.

(c) implies $x = x' + b_1$; we distinguish: (c1) x' is in E^*, (c2) x' is in $D^* - E^*$ and in D, (c3) x' is in $D^* - E^*$ and is not in D.

(c1) implies $x = \pm d'_\nu + b_1$; the negative sign would yield mixed incidence of x and $d_\nu = d'_\nu + b_1$ and hence contradict (3.3); this leaves only $x = d'_\nu + b_1$; hence either $x = h_\nu$ or $x = b_1 - h_\nu = h_1 - h_\nu$.

(c2) cannot occur, since $x' \neq x$ and x' in D imply that x' has two distinct originals in D and therefore x' is in E^*.

(c3), (4.9 ii) and (4.10) imply $x' = \pm(d'_i + d'_j)$; hence $x = \pm(d'_i + d'_j) + b_1$; the negative sign would yield $x = -d_i - d_j + 3b_1$ violating the Dantzig property. This leaves only

$$x = d'_i + d'_j + b_1 = h_i - h_j .$$

This completes the proof of Theorem (4.6).

5. Open questions. While the set of edges of a simplex, which we may briefly call "difference set", is maximal in the sense of statement (3.1), it is, by Theorems (4.2) and (4.6), also maximal in the sense that it contains the largest number of elements for its dimension. Obviously the class of all difference sets of a given dimension can be obtained from a single one of its members by nonsingular linear transformations, and we may consider the set

(5.1) $\quad D = \{e_i - e_j\} \quad (i \neq j;\ i, j = 0, 1, \cdots, n;\ e_0 = o;\ e_i = i\text{th unit vector})$

as a canonical representative of the class.

In regard to computational aspects we refer to [3].

For dimensions $n \geqq 4$ the example (3.4) establishes the existence of other maximal Dantzig sets of necessarily less than $n(n+1)$ elements. A classification of these sets has not been attempted, yet would certainly constitute the next natural step. The problem may be formulated as follows: Determine, for each dimension n, a complete (obviously finite) set of representatives $D_1, D_2, \cdots, D_k$ ($k = k(n)$) of maximal Dantzig sets, in the sense that

(i) two distinct D_i are not related by a linear transformation

(ii) every maximal Dantzig set of dimension n is the image of some D_i under a linear transformation.

6. Interpretations. Geometrically, the statement (3.1) and Theorem (4.6) solve the following problem: Given a set S of $n(n+1)/2$ (free) vectors $\neq o$ in Euclidean space, such that S is of dimension n, and does not contain the negative of any of its vectors; what is a necessary and sufficient condition that S may be so arranged in space as to form a simplex? Statement (3.1) gives the Dantzig property as obvious necessary condition, while Theorem (4.6) proves that it is also sufficient.

The considerations of this note were carried on in vector space in order to assure the benefit of intuition from the geometric picture. It is clear, however, that the study of Dantzig sets belongs properly to group theory; from the number field underlying the vector space only the integers are used, which amounts to actually restricting the considerations to an Abelian group. To interpret the results in terms of this structure, let G be a free Abelian group, and S a set of rank n, in G. The Dantzig property for S is, by § 1, precisely the condition that every set of n linearly independent elements of S span the same group as S. In particular; if S spans G, the Dantzig property means that every set of n linearly independent elements of S is a basis for G. The translation of statement (3.1) and Theorems (4.2) and (4.6) is immediate (compare [4]).

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THE GEORGE WASHINGTON UNIVERSITY